Nomizu's Theorem and its extensions (2)

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Solvmanifolds nilmanifolds

G: a simply connected solvable Lie group
($\mathfrak{g}$: Lie algebra)
Γ a lattice(cocompact discrete subgroup of G)
G/Γ is called solvmanifold.
In particular, if G is nilpotent G/Γ is called nilmanifold.

Theorem (Nomizu)

For a nilmanifold G/Γ the inclusion

$$\bigwedge \mathfrak{g}^* \subset A^*(G/\Gamma)$$

induces an isomorphism

$$H^*(\mathfrak{g}) \cong H^*(G/\Gamma).$$

For a "solvmanifold" G/Γ, the isomorphism

$$H^*(\mathfrak{g}) \cong H^*(G/\Gamma)$$

does not hold.

Theorem (K. 2013)

For a solvmanifold G/Γ, we can obtain an explicite finite-dimensional sub-complex

$$A^*_\Gamma \subset A^*(G/\Gamma)$$

so that the inclusion induces an isomorphism

$$H^*(A^*_\Gamma) \cong H^*(G/\Gamma).$$

Note:
A^*_Γ depends on Γ.

In nilamnifolds case, the cohomology

$$H^*(G/\Gamma)$$

is computed by only G (not Γ)
However, in solvmanifolds case, the cohomology

$$H^*(G/\Gamma)$$

is computed by G and Γ (not only G).

Another setting and way for a better extension of Nomizu's theorem.

$\mathbb{Q}$-algebraic group $\mathbf{G}$

$\Longleftrightarrow$

Subgroup $\mathbf{G} \subset GL_n(\mathbb{R})$ which is an algebraic set defined by polynomials with $\mathbb{Q}$-coefficients.

We denote $\mathbf{G}(\mathbb{Q}) = \mathbf{G} \cap GL_n(\mathbb{Q})$.

Let G be a simply connected nilpotent Lie group with a lattice Γ.

Theorem (Malcev)

- Γ *is finitely generated nilpotent group with* $\mathrm{rank}\,\Gamma = \dim G$.
- G *can be considered as a unipotent* $\mathbb{Q}$*-algebraic group* $\mathbf{U}$.
- $\Gamma \subset \mathbf{U}(\mathbb{Q})$ *and* Γ *is Zariski-dense in* $\mathbf{U}$.

cohomologies

- For a group Γ and a Γ-module V, we consider the group cohomology $H^*(\Gamma, V) = \mathrm{Ext}^*_\Gamma(\mathbb{Q}, V)$.
- For a $\mathbb{Q}$-algebraic group $\mathbf{G}$ and a rational $\mathbf{G}$-module V, consider the rational cohomology $H^*(\mathbf{G}, V) = \mathrm{Ext}^*_{\mathbf{G}}(\mathbb{Q}, V)$.

Theorem (Hochschild)

Let $\mathbf{U}$ *a unipotent* $\mathbb{Q}$*-algebraic group with a Lie algebra* $\mathfrak{u}$*. Then we have an isomorphism*

$$H^*(\mathbf{U}, V) \cong H^*(\mathfrak{u}, V)$$

Thus we have the algebraic presentation of Nomizu's theorem

Theorem

$$H^*(\Gamma) \cong H^*(\mathbf{U})$$

Note: nilmanifold G/Γ is $K(\Gamma, 1)$ and so $H^*(\Gamma) = H^*(G/\Gamma)$.

A group Γ is polycyclic
$\iff \Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}$ s.t. Γ_{i-1}/Γ_i is cyclic.
We define

$$\mathrm{rank}\,\Gamma = \sum_{i=1}^{i=k} \mathrm{rank}\,\Gamma_{i-1}/\Gamma_i$$

It is known that a lattice Γ of a simply connected solvable Lie group G is a torsion-free polycyclic group with $\mathrm{rank}\,\Gamma = \dim G$.

Properties of $\mathbb{Q}$-algebraic groups

Theorem (Mostow)

A $\mathbb{Q}$-algebraic group $\mathbf{G}$ *decomposes as:*

$$\mathbf{G} = \mathbf{T} \ltimes \mathbf{U}(\mathbf{G})$$

where
$\mathbf{T}$ *is a maximal reductive subgroup and*
$\mathbf{U}(\mathbf{G})$ *is the maximal connected normal unipotent subgroup (Unipotent radical).*

Theorem (K. 2014)

Let Γ be a torsion-free polycyclic group. We suppose that $\Gamma \subset \mathbf{G}(\mathbb{Q})$ for a $\mathbb{Q}$-algebraic group $\mathbf{G}$ so that:

- $\mathrm{rank}\,\Gamma = \dim \mathbf{U}(\mathbf{G})$.
- Γ *is Zariski-dense in* $\mathbf{G}$

Then, for any rational $\mathbf{G}$-module V, the inclusion $\Gamma \subset \mathbf{G}$ induces an isomorphism

$$H^*(\mathbf{G}, V) \cong H^*(\Gamma, V)$$

Note:
For any torsion-free polycyclic group Γ, there exists a $\mathbb{Q}$-algebraic group as in the assumption of the Theorem (K). (Mostow, Raghunathan)
Moreover, the minimal one of these groups uniquely exists. (called the algebraic hull of Γ.)

Is this theorem presented geometrically?

Let K be a simplicial complex and V a $\pi_1 K$-module. We can define the V-valued $\mathbb{Q}$-polynomial differential forms on K and they gives the $\mathbb{Q}$-polynomial de Rham complex

$$A^*_{poly}(K, V)$$

For the simplicial cochain complex $C^*(K, V)$, we have the "integration" homomorphism

$$\int : A^*_{poly}(K, V) \to C^*(K, V)$$

which induces a cohomology isomorphism (Sullivan's simplicial de Rham theorem)

For a group Γ, we consider the classifying space $B\Gamma$ as a simplicial complex and its $\mathbb{Q}$-polynomial de Rham complex

$$A^*_{poly}(B\Gamma, V).$$

Invariant differential forms for a $\mathbb{Q}$-algebraic group

For a $\mathbb{Q}$-algebraic group $\mathbf{G}$ with a decomposition

$$\mathbf{G} = \mathbf{T} \ltimes \mathbf{U}(\mathbf{G})$$

and a rational $\mathbf{G}$-module V, the complex of G-invariant differential forms on $\mathbf{U}(\mathbf{G})$ is

$$\left(\bigwedge \mathfrak{u}^* \otimes V \right)^T$$

where $\mathfrak{u}$ is the Lie algebra of $\mathbf{U}(\mathbf{G})$.

Theorem

Let Γ be a group, $\mathbf{G}$ a $\mathbb{Q}$-algebraic group and $\rho : \Gamma \to \mathbf{G}(\mathbb{Q})$ a homomorphism.
Then, we have an explicit homomorphism

$$\left(\bigwedge \mathfrak{u}^* \otimes V\right)^T \to A^*_{poly}(B\Gamma, V)$$

which incuces the map

$$H^*(\mathbf{G}, V) \to H^*(\Gamma, V).$$

Theorem (Simplicial extended Nomizu's theorem, K. 2015)

Let Γ be a torsion-free polycyclic group. We suppose that $\Gamma \subset \mathbf{G}(\mathbb{Q})$ for a $\mathbb{Q}$-algebraic group $\mathbf{G}$ so that:

- $\mathrm{rank}\,\Gamma = \dim \mathbf{U}(\mathbf{G})$.
- *Γ is Zariski-dense in $\mathbf{G}$*

Then we have an explicit homomorphism

$$\psi : \left(\bigwedge \mathfrak{u}^* \otimes V\right)^T \to A^*_{poly}(B\Gamma, V)$$

which induces a cohomology isomorphism.

Take $V = \mathbb{Q}[\mathbf{T}]$ (the ring of polynomial functions on $\mathbf{T}$). Then the de Rham complex $A^*_{poly}(B\Gamma, \mathbb{Q}[\mathbf{T}])$ is a DGA. We have

$$\left(\bigwedge \mathfrak{u}^* \otimes \mathbb{Q}[\mathbf{T}]\right)^T = \bigwedge \mathfrak{u}^*.$$

Hence we have:

Theorem

$\bigwedge \mathfrak{u}^*$ is the (explicit) minimal model of the DGA $A^*_{poly}(B\Gamma, \mathbb{Q}[\mathbf{T}])$.

Malcev completion and Nomizu's theorem

Let Γ be a group.
The Malcev completion of Γ is

$$\mathcal{U}_{\Gamma} = \varprojlim \mathbf{U}$$

where the invers limit runs over the unipotent algebraic groups $\mathbf{U}$ such that there are homomorphism $\Gamma \to \mathbf{U}$ with the Zariski-dense image.
Eg. Γ is a finitely generated nilpotent group. $\mathcal{U}_{\Gamma}$ is the unipotent goup as in Malcev theorem.

Let A^* be a DGA. The 1-minimal model of A^* is a minimal DGA

$$\mathcal{M}^* = \bigwedge \langle x_i \rangle_{i \in I}$$

so that:

- $deg(x_i) = 1$ for any $i \in I$.
- There exists a DGA map $\mathcal{M}^* \to A^*$ which induces an isomorphism

$$H^1(\mathcal{M}^*) \cong H^1(A^*)$$

  and an injection

$$H^2(\mathcal{M}^*) \hookrightarrow H^2(A^*)$$

Theorem (Sullivan, Chen)

Let M be a manifold or simplicial complex and $\Gamma = \pi_1 M$. Then the 1*-minimal model of the DGA $A^*(M)$ is*

$$\bigwedge \mathfrak{u}^*$$

where $\mathfrak{u}$ is the Lie algebra of the Malcev completion $\mathcal{U}_\Gamma$ of Γ.

Problem: What is a map

$$\bigwedge \mathfrak{u}^* \to A^*_{poly}(B\Gamma)?$$

For $\Gamma \to \mathbf{U}$, we have a homomorphism

$$\psi : \bigwedge \mathfrak{u}^* \to A^*_{poly}(B\Gamma)$$

which induces

$$H^*(\mathbf{U}) \to H^*(\Gamma).$$

Taking limit,

$$\bigwedge \mathfrak{u}_\Gamma \to A^*_{poly}(B\Gamma)$$

which induces

$$H^*(\mathcal{U}_\Gamma) \to H^*(\Gamma)$$

where $\mathfrak{u}_\Gamma$ is the Lie algebra of $\mathcal{U}_\Gamma$.

We can show that

$$H^k(\mathcal{U}_\Gamma) \to H(\Gamma)$$

is an isomorphism for $k = 1$ and injective for $k = 2$.
Thus we have:

Theorem

The map

$$\bigwedge \mathfrak{u}_\Gamma \to A^*_{poly}(B\Gamma)$$

induces an isomorphism on the first cohomology and an injection on second cohomology.
Hence $\bigwedge \mathfrak{u}_\Gamma$ is the 1-minimal model of $A^*_{poly}(B\Gamma)$.

Extension

Γ: group,
T reductive $\mathbb{Q}$-algebraic group
$\rho : \Gamma \to T$ homomorphism with Zariski-dense image.
Then the "ρ-relative Malcev completion" of Γ is

$$\mathcal{G}_{\rho,\Gamma} = \varprojlim \mathbf{G}$$

$\varprojlim$ runs over $\mathbf{G} = \mathbf{T} \ltimes \mathbf{U}$ so that: $\exists \Gamma \to \mathbf{G}$ with the Zariski-dense image and composition
$\Gamma \to \mathbf{G} = \mathbf{T} \ltimes \mathbf{U} \to \mathbf{T}$ is ρ.
By construction

$$\mathcal{G}_{\rho,\Gamma} = \mathbf{T} \ltimes \mathcal{U}_{\rho,\Gamma}.$$

Theorem (K. 2015, cf. Hain)

K simplicial complex.
$\Gamma = \pi_1 K$
Consider the DGA

$$A^*_{poly}(K, \mathbb{Q}[T])$$

Then we have an explicit homomorphism

$$\bigwedge \mathfrak{u}^*_{\rho,\Gamma} \to A^*_{poly}(K, \mathbb{Q}[T])$$

which induces an isomorphism on the first cohomology and an injection on second cohomology where $\mathfrak{u}_{\rho,\Gamma}$ is the lie algebra of $\mathcal{U}_{\rho,\Gamma}$.
Hence $\bigwedge \mathfrak{u}^*_{\rho,\Gamma}$ is the 1-minimal model of $A^*_{poly}(K, \mathbb{Q}[T])$.

Proof

For $\Gamma \to \mathbf{T} \ltimes \mathbf{U}$, we have a homomorphism

$$\psi : \bigwedge \mathfrak{u}^* \to A^*_{poly}(B\Gamma, \mathbb{Q}[\mathbf{T}])$$

which induces

$$H^*(\mathbf{G}, \mathbb{Q}[\mathbf{T}]) \to H^*(\Gamma, \mathbb{Q}[\mathbf{T}]).$$

Taking limit,

$$\bigwedge \mathfrak{u}_{\rho,\Gamma} \to A^*_{poly}(B\Gamma, \mathbb{Q}[\mathbf{T}])$$

which induces

$$H^*(\mathcal{G}_{\rho,\Gamma}, \mathbb{Q}[\mathbf{T}]) \to H^*(\Gamma, \mathbb{Q}[\mathbf{T}])$$