ON THE COHOMOLOGY OF FREE LOOP SPACES AND HOMOTOPY FIXED POINTS

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Reference

- D. Kishimoto and A. Kono, *On the cohomology of free and twisted loop spaces*, J. Pure Appl. Algebra **214** (2010), no. 5, 646-653.
- K. Kuribayashi, *Module derivations and the adjoint action of a finite loop space*, J. Math. Kyoto Univ. **39** (1999), 67-85.

Spaces and maps will be pointed.

Part 1. **Free loop spaces**

1. Motivation

Aim : Give an explicit description of $H^*(LX)$.

The most popular way to compute $H^*(LX)$ is the Eilenberg-Moore spectral sequence of a homotopy pullback

$$\begin{array}{ccc} LX & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$

Good points are that things are purely algebraic and $\exists$helpful tools. **Bad points** are that the extensions are too hard and results are less geometric.

Our policy : Don't use spectral sequences.

2. Free cohomology suspension

The coefficient of the cohomology will be a ring R. Let X be a simply connected space.

Observation on ΩX : The cohomology suspension $\sigma(x)$ of $x \in \overline{H}^n(X)$, equivalently $x : X \to K(R,n)$, is

$$\Omega x : \Omega X \to \Omega K(R,n) = K(R,n-1).$$

$\exists$commutative diagram

$$\begin{array}{ccccc} \Sigma\Omega X & \xrightarrow{\bar{\omega}} & X & \xrightarrow{x} & K(R,n) \\ \downarrow{\scriptstyle \Sigma\Omega x} & & & & \Vert \\ \Sigma\Omega K(R,n) & & \xrightarrow{\bar{\omega}} & & K(R,n), \end{array}$$

where $\bar{\omega}(t,\ell)=\ell(t)$ is the evaluation map. Then

$$\bar{\omega}^*(x)=s\otimes\sigma(x)$$

for the dual $s\in H^1(S^1)$ of the Hurewicz image of $[1_{S^1}]\in\pi_1(S^1)$.

Let $\hat{\omega}:S^1\times LX\to X$ be the evaluation $\hat{\omega}(t,\ell)=\ell(t)$.

Definition . The *free cohomology suspension*

$$\hat{\sigma}:\overline{H}^*(X)\to H^{*-1}(LX)$$

is defined as

$$\hat{\omega}^*(x)=s\otimes\hat{\sigma}(x)+1\otimes x,$$

where we regard $H^*(X)\subset H^*(LX)$ by the evaluation $LX\to X$ at the basepoint of S^1.

Remark . Kuribayashi called $\hat{\sigma}$ a module derivation and used it to solve the extension of the above Eilenberg-Moore spectral sequence.

Proposition . (1) *For $f:X\to Y$,*

$$Lf^*\circ\hat{\sigma}=\hat{\sigma}\circ f^*.$$

(2) *For the inclusion $i:\Omega X\to LX$,*

$$i^*\circ\hat{\sigma}=\sigma.$$

(3) *$\hat{\sigma}$ is a derivation.*

(4) *$\hat{\sigma}$ commutes with Steenrod operations.*

Proof. (1) follows from naturality of $\hat{\omega}$. We get (2) by the above observation on ΩX. For $x,y\in\overline{H}^*(X)$,

$$\begin{aligned}\hat{\omega}(xy)&=s\otimes\hat{\sigma}(xy)+1\otimes xy\\&=\hat{\omega}^*(x)\hat{\omega}^*(y)=(s\otimes\hat{\sigma}(x)+1\otimes x)(s\otimes\hat{\sigma}(y)+1\otimes y)\\&=s\otimes(\hat{\sigma}(x)y+(-1)^{|x|}x\hat{\sigma}(y))+1\otimes xy,\end{aligned}$$

implying (3). For any Steenrod operation α, we have $\alpha(s)=0$, and then for a Steenrod operation α, we have

$$\begin{aligned}\alpha(\hat{\omega}^*(x))&=s\otimes\alpha(\hat{\sigma}(x))+1\otimes\alpha(x)\\&=\hat{\omega}^*(\alpha(x))=s\otimes\hat{\sigma}(\alpha(x))+1\otimes\alpha(x),\end{aligned}$$

implying (4). □

Theorem . *If $H^*(X)=R[X_1,\dots,x_n]$, then*

$$H^*(LX)=R[x_1,\dots,x_n]\otimes\Delta(\hat{\sigma}(x_1),\dots,\hat{\sigma}(x_n)).$$

Proof. By the Borel transgression theorem, we have

$$H^*(\Omega X) = \Delta(\sigma(x_1), \dots, \sigma(x_n)).$$

Then since $\hat{\sigma}$ restricts to σ, the result follows from the Leray-Hirsch theorem applied to a fiber sequence $\Omega X \to LX \to X$. □

3. Example calculation

Let us calculate $H^*(LBG_2; \mathbb{Z}/2)$.

Data : $H^*(BG_2) = \mathbb{Z}/2[x_4, x_6, x_7]$, $|x_i| = i$.

	x_4	x_6	x_7
Sq^1	0	x_7	0
Sq^2	x_6	0	0
Sq^4	x_4^2	x_4x_6	x_4x_7

Theorem . *For* $\hat{x}_i = \hat{\sigma}(x_i)$,

$$H^*(LBG_2) = \mathbb{Z}/2[x_4, x_6, x_7, \hat{x}_3, \hat{x}_5]/(\hat{x}_5^2 + \hat{x}_3x_7 + x_4\hat{x}_3^2, \hat{x}_3^4 + \hat{x}_5x_7 + x_6\hat{x}_3^2).$$

Proof. By the above theorem,

$$H^*(LBG_2) = \mathbb{Z}/2[x_4, x_6, x_7] \otimes \Delta(\hat{x}_3, \hat{x}_5, \hat{x}_6).$$

Then our task is to compute $\hat{x}_i^2$. By the Adem relation,

$$\mathrm{Sq}^3 = \mathrm{Sq}^1\mathrm{Sq}^2, \quad \mathrm{Sq}^5 = \mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^1, \quad \mathrm{Sq}^6 = \mathrm{Sq}^5\mathrm{Sq}^1 + \mathrm{Sq}^2\mathrm{Sq}^4,$$

and thus

$$\begin{aligned}
\hat{x}_3^2 &= \mathrm{Sq}^3\hat{x}_3 = \hat{\sigma}(\mathrm{Sq}^3x_4) = \hat{\sigma}(x_7) = \hat{x}_6,\\
\hat{x}_5^2 &= \mathrm{Sq}^5\hat{x}_5 = \hat{\sigma}(\mathrm{Sq}^5x_6) = \hat{\sigma}(x_4x_7) = \hat{x}_3x_7 + x_4\hat{x}_6,\\
\hat{x}_6^2 &= \mathrm{Sq}^6\hat{x}_6 = \hat{\sigma}(\mathrm{Sq}^6x_7) = \hat{\sigma}(x_6x_7) = \hat{x}_5x_7 + x_6\hat{x}_6.
\end{aligned}$$

□

4. Invariant theory

There is a close relationship between the polynomial invariants of reflection groups and the cohomology of Lie groups as follows. Let V be a vector space over a field $\Bbbk$ of dimension n. If $V = \langle x_1, \dots, x_n\rangle$, we put

$$\Bbbk[V] = \Bbbk[x_1, \dots, x_n]$$

which is independent of a choice of $x_1, \dots, x_n$. Note that a group action on V extends canonically to $\Bbbk[V]$.

Theorem ((a small part of) Shephard-Todd)**.** *If W is a finite group generated by reflections on V and* $\operatorname{char}\Bbbk \nmid |W|$*, then*

$$\Bbbk[V]^W = \Bbbk[q_1, \dots, q_n]$$

for some $q_1, \dots, q_n \in \Bbbk[V]$.

Let G be a compact, connected Lie group with the Weyl group $W(G)$. Then $W(G)$ is generated by reflections on $H^2(BT;\Bbbk)$. Then if $\operatorname{char}\Bbbk \nmid |W(G)|$, there is a natural isomorphism

$$H^*(BG;\Bbbk) \xrightarrow{\cong} H^*(BT;\Bbbk)^{W(G)}.$$

In fact, the above holds if $H_*(G;\mathbb{Z})$ has no p-torsion, where $p = \operatorname{char}\Bbbk$.

The Shephard-Todd theorem is generalized to polynomial tensor exterior algebras as follow. Fix an isomorphism $f : V \xrightarrow{\cong} \widehat{V}$. Then a group action on V is translated to $\widehat{V}$ through f and extended to the group action on $\Bbbk[V] \otimes \Lambda(\widehat{V})$. We also have a derivation

$$\bar{f} : \Bbbk[V] \to \Bbbk[V] \otimes \Lambda(\widehat{V})$$

extending f.

Theorem (Solomon)**.** *If W is a finite group generated by reflections on V and* $\operatorname{char}\Bbbk \nmid |W|$*, then*

$$(\Bbbk[V] \otimes \Lambda(\widehat{V}))^W = \Bbbk[q_1, \dots, q_n] \otimes \Lambda(\bar{f}(q_1), \dots, \bar{f}(q_n)),$$

where $\Bbbk[V]^W = \Bbbk[q_1, \dots, q_n]$.

This generalization of the Shephard-Todd theorem applies to free loop spaces of the classifying spaces of Lie groups.

Theorem . *Let G be a compact, connected Lie group. If* $\operatorname{char}\Bbbk \nmid |W(G)|$*, there is a natural isomorphism*

$$H^*(LBG;\Bbbk) \xrightarrow{\cong} H^*(LBT;\Bbbk)^{W(G)}.$$

Proof. In Solomon's theorem, we put $V = H^2(BT)$ and $f = \hat{\sigma}$. Then the result follows. ☐

Remark . We don't have the above isomorphism if $\operatorname{char}\Bbbk \mid |W(G)|$ but $H_*(G;\mathbb{Z})$ is torsion free. For example, put $G = \mathrm{Sp}(1)$. Then $H^2(BT) = \langle t \rangle$ and $W(\mathrm{Sp}(1))$ is generated by a reflection τ with $\tau(t) = -t$. Then we have $H^*(B\mathrm{Sp}(1);\mathbb{Z}/2) = \mathbb{Z}/2[q]$ such that q pulls back to t^2 in $H^*(BT;\mathbb{Z}/2)$. Thus since $\hat{\sigma}(t^2) = 0$, $H^*(LB\mathrm{Sp}(1);\mathbb{Z}/2) \to H^*(LBT;\mathbb{Z}/2)^{W(\mathrm{Sp}(1))}$ is not injective.

Part 2. Homotopy fixed points

5. Review of the free cohomology suspension

Let X be a simply connected space, and let $\hat{\omega} : S^1 \times LX \to X$ be the evaluation $\hat{\omega}(t, \ell) = \ell(t)$. Er have defined the free cohomology suspension $\hat{\sigma} : \overline{H}^*(X) \to H^{*-1}(LX)$ as

$$\hat{\omega}^*(x) = s \otimes \hat{\sigma}(x) + 1 \otimes x,$$

where s is the dual of the Hurewicz image of $[1_{S^1}] \in \pi_1(S^1)$, and we have seen the following properties.

(1) For the inclusion $i : \Omega X \to LX$,

$$i^* \circ \hat{\sigma} = \sigma.$$

(2) $\hat{\sigma}$ is a derivation.

(3) $\hat{\sigma}$ commutes with Steenrod operations.

6. Homotopy fixed points

The homotopy fixed points of a self-map $\phi : X \to X$ is defined as the homotopy pullback

$$\begin{array}{ccc} X^{\mathrm{h}\phi} & \longrightarrow & X \\ \downarrow & & \downarrow {\scriptstyle 1\times\Phi} \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$

Namely,

$$X^{\mathrm{h}\phi} = \{\ell : [0,1] \to X \mid \ell(1) = \phi(\ell(0))\}.$$

<u>Aim</u> : Describe $H^*(X^{\mathrm{h}\phi})$ without spectral sequences.

To this end, we would like to generalize the free cohomology suspension. But $X^{\mathrm{h}\phi}$ includes non-closed paths, we don't have the evaluation $S^1 \times X^{\mathrm{h}\phi} \to X$. So we force to close elements of $X^{\mathrm{h}\phi}$.

7. Mapping torus

Definition . The mapping torus of $\phi : X \to X$ is defined as

$$M_\phi = [0,1] \times X/(0,x) \sim (1, \phi(x)).$$

Since ϕ is pointed, we may regard $S^1 = \{(t, x_0) \in M_\phi\} \subset M_\phi$ for the basepoint x_0 of X. Let $\iota : X \to M_\phi$ be the inclusion $\iota(x) = (1, x)$.

Proposition . *for a self-map $\psi : Y \to Y$ and a map $f : X \to Y$ satisfying $\psi \circ f \simeq f \circ \phi$, there is a natural map $M(f) : M_\phi \to M_\psi$.*

Consider the Mayer-Vietoris exact sequence for the covering

$$M_\phi = \{(t,x) \in M_\phi \mid 0 \le t \le \tfrac{1}{4} \text{ or } \tfrac{3}{4} \le t \le 1\} \cup \{(t,x) \in M_\phi \mid \tfrac{1}{4} \le t \le \tfrac{3}{4}\}.$$

Then we get an exact sequence

$$\cdots \to H^*(M_\phi) \xrightarrow{\iota^*} H^*(X) \xrightarrow{\phi^*-1} H^*(X) \to H^{*+1}(M_\phi) \to \cdots .$$

Let $\mathcal{A}'_p$ be the subalgebra of $\mathcal{A}_p$ generated by $\mathcal{P}^i$ for p odd and Sq^{2i} for $p = 2$.

Proposition . *Let $R = \mathbb{Z}/p$. If $H^{\mathrm{odd}}(X) = 0$ and $\phi^* = 1$, then $\iota^* : H^*(M_\phi) \to H^*(X)$ has a section as $\mathcal{A}'_p$-modules.*

8. Twisted cohomology suspension

Define a map $\delta : X^{\mathrm{h}\phi} \to LM_\phi$ as

$$\delta(\ell) = [t \mapsto (t, \ell(t))].$$

Every element of $X^{\mathrm{h}\phi}$, possibly non-closed, is closed by δ.

Definition . The twisted cohomology suspension

$$\hat{\sigma}_\phi : \overline{H}^*(M_\phi) \to H^{*-1}(X^{\mathrm{h}\phi})$$

is defined as the composite

$$\overline{H}^*(M_\phi) \xrightarrow{\hat{\sigma}} H^{*-1}(LM_\phi) \xrightarrow{\delta^*} H^{*-1}(X^{\mathrm{h}\phi}).$$

Proposition . (1) *For the inclusion $i : \Omega X \to X^{\mathrm{h}\phi}$ and the projection $q : M_\phi \to M_\phi/S^1$,*

$$i^* \circ \hat{\sigma}_\phi \circ q^* = \sigma \circ \iota^* \circ q^*.$$

(2) *For $\psi : Y \to Y$ and $f : X \to Y$ with $f \circ \phi \simeq \psi \circ f$,*

$$\bar{f}^* \circ \hat{\sigma}_\phi = \hat{\sigma}_\psi \circ M(f)^*,$$

where $\bar{f} : X^{\mathrm{h}\phi} \to Y^{\mathrm{h}\psi}$ is the induced map.

(3) *Let $\omega : X^{\mathrm{h}\phi} \to X$ be the evaluation at 0. For $\omega_\phi = \iota \circ \phi \circ \omega$,*

$$\hat{\sigma}_\phi(xy) = \hat{\sigma}_\phi(x)\omega_\phi^*(y) + (-1)^{|x|}\omega_\phi^*(x)\hat{\sigma}_\phi(y).$$

(4) *$\hat{\sigma}_\phi$ commutes with Steenrod operations.*

Proof. Define $h : [0,1] \times S^1 \times \Omega X \to M_\phi/S^1$ as

$$h(s,t,\ell) = \begin{cases} (2st, \ell((1-s)t)) & 0 \le t \le \frac{1}{2} \\ (\min\{2st, 1\}, \ell((1+s)t - s)) & \frac{1}{2} \le t \le 1. \end{cases}$$

Using this homotopy, we get a homotopy commutative diagram

$$\begin{array}{ccccccc} S^1 \times \Omega X & & \xrightarrow{\quad\bar{\omega}\quad} & & X & \xrightarrow{\iota} & M_\phi \\ \downarrow{\scriptstyle 1\times i} & & & & & & \downarrow{\scriptstyle q} \\ S^1 \times X^{\mathrm{h}\phi} & \xrightarrow{1\times\delta} & S^1 \times LM_\phi & \xrightarrow{\hat{\omega}} & M_\phi & \xrightarrow{q} & M_\phi/S^1. \end{array}$$

Then (1) and (2) follows.

∃commutative diagram

$$\begin{array}{ccccc} X^{\mathrm{h}\phi} & \xrightarrow{\mathrm{incl}} S^1 \times X^{\mathrm{h}\phi} & \xrightarrow{1\times\delta} & S^1 \times LM_\phi \\ \downarrow{\scriptstyle \omega} & & & \downarrow{\scriptstyle \hat{\omega}} \\ X & \xrightarrow{\quad\phi\quad} X & \xrightarrow{\quad\iota\quad} & M_\phi. \end{array}$$

Then

$$(\hat{\omega} \circ (1 \times \delta))^*(x) = s \otimes \hat{\sigma}_\phi(x) + 1 \otimes \omega_\phi^*(x),$$

implying (3) and (4). □

Theorem . *If $H^*(X) = R[x_1, \ldots, x_n]$ and ∃section α of $\iota^* : H^*(M_\phi) \to H^*(X)$, then*

$$H^*(X^{\mathrm{h}\phi}) \cong R[\omega_\phi^*(x_1), \ldots, \omega_\phi^*(x_n)] \otimes \Delta(\hat{\sigma}_\phi(\alpha(x_1)), \ldots, \hat{\sigma}_\phi(\alpha(x_n))).$$

*Moreover, if α respects $\mathcal{A}'_p$ (*resp.* $\mathcal{A}_p$), the above identification is over $\mathcal{A}'_p$ (*resp.* $\mathcal{A}_p$).*

9. Applications

Let G be a connected Lie group and let $\phi^q : BG_p \to BG_p$ be the unstable Adams operation for a prime power q with $p \nmid q$. Let $G(q)$ be the Chevalley group of type G over a field $\mathbb{F}_q$. Then we have

$$BG(q)_p \simeq BG_p^{\mathrm{h}\phi^q}.$$

Theorem . *If $H^*(G; \mathbb{Z})$ has no p-torsion and $q \equiv 1 \mod p$, then*

$$H^*(G(q); \mathbb{Z}/p) \cong H^*(LBG; \mathbb{Z}/p)$$

as $\mathcal{A}'_p$-modules. Moreover, if $q \equiv 1 \mod p^2$, the above congruence is over $\mathcal{A}_p$.

Proof. If $H^*(G; \mathbb{Z})$ has no p-torsion, $H^{\mathrm{odd}}(BG; \mathbb{Z}/p) = 0$, implying the first assertion. The second assertion follows analogously. □

For an odd prime power q, let us next calculate $H^*(G_2(q); \mathbb{Z}/2)$. We construct a section of $\iota^* : H^*(M_{\phi^q}) \to H^*(BG_2)$. Since $H^4(M_{\phi^q}) \cong \mathbb{Z}/2$, we get $\bar{x}_4 \in H^4(M_{\phi^q})$ with $\iota^*(\bar{x}_4) = x_4$. Put

$$\mathrm{Sq}^2 \bar{x}_4 = \bar{x}_6, \quad \mathrm{Sq}^1 \bar{x}_6 = \bar{x}_7.$$

Then $\iota^*(\bar{x}_i) = x_i$ for $i = 6, 7$. We can now define a section α as

$$\alpha(x_i) = \bar{x}_i \quad \text{for } i = 4, 6, 7.$$

We show that α respects $\mathcal{A}_2$. Since $q^2 \equiv 1 \mod 4$, we have $\mathrm{Sq}^1 \bar{x}_4 = 0$ by considering the integral cohomology, which implies

$$\mathrm{Sq}^2 \bar{x}_6 = \mathrm{Sq}^2 \mathrm{Sq}^2 \bar{x}_4 = \mathrm{Sq}^3 \mathrm{Sq}^1 \bar{x}_4 = 0, \quad \mathrm{Sq}^2 \bar{x}_7 = \mathrm{Sq}^2 \mathrm{Sq}^3 \bar{x}_4 = (\mathrm{Sq}^1 \mathrm{Sq}^4 + \mathrm{Sq}^4 \mathrm{Sq}^1) \bar{x}_4 = 0.$$

Since $H^9(BG_2) = 0$, $\iota^* : H^{10}(M_{\phi^q}) \to H^{10}(BG_2)$ is monic, and then

$$\mathrm{Sq}^4 \bar{x}_6 = \bar{x}_4 \bar{x}_6, \quad \mathrm{Sq}^4 \bar{x}_7 = \mathrm{Sq}^4 \mathrm{Sq}^3 \bar{x}_4 = \mathrm{Sq}^1 \mathrm{Sq}^4 \mathrm{Sq}^2 \bar{x}_4 = \mathrm{Sq}^1 \mathrm{Sq}^4 \bar{x}_6 = \mathrm{Sq}^1(\bar{x}_4 \bar{x}_6) = \bar{x}_4 \bar{x}_7.$$

Thus we have seen that α respects $\mathcal{A}_2$.

Theorem . *$H^*(G_2(q);\mathbb{Z}/2) \cong H^*(LBG_2;\mathbb{Z}/2)$ over $\mathcal{A}_2$-algebras.*

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