Products in equivariant homology

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(非) 可換代数とトポロジー, 信州大学, 2016 年 2 月 20 日

Outline

Let h be a ring spectrum. The goal of this talk is to generalise the cross product in the homology

$$\times : h_s(X) \otimes h_t(Y) \to h_{s+t}(X \times Y)$$

to the spaces over the Borel construction of a manifold with a nice Lie group action: for $X \to M_G, Y \to M_G$, we define

$$\mu : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y),$$

and its secondary version.

Outline of talk:

1. External products for fibre squares
2. Examples in String topology
3. Vanishing of the product
4. A secondary product
5. Computational examples

Preprint at arXiv:1506.00441

(このバージョンは古いのでご覧になりたい方はお知らせください)

External product for fibre square

Given a fibre square (or homotopy pullback)

$$\begin{tikzcd} X \times_B Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

with certain conditions on B.
Under certain conditions, one can define homomorphisms of the form

$$h_s(X) \otimes h_t(Y) \to h_{s+t+*}(X \times_B Y)$$

with degree shifts.

External product for fibre square

The key idea is that a fibre square

$$\begin{tikzcd} X \times_B Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

is equivalent to the pullback of fibrations

$$\begin{tikzcd} \Omega B \arrow[r] \arrow[d, equal] & X \times_B Y \arrow[r, "\hat{\Delta}"] \arrow[d] & X \times Y \arrow[d, "f \times g"] \\ \Omega B \arrow[r] & B \arrow[r, "\Delta"] & B \times B \end{tikzcd}$$

and we use a *wrong-way map* $\hat{\Delta}^\natural : h_*(X \times Y) \to h_{*+\mathrm{shift}}(X \times_B Y)$.

When $B = M$ is an oriented manifold

External product for fibre square

Let $B = M$ be an oriented manifold. Consider the pullback diagram:

$$\begin{tikzcd} X \times_M Y \arrow[r, "\hat{\Delta}"] \arrow[d] & X \times Y \arrow[d, "f \times g"] \\ M \arrow[r, "\Delta"] & M \times M. \end{tikzcd}$$

The diagonal $\Delta : M \to M \times M$ is a finite codimensional embedding with the normal bundle isomorphic to TM. We have the Gysin map
$\Delta^! : h_*(M \times M) \to h_{*-\dim(M)}(M)$.
We can pull it back to define a Gysin map for $\hat{\Delta}$:

$$\hat{\Delta}^! : h_*(X \times Y) \to h_{*-\dim(M)}(X \times_M Y).$$

External product for fibre square

Composing

$$\hat{\Delta}^{!} : h_{s+t}(X \times Y) \to h_{s+t-\dim(M)}(X \times_M Y).$$

with the cross product

$$\times : h_s(X) \otimes h_t(Y) \to h_{s+t}(X \times Y).$$

we define

$$\mu_M : h_s(X) \otimes h_t(Y) \to h_{s+t-\dim(M)}(X \times_M Y).$$

Example: Intersection product

The trivial diagram

$$\begin{tikzcd} M \arrow[r, equal] \arrow[d, equal] & M \arrow[d, equal] \\ M \arrow[r, equal] & M \end{tikzcd}$$

gives rise to the intersection product
(dual to the cup product on cohomology)

$$h_s(M) \otimes h_t(M) \to h_{s+t-\mathrm{dim}(M)}(M).$$

Example: String product in M

Consider the following pullback, where $LM = \mathrm{Map}(S^1, M)$

$$\begin{tikzcd}
\mathrm{Map}(S^1 \vee S^1, M) \arrow[r] \arrow[d] & LM \arrow[d, "ev"] \\
LM \arrow[r, "ev"] & M
\end{tikzcd}$$

Composing μ_M with the concatenation of loops, we obtain

$$h_s(LM) \otimes h_t(LM) \to h_{s+t-\mathrm{dim}(M)}(LM),$$

which is equivalent to the Chas-Sullivan product when $h_* = H_*$.
(Cohen-Jones's definition of the string product)

When $B = BG$ is the classifying space of a compact Lie group

Grothendieck bundle transfer

Let $B = BG$ be the classifying space of a compact Lie group G. We require a technical condition: the universal adjoint bundle $\mathfrak{g} \hookrightarrow ad(EG) \to BG$ is oriented.
Consider the pullback diagram

$$\begin{tikzcd}
\Omega BG \arrow[r] \arrow[d, equal] & X \times_{BG} Y \arrow[r, "\hat{\Delta}"] \arrow[d] & X \times Y \arrow[d] \\
\Omega BG \arrow[r] \arrow[d, equal] & BG \arrow[r, "\Delta"] \arrow[d, equal] & BG \times BG \arrow[d, equal] \\
(G \times G)/\Delta G \arrow[r] & E(G \times G)/\Delta G \arrow[r] & E(G \times G)/(G \times G)
\end{tikzcd}$$

Grothendieck bundle transfer

We need an wrong-way map for $\hat{\Delta} : X \times_{BG} Y \to X \times Y$.

Let $F \to E \to B$ be a fiber bundle with a compact Lie structure group, where the fibre F is a compact manifold. By fibre-wise Pontrjagin-Thom construction, we obtain a map between Thom spectra $B^0 \to E^{-t}$, where t is the bundle of tangents. Applying this to our setting, we have

$$\hat{\Delta}^{\natural} : h_{s+t}(X \times Y) \to h_{s+t+\dim(G)}(X \times_{BG} Y).$$

Composing with the cross product, we define

$$\mu_{BG} : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)}(X \times_{BG} Y)$$

Example: Equivariant cross product

Consider the following pullback diagram

$$\begin{tikzcd}
(X \times Y)_G \arrow[r] \arrow[d] & X_G \arrow[d] \\
Y_G \arrow[r] & (*)_G = BG
\end{tikzcd}$$

where Z_G is the Borel construction.
For $h = H$, we have

$$\mu_{BG} : H_s^G(X) \otimes H_t^G(Y) \to H^G_{s+t+\dim(G)}(X \times Y).$$

Example: String product in BG

Consider the following fibre square

$$\begin{tikzcd} \mathrm{Map}(S^1 \vee S^1, BG) \arrow[r] \arrow[d] & LBG \arrow[d, "ev"] \\ LBG \arrow[r, "ev"] & BG \end{tikzcd}$$

where $LBG \simeq \mathrm{Map}(S^1, BG) \simeq BLG$.
Composing μ_{BG} with the concatenation of loops $\mathrm{Map}(S^1 \vee S^1, BG) \to LBG$, we obtain

$$h_s(LBG) \otimes h_t(LBG) \to h_{s+t+\dim(G)}(LBG)$$

which reduces to the Chataur-Menichi product when $h_* = H_*$.

Two external products

When $B = M$ is an oriented manifold

$$\mu_M : h_s(X) \otimes h_t(Y) \to h_{s+t-\dim(M)}(X \times_M Y)$$

When $B = BG$ is the classifying space of a compact Lie group

$$\mu_{BG} : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)}(X \times_{BG} Y)$$

Can we unify the two constructions?

When $B = M_G$ the Borel construction of a G-action on M

Mixed external product

We combine the previous two constructions and define an external product in an equivariant setting. Let G acts on M orientation preservingly (meaning $(TM)_G$ is h-oriented) and M_G be the Borel construction. For a fibre square

$$\begin{tikzcd} X \times_{M_G} Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & M_G \end{tikzcd}$$

we will define

$$\mu_{M_G} : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y).$$

Definition: Mixed product

The problem when defining

$$\mu_{M_G} : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y).$$

is that we cannot define a wrong-way map for the diagonal map

$$M_G \to M_G \times M_G$$

since its fibre is not finite dimensional and it is not a finite codimensional embedding.

We will decompose the diagonal map into two steps and define wrong-way maps step-by-step.

Definition: Mixed product

The diagonal map $M_G \to M_G \times M_G$ decomposes into two steps:

$$M_G \xrightarrow{\Delta_G} (M \times M)_{\Delta G} \xrightarrow{q} M_G \times M_G,$$

where Δ_G is the equivariant diagonal (which is codimension $\dim(M)$) and

$$(G \times G)/\Delta G \hookrightarrow (M \times M)_{\Delta G} \xrightarrow{q} M_G \times M_G$$

is the homogeneous fibration. That is, the pullback of

$$(G \times G)/\Delta G \hookrightarrow E(G \times G)/\Delta G \to E(G \times G)/(G \times G)$$

Definition: Mixed product

Consider the ladder of pullbacks:

$$\begin{tikzcd}
X \times_{M_G} Y \arrow[r, "\hat{\Delta}_G"] \arrow[d] & X \times_{BG} Y \arrow[r, "\hat{q}"] \arrow[d] & X \times Y \arrow[d] \\
M_G \arrow[r, "\Delta_G"] & (M \times M)_G \arrow[r, "q"] & M_G \times M_G .
\end{tikzcd}$$

Then define the product μ_{M_G} as the composition:

$$h_s(X) \otimes h_t(Y) \to h_{s+t}(X \times Y)$$

$$\xrightarrow{\hat{q}^\natural} h_{s+t+\dim(G)}(X \times_{BG} Y) \xrightarrow{\hat{\Delta}_G^!} h_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y).$$

Mixed product unifies the two

The mixed product

$$\mu_{M_G} : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y)$$

reduces to

$$\mu_{BG} : h_s(X) \otimes h_t(Y) \to h_{s+t+\dim(G)}(X \times_{BG} Y)$$

when $M = pt$ and

$$\mu_{M} : h_s(X) \otimes h_t(Y) \to h_{s+t-\dim(M)}(X \times_{M} Y).$$

when $G = *$.

Example: String product in M_G

Consider the following pullback, where $L(M_G) = \mathrm{Map}(S^1, M_G)$

$$\begin{tikzcd}
\mathrm{Map}(S^1 \vee S^1, M_G) \arrow[r] \arrow[d] & L(M_G) \arrow[d, "ev"] \\
L(M_G) \arrow[r, "ev"] & M_G
\end{tikzcd}$$

Composing μ_{M_G} with the concatenation of loops, we obtain

$$h_s(L(M_G)) \otimes h_t(L(M_G)) \to h_{s+t+\dim(G)-\dim(M)}(L(M_G)).$$

This reduces to Chas-Sullivan when $G = *$ and to Chataur-Menichi when $M = *$.

Example: String coproduct in M_G

From the following pullback

$$\begin{tikzcd}
\mathrm{Map}(S^1 \vee S^1, M_G) \arrow[r] \arrow[d] & L(M_G) \arrow[d, "(ev_0, ev_{1/2})"] \\
M_G \arrow[r, "\Delta_{M_G}"] & M_G \times M_G,
\end{tikzcd}$$

we obtain $h_*(L(M_G)) \xrightarrow{\Delta^!_G \circ p^\natural} h_{*+\dim(G)-\dim(M)}(\mathrm{Map}(S^1 \vee S^1, M_G))$.
Composing it with the inclusion, we obtain

$$h_*(L(M_G)) \to h_{*+\dim(G)-\dim(M)}(L(M_G) \times L(M_G)).$$

This reduces to Chataur-Menichi when $M = *$.

Example: Equivariant intersection

When applied to the identity square

$$\begin{tikzcd} M_G \arrow[r] \arrow[d] & M_G \arrow[d] \\ M_G \arrow[r] & M_G, \end{tikzcd}$$

it defines an *equivariant intersection product*

$$\mu_{M_G} : h_s(M_G) \otimes h_t(M_G) \to h_{s+t+\dim(G)-\dim(M)}(M_G).$$

Properties of the product

Properties of μ_{M_G}

Proposition

- Natural with respect to a homomorphism between homology theories
- Compatible with the group restriction: Let $H \subset G$ be a closed subgroup and $i : M_H \to M_G$ be the induced map. Then, $\mu_{M_H} \circ (i^\natural \otimes i^\natural) = i^\natural \circ \mu_{M_G}$.

$$
\begin{tikzcd}
X' \times_{M_H} Y' \arrow[r] \arrow[d] & X' \arrow[d] & \xrightarrow{i} & X \times_{M_G} Y \arrow[r] \arrow[d] & X \arrow[d] \\
Y' \arrow[r] & M_H & & Y \arrow[r] & M_G
\end{tikzcd}
$$

- Compatible with the induction $(M \times G/H)_G \simeq M_H$.

Vanishing of μ_{M_G}

From now on, we specialises the case when $h_* = H_*(\ ; R)$ with a ring R.
An easy but interesting property of μ_{M_G} is that it vanishes in higher degrees:

Theorem

Let d_X (resp. d_Y) be the homological dimension of the homotopy fibre of the composition $X \to M_G \to BG$. Then,

$$\mu_{M_G} : H_s(X; R) \otimes H_t(Y; R) \to H_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y; R)$$

vanishes if $s > d_X - \dim(G)$ or $t > d_Y - \dim(G)$.

When applied to special cases, it has non-trivial consequences.

Vanishing of string product

Chataur-Menichi defined *string operations* for LBG:
For a surface $F_{g,p+q}$ of genus g with *p-incoming* and *q-outgoing* boundary circles, they defined a homomorphism

$$\mu(F_{g,p+q}) : H_*(LBG)^{\otimes p} \to H_{\dim(G)(2g+p+q-2)}(LBG)^{\otimes q}$$

which is compatible with the gluing of the surfaces.
(when $g = 0, p = 2, q = 1$, it gives the product we saw earlier)
A consequence of our vanishing theorem is:

Corollary

$\mu(F_{g,p+q})$ is trivial unless $g = 0$ and $p = 1$, or $* = 0$.

Note that any map $f : X \to M_G$ is converted to an equivariant map by

$$\begin{tikzcd} \hat{X} \arrow[r, "\hat{f}"] \arrow[d] & M \arrow[r] \arrow[d] & EG \arrow[d] \\ X \arrow[r, "f"] & M_G \arrow[r] & BG, \end{tikzcd}$$

which identifies $f = \hat{f}_G : \hat{X}_G \to M_G$ with fibre $\hat{X}$.
Hence, the initial diagram is equivalent to the Borel construction of

$$\begin{tikzcd} \hat{X} \times_M \hat{Y} \arrow[r] \arrow[d] & \hat{X} \arrow[d] \\ \hat{Y} \arrow[r] & M \end{tikzcd}$$

$$\mu_{M_G} : H^G_s(\hat{X}; R) \otimes H^G_t(\hat{Y}; R) \to H^G_{s+t+\dim(G)-\dim(M)}(\hat{X} \times_M \hat{Y}; R)$$

vanishes if $s > \dim(\hat{X}) - \dim(G)$ or $t > \dim(\hat{Y}) - \dim(G)$.

Secondary product

Secondary product

Vanishing of

$$\mu_{M_G} : H_s(X) \otimes H_t(Y) \to H_{s+t+\dim(G)-\dim(M)}(X \times_{M_G} Y).$$

for $s > d_X - \dim(G)$ or $t > d_Y - \dim(G)$ suggests that we may define a "secondary" product.
In fact, we can define

$$\overline{\mu_{M_G}} : H_s(X) \otimes H_t(Y) \to H_{s+t+\dim(G)-\dim(M)+1}(X \times_{M_G} Y).$$

for $s > d_X - \dim(G)$ and $t > d_Y - \dim(G)$.

Given

$$\begin{tikzcd} X \times_{M_G} Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & M_G \end{tikzcd}$$

consider the following diagram with all the squares pullback:

$$\begin{tikzcd} X_s \times_{BG} Y_t \arrow[r] \arrow[d] & \tilde{X}_s \arrow[r] \arrow[d] & X_s \arrow[d] \\ \tilde{Y}_t \arrow[r] \arrow[d] & X \times_{BG} Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y_t \arrow[r] & Y \arrow[r] & BG, \end{tikzcd}$$

where X_s and Y_t are s- and t-skeleta.

Secondary product

Now we take the homotopy pushforward of the upper-left corner

$$
\begin{tikzcd}
X_s \times_{BG} Y_t \arrow[rr] \arrow[dd] & & \tilde{X}_s \arrow[dl] \arrow[dd] \\
& R \arrow[dr, dashed, "r"] & \\
\tilde{Y}_t \arrow[ur] \arrow[rr] & & X \times_{BG} Y
\end{tikzcd}
$$

Since $\tilde{X}_s$ and $\tilde{Y}_t$ have low homological degrees by assumption, we obtain a well-defined map by the composition

$$H_*(X_s \times_{BG} Y_t) \xrightarrow{\text{sus}} H_{*+1}(R) \xrightarrow{r_*} H_{*+1}(X \times_{BG} Y)$$

Secondary product

Our secondary product is defined to be the composition

$$\begin{aligned}\overline{\mu_{M_G}} : & H_s(X) \otimes H_t(Y) \xrightarrow{\text{lift}} H_s(X_s) \otimes H_t(Y_t) \xrightarrow{\mu_{BG}} H_{s+t+\dim(G)}(X_s \times_{BG} Y_t) \\ & \to H_{s+t+\dim(G)+1}(X \times_{BG} Y) \xrightarrow{\hat{\Delta}_G^!} H_{s+t+\dim(G)-\dim(M)+1}(X \times_{M_G} Y).\end{aligned}$$

This can be shown to be well-defined. (it does not depend on the choices of skeleta and lift of classes)

Proposition

$\overline{\mu_{M_G}}$ is compatible with restriction wrt a closed subgroup $H \subset G$.

Application: String Product in BG

Again from the diagram

$$\begin{tikzcd} \mathrm{Map}(S^1 \vee S^1, BG) \arrow[r] \arrow[d] & LBG \arrow[d, "ev"] \\ LBG \arrow[r, "ev"] & BG \end{tikzcd}$$

we obtain

$$H_s(LBG) \otimes H_t(LBG) \to H_{s+t+\dim(G)+1}(LBG),$$

which is a secondary product of Chataur-Menichi's string product.
This product does not usually vanish!

Application: Intersection product

When $X = Y = M_G$, that is, for the identity square

$$\begin{tikzcd} M_G \arrow[r, equal] \arrow[d, equal] & M_G \arrow[d, equal] \\ M_G \arrow[r, equal] & M_G, \end{tikzcd}$$

$\overline{\mu_{M_G}}$ specialises to a product in $H_*^G(M)$:

$$\overline{\mu_{M_G}} : H_s^G(M) \otimes H_t^G(M) \to H^G_{s+t+\dim(G)-\dim(M)+1}(M),$$

which can be thought of as a secondary equivariant intersection product.
For any G, $\exists M$ s.t. this product is non-trivial!

Application: Tate cohomology

When $M = pt$, we have

$$H_s(BG) \otimes H_t(BG) \to H_{s+t+\dim(G)+1}(BG)$$

Theorem

It coincides with the product in Tate cohomology when G is finite.

1. Tate cohomology: cup product in homology of finite groups
2. Kreck product: generalisation to homology of compact Lie groups
3. Our product: generalisation to equivariant homology of manifolds

Note: Greenlees-May[1995] defined a Tate cohomology spectra for compact Lie groups

Tate cohomology

- G: finite group
- $\to P_k \to P_{k-1} \to \cdots P_0 \to \mathbb{Z} \to 0$: the standard $\mathbb{Z}[G]$-resolution ($P_k := \mathbb{Z}[G^{k+1}]$)
- Then, $0 \to \mathbb{Z} \to \mathrm{Hom}_{\mathbb{Z}[G]}(P_0, \mathbb{Z}[G]) \to \mathrm{Hom}_{\mathbb{Z}[G]}(P_1, \mathbb{Z}[G]) \to \cdots$ is exact
- set $P_{-i} := \mathrm{Hom}_{\mathbb{Z}[G]}(P_{i-1}, \mathbb{Z}[G])$ and

$$P_* := \to P_k \to P_{k-1} \to \cdots \to P_0 \to P_{-1} \to P_{-2} \cdots$$

- The *Tate cohomology* $\hat{H}^*(G) := H(\mathrm{Hom}_{\mathbb{Z}[G]}(P_*, \mathbb{Z}))$

Tate cohomology

It is easy to see

$$\hat{H}^i(G) \simeq \begin{cases} H^i(G) & (i \geq 1) \\ \mathbb{Z}/|G|\mathbb{Z} & (i = 0) \\ 0 & (i = -1) \\ H_{-i-1}(G) & (i \leq -2). \end{cases}$$

It amalgamates homology and cohomology into one object.

Cup product in Tate cohomology

By $P_{i+j} \to P_i \otimes P_j$, the cup product is defined:

$$\hat{H}^l(G) \otimes \hat{H}^k(G) \to \hat{H}^{k+l}(G),$$

which gives a product in homology of degree $+1$:

$$H_{-l-1}(G) \otimes H_{-k-1}(G) \to H_{-k-l-1}(G)$$

through $\hat{H}^n(G) = H_{-n-1}(G)$ for $n \leq -2$. Note that we have to assume that $-l-1, -k-1 > 0$.

Computational Examples

Simplest example

Let $X = Y = M = pt$. Then the defining pull-push diagram for

$$\overline{\mu_{M_G}} : H_0(pt) \otimes H_0(pt) \to H_{\dim(G)+1}(BG)$$

is identified with the first stage of the Ganea construction:

$$\begin{tikzcd}
G \arrow[rr] \arrow[dd] & & pt \arrow[dl] \arrow[dd] \\
& \Sigma G \arrow[dr, dashed, "r"] & \\
pt \arrow[ur] \arrow[rr] & & BG
\end{tikzcd}$$

Second computation

When $G = S^1, M = pt$

$$H_*^G(pt;\mathbb{Z}) = H_*(BS^1;\mathbb{Z}) \simeq \mathbb{Z}\langle a_{2k}\rangle \quad (k \geq 0),$$

where a_{2k} is represented by $\mathbb{C}P^k \hookrightarrow \mathbb{C}P^\infty = BS^1$.

The product $H_s(BS^1) \otimes H_t(BS^1) \to H_{s+t+2}(BS^1)$ is given by

$$a_{2i} * a_{2j} = a_{2(i+j+1)}.$$

$$\begin{tikzcd}
S^{2i+1} \times_{S^1} S^{2j+1} \arrow[rr] \arrow[dd] & & \mathbb{C}P^j \arrow[dl] \arrow[dd] \\
& \mathbb{C}P^{i+j+1} \arrow[dr, dashed] & \\
\mathbb{C}P^i \arrow[ur] \arrow[rr] & & BS^1
\end{tikzcd}$$

Computation for classical groups

Proposition

The product is torsion for all compact Lie groups of rank greater than 1. The product vanishes for all compact connected classical Lie groups of rank greater than 1.
For $H_*(BSp(1);\mathbb{Z}) \simeq \mathbb{Z}\langle a_{4k}\rangle \quad (k \geq 0)$,

$$a_{4i} * a_{4j} = a_{4(i+j+1)}.$$

For $H_*(BSO(3);\mathbb{Z}) \simeq \mathbb{Z}\langle b_{4k}\rangle \oplus \text{2-torsion} \quad (k \geq 0)$,

$$b_{4i} * b_{4j} = 2b_{4(i+j+1)}$$

and all the other products vanish.

Computation for $\mathbb{C}P^1$

Let $S^1 \curvearrowright \mathbb{C}P^1$ by the standard action. Then, $H_*^{S^1}(\mathbb{C}P^1) = \mathbb{Z}\langle \alpha_{2k}, \beta_{2k+2} \rangle$, where

$$\alpha_{2k} : S^{2n+1} \times_{S^1} pt \to ES^1 \times_{S^1} \mathbb{C}P^1$$
$$\beta_{2k+2} : S^{2n+1} \times_{S^1} \mathbb{C}P^1 \to ES^1 \times_{S^1} \mathbb{C}P^1$$

$\overline{\mu_{M_G}} : H_s^{S^1}(\mathbb{C}P^1) \otimes H_t^{S^1}(\mathbb{C}P^1) \to H_{s+t}^{S^1}(\mathbb{C}P^1)$ is computed as

Proposition

$\alpha_{2i} * \alpha_{2j} = 0, \alpha_{2i} * \beta_{2j+2} = \alpha_{2(i+j+1)}, \beta_{2i+2} * \beta_{2j+2} = \beta_{2(i+j+1)+2}$

Future work

- Find other applications (e.g., obstruction for group action)
- Develop computational method (e.g., Eilenberg-Moore SS)
  - secondary product for $H_*(BG)$
  - secondary product for $H_*^T(M)$ for toric and flag manifolds
- Relation with other structures (co-product, Steenrod co-operations)
- Compare the product to Félix-Thomas's string topology for Gorenstein spaces
- Compare the product to Behrend-Ginot-Noohi-Xu's string topology for differential stacks
- Compare the secondary product to Greenlees-May's Tate cohomology
- Extend to more general groups (e.g., p-compact groups)

Thank you!