The Lefschetz condition on projectivizations of complex vector bundles

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Abstract

We consider a condition under which the projectivization $P(E^k)$ of a complex k-bundle $E^k \to M$ over an even-dimensional manifold M can have the hard Lefschetz property, affected by [4]. It depends strongly on the rank k of the bundle E^k. Our approach is purely algebraic by using rational Sullivan minimal models [2]. We will give some examples.

The contents of this paper is according to [6]. A Poincaré duality space Y of the formal dimension

$$fd(Y) = \max\{i; H^i(Y;\mathbb{Q}) \neq 0\} = 2m$$

is said to be *cohomologically symplectic* (c-symplectic) if $u^m \neq 0$ for some $u \in H^2(Y;\mathbb{Q})$ and, furthermore, is said to have the *hard Lefschetz property* (or simply the Lefschetz property) with respect to the c-symplectic class u, if the maps

$$\cup u^j : H^{m-j}(Y;\mathbb{Q}) \to H^{m+j}(Y;\mathbb{Q}) \quad 0 \leq j \leq m$$

are monomorphisms (then called the *Lefschetz maps*) [7]. (Then we often say that $H^*(Y;\mathbb{Q})$ is a Lefschetz algebra [4].) For example, a compact Kähler manifold has the hard Lefschetz property [7], [3, Theorem 4.35].

Let M be an even-dimensional manifold and $\xi : E^k \to M$ be a complex k-bundle over M. The projectivization of the bundle ξ

$$P(\xi): \ \mathbb{C}P^{k-1} \xrightarrow{j} P(E^k) \to M$$

satisfies the rational cohomology algebra condition $(*)$:

$$H^*(P(E^k);\mathbb{Q}) = \frac{H^*(M;\mathbb{Q})[x]}{(x^k + c_1x^{k-1} + \cdots + c_jx^{k-j} + \cdots + c_{k-1}x + c_k)}$$

where c_i are the Chern classes of ξ and x is a degree 2 class generating the cohomology of the complex projective space fiber (Leray-Hirsch theorem) [1], [4], [7, p.122]. The manifold $P(E^k)$ appears as the exceptional divisor in blow-up construction for a certain embedding of M [5], [7, Chap.4]. When M is a non-toral symplectic nilmanifold of dimension $2n$, there is a bundle E^n such that $P(E^n)$ is not Lefschetz [8], [4, Example 4.4]. In general, for a $2k$-dimensional manifold M and a fibration

$$\mathbb{C}P^{k-1} \to E \to M,$$

the total space E is Lefschetz if and only if M is Lefschetz [4, Remark 4.2]. We consider the following

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Problem 1 *Suppose that the projectivization $P(E^k)$ of a k-dimensional vector bundle $E^k \to M$ is c-symplectic with respect to $\tilde{x}$ where $j^*(\tilde{x}) = x$; i.e., $\tilde{x}^m \neq 0$ when $\dim P(E^k) = 2m$. What rational homotopical conditions on M are necessary for $P(E^k)$ to have the Lefschetz property with respect to $\tilde{x}$?*

Proposition 2 *Let M be a an even dimensional manifold.*

(1) For a sufficiently large k, there is a k-dimensional vector bundle $E^k \to M$ such that $P(E^k)$ is c-symplectic with respect to x.

(2) If $P(E^k)$ is c-symplectic with respect to x, then there is a vector bundle $E^m \to M$ such that $P(E^m)$ is c-symplectic with respect to x for any $m > k$.

Definition 3 An even-dimensional manifold (or more general Poincaré duality space) M is said to be *projective* (k)*-Lefschetz* if there exists a bundle E^k such that the projectivization $P(E^k)$ is c-symplectic with respect to $\tilde{x}$ and has the Lefschetz property with respect to $\tilde{x}$. Then we often say simply that M is *projective Lefschetz*. In particular, we say that M is *projective non-Lefschetz* if $P(E^k)$ cannot have the Lefschetz property for any k and E^k.

We recall D.Sullivan's rational model and we give some examples that indicate how the rational cohomology algebra of M determines the projective (n)-Lefschetzness of M when M is the product of (at most four) spheres.

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