自由群のFricke指標環とJohnson準同型について

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Automorphism groups of free groups

- $F_n := \langle x_1, \ldots, x_n \rangle$: Free group of rank $n \geq 2$
- $H := F_n/[F_n, F_n] \cong \mathbb{Z}^{\oplus n}$: Abelianization of F_n

$$\rho : \mathrm{Aut}\, F_n \xrightarrow{\text{surj.}} \mathrm{Aut}(H)$$

$$\| $$

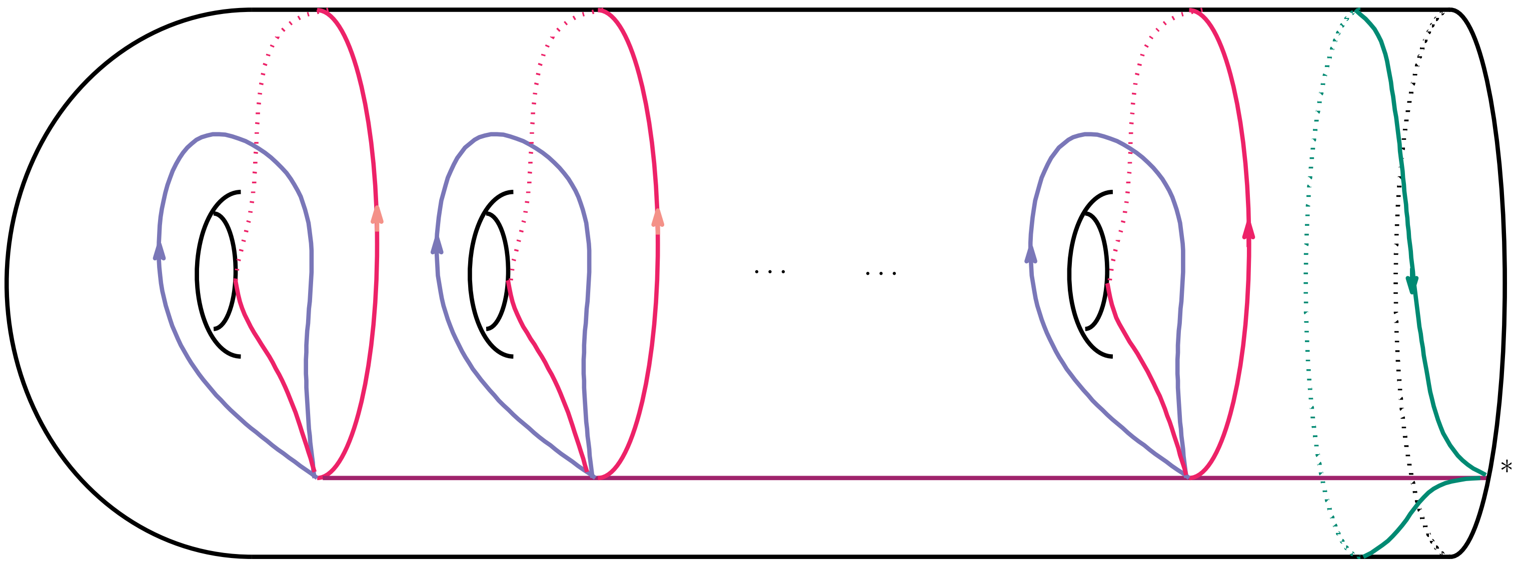
Nielsen, 1924 $\qquad\qquad \mathrm{GL}(n, \mathbb{Z})$

- $\mathrm{IA}_n := \mathrm{Ker}(\mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbb{Z}))$

Free group analogue of the Torelli group

Mapping class groups of surfaces

- $\Sigma_{g,1} :=$



$$\pi_1(\Sigma_{g,1}, *) \cong F_{2g}$$

- $\mathcal{M}_{g,1} := \mathrm{Diff}^+(\Sigma_{g,1}, \partial)/\mathrm{isotopy}$

Theorem (Dehn, Nielsen) $g \geq 1$

$\exists \iota : \mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}\, F_{2g}$ s.t.,

$$\mathrm{Im}(\iota) = \{\sigma \in \mathrm{Aut}\, F_{2g} \mid \zeta^{\sigma} = \zeta\}$$

- **Torelli group**

$$H_1(\Sigma_{g,1}, \mathbb{Z}) \curvearrowleft \mathcal{I}_{g,1} := \mathrm{IA}_{2g} \cap \mathcal{M}_{g,1}$$

trivially

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{IA}_{2g} & \to & \mathrm{Aut}\, F_{2g} & \xrightarrow{\rho} & \mathrm{GL}(2g, \mathbb{Z}) & \to & 1 \\ & & \uparrow & & \iota \uparrow & & \uparrow & & \\ 1 & \to & \mathcal{I}_{g,1} & \to & \mathcal{M}_{g,1} & \to & \mathrm{Sp}(2g, \mathbb{Z}) & \to & 1 \end{array}$$

Andreadakis-Johnson filtration of Aut F_n

- Lower central series of F_n

$$F_n = \Gamma_n(1) \supset \Gamma_n(2) \supset \Gamma_n(3) \supset \cdots$$

$$[\Gamma_n(k), \Gamma_n(l)] \subset \Gamma_n(k+l)$$

Fact (Magnus, Witt, Hall)

$\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$ is a free abelian group of finite rank.

- **Andreadakis-Johnson filtration** $\quad k \geq 1$

$$\mathcal{A}_n(k) := \mathrm{Ker}(\mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)))$$

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

Theorem (Andreadakis, 1965)

(1) $[\mathcal{A}_n(k), \mathcal{A}_n(l)] \subset \mathcal{A}_n(k+l)$

(2) $\displaystyle\bigcap_{k \geq 1} \mathcal{A}_n(k) = 1$

Johnson homomorphisms

- $H^* := \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$
- The **k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$**

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to \mathrm{Hom}_{\mathbb{Z}}(H, \mathcal{L}_n(k+1)) = H^* \otimes \mathcal{L}_n(k+1)$$

$$\sigma \mod(\mathcal{A}_n(k+1)) \mapsto (x \mapsto x^{-1}x^{\sigma})$$

Fact.

Each of τ_k is injective and $\mathrm{GL}(n, \mathbb{Z})$-equivariant.

S. Morita, R. Hain, F. Cohen, B. Farb, ...

The first Johnson homomorphisms

Theorem. (Cohen-Pakianathan, Farb, Kawazumi)

$$\tau_1 : \mathrm{gr}^1(\mathcal{A}_n) \to H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(2)$$

is surjective, and the abelianization of IA_n.

Theorem. (Kawazumi) For $n \geq 3$,

$$\mathrm{IA}_n \to \mathrm{gr}^1(\mathcal{A}_n) \xrightarrow{\tau_1} (H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(2)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

extends to $\mathrm{Aut}\, F_n$ as a crossed homomorphism.

- Cf. (Day, 2009) An extension of τ_k for $k \geq 1$.

- Johnson filtration : $\mathcal{M}_{g,1}(k) := \mathcal{A}_{2g}(k) \cap \mathcal{M}_{g,1}$
- $\mathrm{gr}^k(\mathcal{M}_{g,1}) := \mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1)$

Theorem. (Johnson, 1983)

$$\tau_1 : \mathrm{gr}^1(\mathcal{M}_{g,1}) \to \Lambda^3 H$$

detects the free part of the abelianization of $\mathcal{I}_{g,1}$.

Theorem. (Morita, 1993) For $g \geq 3$,

$$\mathcal{I}_{g,1} \to \mathrm{gr}^1(\mathcal{M}_{g,1}) \xrightarrow{\tau_1} \Lambda^3 H \otimes_{\mathbb{Z}} \mathbb{Q}$$

extends to $\mathcal{M}_{g,1}$ as a crossed homomorphism.

The first cohomology groups

Theorem. (Morita, 1989) $g \geq 3$

$$H^1(\mathcal{M}_{g,1}, \Lambda^3 H) = \mathbb{Z}^{\oplus 2}$$

Theorem. (S., 2009) $n \geq 6$,

$$H^1(\mathrm{Aut}\, F_n, H^* \otimes_{\mathbb{Z}} \Lambda^2 H) = \mathbb{Z}^{\oplus 2}$$

Fricke characters of F_n

- $R(F_n) := \mathrm{Hom}(F_n, \mathrm{SL}(2,\mathbb{C}))$

$$= \left\{ (a_i, b_i, c_i, d_i)_{1 \le i \le n} \in \mathbb{C}^{4n} \,|\, a_i d_i - b_i c_i = 1 \right\}$$

- $\mathcal{F}(n,\mathbb{C}) := \{\chi : R(F_n) \to \mathbb{C}\}$: $\mathbb{C}$-algebra

$$\chi,\ \chi' \in \mathcal{F}(n,\mathbb{C}), \quad \rho \in R(F_n), \quad \lambda \in \mathbb{C}$$

$$(\chi + \chi')(\rho) := \chi(\rho) + \chi'(\rho)$$
$$(\chi\chi')(\rho) := \chi(\rho)\chi'(\rho)$$
$$(\lambda\chi)(\rho) := \lambda\chi(\rho)$$

- $\sigma \in \mathrm{Aut}\, F_n, \quad \rho \in R(F_n),$

$$(\rho \cdot \sigma)(x) := \rho(x^{\sigma^{-1}}), \qquad x \in F_n$$

$R(F_n)$ and $\mathcal{F}(n, \mathbb{C}) \curvearrowleft \mathrm{Aut}\, F_n$

- Define a Fricke character $\mathrm{tr}\, x \in \mathcal{F}(n, \mathbb{C})$ by

$$(\mathrm{tr}\, x)(\rho) := \mathrm{tr}\, \rho(x)$$

for any $\rho \in R(F_n)$.

- $\sigma \in \mathrm{Aut}\, F_n, \quad (\mathrm{tr}\, x)^{\sigma} = \mathrm{tr}\, x^{\sigma}$
- $\rho \in R(F_n), \ (\mathrm{tr}\, 1_{F_n})(\rho) = 2$

Formulae for $\mathrm{tr}\, x$

- $\mathrm{tr}\, x^{-1} = \mathrm{tr}\, x$
- $\mathrm{tr}\, xy = \mathrm{tr}\, yx,$
- $\mathrm{tr}\, xy + \mathrm{tr}\, xy^{-1} = (\mathrm{tr}\, x)(\mathrm{tr}\, y)$
- $$\begin{aligned}\mathrm{tr}\, xyz + \mathrm{tr}\, yxz = {} & (\mathrm{tr}\, x)(\mathrm{tr}\, yz) + (\mathrm{tr}\, y)(\mathrm{tr}\, xz) \\ & + (\mathrm{tr}\, z)(\mathrm{tr}\, xy) - (\mathrm{tr}\, x)(\mathrm{tr}\, y)(\mathrm{tr}\, z)\end{aligned}$$
- $$\begin{aligned}2\mathrm{tr}\, xyzw = {} & (\mathrm{tr}\, x)(\mathrm{tr}\, yzw) + (\mathrm{tr}\, y)(\mathrm{tr}\, zwx) \\ & + (\mathrm{tr}\, z)(\mathrm{tr}\, wxy) + (\mathrm{tr}\, w)(\mathrm{tr}\, xyz) \\ & + (\mathrm{tr}\, xy)(\mathrm{tr}\, zw) - (\mathrm{tr}\, xz)(\mathrm{tr}\, yw) + (\mathrm{tr}\, xw)(\mathrm{tr}\, yz) \\ & - (\mathrm{tr}\, x)(\mathrm{tr}\, y)(\mathrm{tr}\, zw) - (\mathrm{tr}\, y)(\mathrm{tr}\, z)(\mathrm{tr}\, xw) \\ & - (\mathrm{tr}\, x)(\mathrm{tr}\, w)(\mathrm{tr}\, yz) \\ & - (\mathrm{tr}\, z)(\mathrm{tr}\, w)(\mathrm{tr}\, xy) + (\mathrm{tr}\, x)(\mathrm{tr}\, y)(\mathrm{tr}\, z)(\mathrm{tr}\, w)\end{aligned}$$

- $\mathfrak{X}_{\mathbb{Q}}(F_n) := \langle \operatorname{tr} x \,|\, x \in F_n \rangle_{\mathbb{Q}} \subset \mathcal{F}(n, \mathbb{C})$: $\mathbb{Q}$-subalgebra

Theorem (Horowitz, 1972) For $n \geq 1$,

As a ring, $\mathfrak{X}_{\mathbb{Q}}(F_n)$ is generated by $n + \binom{n}{2} + \binom{n}{3}$ elements

- $\operatorname{tr} x_i, \quad 1 \leq i \leq n$
- $\operatorname{tr} x_i x_j, \quad 1 \leq i < j \leq n$
- $\operatorname{tr} x_i x_j x_k, \quad 1 \leq i < j < k \leq n$

- $\mathbb{Q}$-polynomial ring

$$\mathbb{Q}[t] := \mathbb{Q}[t_i, t_{pq}, t_{stu} \,|\, 1 \le i \le n,\ \ 1 \le p < q \le n,\ \ 1 \le s < t < u \le n]$$

- $\pi : \mathbb{Q}[t] \to \mathcal{F}(n, \mathbb{C})$: ring homomorphism

$$1 \mapsto \frac{1}{2}(\mathrm{tr}\, 1_{F_n}), \quad t_{i_1 \cdots i_l} \mapsto \mathrm{tr}\, x_{i_1} \cdots x_{i_l}$$

$$\mathrm{Im}(\pi) = \mathfrak{X}_{\mathbb{Q}}(F_n)$$

- $I := \mathrm{Ker}(\pi)$
$$= \{f \in \mathbb{Q}[t] \mid f(\mathrm{tr}\,\rho(x_{i_1}\cdots x_{i_l})) = 0, \ \ \forall \rho \in R(F_n)\}$$

- The ring of Fricke characters of F_n over $\mathbb{Q}$

$$\mathfrak{X}_{\mathbb{Q}}(F_n) \cong \mathbb{Q}[t]/I$$

Theorem (Horowitz, 1972)

(1) For $n = 1,\ 2,\quad I = (0)$

(2) For $n = 3,\quad I = (t_{123}^2 - P_{123}(t)t_{123} + Q_{123}(t))$

$$P_{abc}(t) := t_{ab}t_c + t_{ac}t_b + t_{bc}t_a,$$
$$Q_{abc}(t) := t_a^2 + t_b^2 + t_c^2 + t_{ab}^2 + t_{ac}^2 + t_{bc}^2 - t_at_bt_{ab} - t_at_ct_{ac} - t_bt_ct_{bc} + t_{ab}t_{bc}t_{ac} - 4$$

- $t'_{i_1\cdots i_l} := t_{i_1\cdots i_l} - 2 \in \mathbb{Q}[t]$

$\forall\ f \in \mathbb{Q}[t]$, f is considered as a polynomial of $t'_{i_1\cdots i_l}$s.

- $J_0 := (t'_i,\ t'_{pq},\ t'_{stu} \mid i;\ p < q;\ s < t < u) \subset \mathbb{Q}[t]$

$$I \subset J_0, \quad \text{and} \quad J := J_0/I \subset \mathbb{Q}[t]/I.$$

Lemma. (For $n = 3$, Magnus)

The ideal J is Aut F_n-invariant.

- A descending filtration

$$J \supset J^2 \supset J^3 \supset \cdots$$

of Aut F_n-invariant ideals of $\mathbb{Q}[t]/I$

- $\mathrm{gr}^k(J) := J^k/J^{k+1}$: $\mathbb{Q}$-vector space of finite dim.

We want to extract group theoretic properties of Aut F_n from

$$\mathrm{gr}^k(J) := J^k/J^{k+1} \curvearrowleft \mathrm{Aut}\, F_n.$$

Theorem (Hatakenaka-S., 2012)

(1) A set

$$T := \{t'_i \mid 1 \leq i \leq n\} \cup \{t'_{pq} \mid 1 \leq p < q \leq n\} \cup \{t'_{stu} \mid 1 \leq s < t < u \leq n\}$$

is a basis of $\mathrm{gr}^1(J)$.

(2) We have obtained a basis of $\mathrm{gr}^2(J)$.

It seems too hard to write down a basis of $\mathrm{gr}^k(J)$ explicitly for $k \geq 3$.

New filtration of $\mathrm{Aut}\, F_n$

- $k \geq 1$,

$$\mathcal{E}_n(k) := \mathrm{Ker}(\mathrm{Aut}\, F_n \to \mathrm{Aut}(J/J^{k+1}))$$

Then we have a descending filtration

$$\mathcal{E}_n(1) \supset \mathcal{E}_n(2) \supset \cdots \supset \mathcal{E}_n(k) \supset \cdots$$

Theorem (Hatakenaka-S., 2012)

(1) $[\mathcal{E}_n(k), \mathcal{E}_n(l)] \subset \mathcal{E}_n(k+l)$ for any $k, l \geq 1$.

(2) $\mathcal{E}_n(1) = \mathrm{Inn}\, F_n \cdot \mathcal{A}_n(2)$.

(3) $\mathcal{A}_n(2k) \subset \mathcal{E}_n(k)$.

Graded quotients

- $\mathrm{gr}^k(\mathcal{E}_n) := \mathcal{E}_n(k)/\mathcal{E}_n(k+1)$

Theorem (Hatakenaka-S., 2012)

(1) $\mathrm{gr}^k(\mathcal{E}_n)$ is torsion-free.

(2) $\dim_{\mathbb{Q}}(\mathrm{gr}^k(\mathcal{E}_n) \otimes_{\mathbb{Z}} \mathbb{Q}) < \infty$.

In order to show this, we construct and use a Johnson homomorphism like homomorphism:

$$\eta_k : \mathrm{gr}^k(\mathcal{E}_n) \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^{k+1}(J))$$

$$\sigma \mapsto (f \mapsto f^\sigma - f)$$

The main theorem

Theorem (S., 2013) For $n \geq 3$,

$$\mathcal{E}_n(1) \to \mathrm{gr}^1(\mathcal{E}_n) \xrightarrow{\eta_1} \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J))$$

extends to $\mathrm{Aut}\, F_n$ as a crossed homomorphism.

- We showed that η is non-trivial in H^1.

$$H^1(\mathrm{Aut}\, F_n, \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J))) = ?$$

The keypoint of the proof

- We show that there exists a split exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J))$$

$$\to \mathrm{Aut}\,(J/J^3) \to \mathrm{Aut}\,(J/J^2) \to 1.$$

- We obtain a crossed homomorphism

$$\mathrm{Aut}\, F_n \to \mathrm{Aut}\,(J/J^3) \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^2(J))$$