Topology of Polyhedral products and Golod property of Stanley-Reisner ring, II

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Plan of talks

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- First day. Main results and topological background.
- Today. Bridge between algebra and topology
  - Tor algebra
  - Cellular cochain complex
  - Hochster's theorem
- Third day. Sketch of Proofs.

Tor algebra

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Definition (1.1)

Let K be a simplicial complex on the vertex set $[m] = \{1, \ldots, m\}$. The Stanley-Reisner ring of K is the following quotient algebra of the polynomial ring on m generators:

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \ldots, v_m]/(v_I \mid I \notin K).$$

The Stanley-Reisner ring $\mathbb{Z}[K]$ is a $\mathbb{Z}[m]$-module via the quotient projection $\mathbb{Z}[m] \to \mathbb{Z}[K]$.

A free resolution of $\mathbb{Z}[K]$ is an exact sequence of finitely generated $\mathbb{Z}[m]$-modules:

$$0 \to R^{-m} \to \cdots \to R^{-1} \to R^0 \to \mathbb{Z}[K] \to 0,$$

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Definition (1.2)

The $-i$ th Tor group $\mathrm{Tor}^{-i}_{\mathbb{Z}[m]}(\mathbb{Z}[K], \mathbb{Z})$ is defined as the $-i$ th cohomology groups of the complex:

$$0 \to R^{-m} \otimes_{\mathbb{Z}[m]} \mathbb{Z} \to \cdots \to R^{-1} \otimes_{\mathbb{Z}[m]} \mathbb{Z} \to R^0 \otimes_{\mathbb{Z}[m]} \mathbb{Z} \to 0.$$

We define

$$\mathrm{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[K], \mathbb{Z}) = \bigoplus_{i=0}^{m} \mathrm{Tor}^{-i}_{\mathbb{Z}[m]}(\mathbb{Z}[K], \mathbb{Z})$$

which has double gradings.

Theorem (Baskakov-Buchstaber-Panov '04, Franz '06)

The cohomology ring of the moment angle complex $Z_K = Z_K(D^2, S^1)$ is given by the isomorphisms

$$\begin{aligned} H^*(Z_K;\mathbb{Z}) &\cong \mathrm{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[K],\mathbb{Z}) \\ &\cong H[\Lambda[u_1,\dots,u_m]\otimes\mathbb{Z}[K],d], \end{aligned}$$

where the latter ring is the cohomology of differential graded algebra whose grading and differential are given by

$$\deg u_1 = (-1,2),\ \deg v_i = (0,2); \quad du_i = v_i,\ dv_i = 0.$$

The cohomology ring $H^*(Z_K;\mathbb{Z})$ also has its own bigrading, which will be given later, and the isomorphism above is that of bigraded algebras.

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$$\begin{aligned} 0 \to \Lambda^m[u_1, \ldots, u_m] \otimes \mathbb{Z}[m] \to \ldots \\ \to \Lambda^1[u_1, \ldots, u_m] \otimes \mathbb{Z}[m] \to \mathbb{Z}[m] \to \mathbb{Z} \to 0 \end{aligned}$$

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where $\Lambda^i[u_1, \dots, u_m]$ is the subalgebra of $\Lambda[u_1, \dots, u_m]$ spanned by monomials of length i. Therefore we have

$$\mathrm{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[K], \mathbb{Z}) \cong H[\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K], d]$$

and the latter has an algebra structure. By this isomorphism we endow $\mathrm{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[K], \mathbb{Z})$ with a bigraded algebra structure.

We introduce a factor algebra

$$R^*(K) = \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K]/(v_i^2 = u_i v_i = 0,\ i = 1, \dots, m)$$

with the same grading and differential as in $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K]$.

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The algebra $R^*(K)$ has a finite additive basis consisting of the monomials of the form $u_\tau v_\sigma$ where $\sigma \in K$ and $\tau \subset [m] \setminus \sigma$. Therefore we have an additive inclusion

$$\iota : R^*(K) \to \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K]$$

which satisfies $\rho \cdot \iota = id$.

Lemma (1.3)

The quotient map $\rho : \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K] \to R^*(K)$ induces an isomorphism in cohomology.

By this lemma we have $\mathrm{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[K], \mathbb{Z}) \cong H[R^*(K), d]$ as algebras.

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To prove that ρ is an isomorphism, we show that all maps $\rho_j : R^*(K)_j \to R^*(K)_{j+1}$ are isomorphic for $j = 0, 1, \cdots, m-1$. $\square$

Proof of Lemma 1.3.

$\rho_j : R^*(K)_j \to R^*(K)_{j+1}$ is surjective and its kernel is

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Since, for $f, g \in R^*(K)_j$, we have

$$d(u_{j+1}v_{j+1}f + v_{j+1}^2 g) = -u_{j+1}v_{j+1}df + v_{j+1}^2(f + dg),$$

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it is easy to see that the $H[u_{j+1}v_{j+1}R^*(K)_j + v_{j+1}^2 R^*(K)_j, d] = 0$. By the long exact sequence associated with the short exact sequence of cochain complexes

$$0 \to u_{j+1}v_{j+1}R^*(K)_j + v_{j+1}^2 R^*(K)_j \to R^*(K)_j \to R^*(K)_{j+1} \to 0$$

and the fact above we see that $\rho_j : R^*(K)_j \to R^*(K)_{j+1}$ is isomorphic. $\square$

Cellular cochain complex

Recall that

$$Z_K = Z_K(D^2, S^1) = \bigcup_{\sigma \in K} (D^2, S^1)^\sigma = \bigcup_{\sigma \in K} (D^2)^\sigma \times (S^1)^{[m] \setminus \sigma}.$$

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Thus

Lemma (2.1)

$$Z_K = \bigcup_{\sigma \in K,\ \tau \subset [m]\setminus\sigma} T(\sigma, \tau).$$

The cellular chain complex $C_*(Z_K)$ is a chain complex whose ith chain $C_i(Z_K)$ has a free basis $T(\sigma,\tau),\ \sigma\in K,\ \tau\subset[m]\setminus\sigma$ with $i=2|\sigma|+|\tau|$. Its boundary operator ∂ is given by

$$\partial T(\sigma,\tau)=\sum_{j\in\sigma} sgn(j,\tau)T(\sigma\setminus\{j\},\tau\cup\{j\}),$$

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Its dual cochain complex is the cellular cochain complex $C^*(Z_K)$, which has an additive basis consisting of the cochains $T(\sigma,\tau)^*$. The coboundary operator is the dual $\delta = \partial^*$. It has a natural bigrading defined by $\mathrm{bideg}\, T(\sigma,\tau)^* = (-|\tau|, 2|\sigma| + 2|\tau|)$, so that $\mathrm{bideg} D = (0,2)$, $\mathrm{bideg}\, T = (-1,2)$ and $\mathrm{bideg} 1 = (0,0)$.

$$C^*(Z_K) = \oplus_{j=0}^{m} C^{*,2j}(Z_K)$$

Since the cohomology of $C^*(Z_K)$ is $H^*(Z_K;\mathbb{Z})$, the cohomology of Z_K acquires an additional grading.

$$H^k(Z_K;\mathbb{Z}) = \oplus_{-i+2j=k} H^{-i,2j}(Z_K),$$

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Lemma (2.2)

The map

$$g : R^*(K) \to C^*(Z_K), \quad u_\tau v_\sigma \to T(\sigma,\tau)^*$$

is an isomorphism of bigraded differential modules. In particular, we have an additive isomorphism

$$H[R^*(K)] \cong H^*(Z_K).$$

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$$d(u_\tau v_\sigma) = \sum_{j \in \tau} sgn(j, \tau) u_{\tau \setminus \{j\}} v_{\sigma \cup \{j\}}.$$

Here we remark that $v_{\sigma \cup \{j\}} = 0$ if $\sigma \cup \{j\} \notin K$ by definition.

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Here we remark that $v_{\sigma \cup \{j\}} = 0$ if $\sigma \cup \{j\} \notin K$ by definition. On the other hand in $C^*(Z_K)$ we have

$$\delta T(\sigma,\tau)^* = \sum_{j \in \tau,\ \sigma \cup \{j\} \in K} sgn(j,\tau) T(\sigma \cup \{j\}, \tau \setminus \{j\})^*$$

since

$$\partial T(\sigma,\tau) = \sum_{j \in \sigma} sgn(j,\tau) T(\sigma \setminus \{j\}, \tau \cup \{j\}).$$

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The standard definition of the multiplication in cohomology of a cell complex X via cellular cochain complex is as follows. Consider a composite map of cellular cochain complexes:

$$C^*(X) \otimes C^*(X) \xrightarrow{\times} C^*(X \times X) \xrightarrow{\tilde{\Delta}^*} C^*(X).$$

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Here the map $\times$ assigns to a cellular cochain $c_1 \otimes c_2 \in C^p(X) \otimes C^q(X)$ the cochain $c_1 \times c_2 \in C^{p+q}(X \times X)$ whose value on a cell $e_1 \times e_2 \in X \times X$ is $(-1)^{pq} c_1(e_1) c_2(e_2)$.

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In cohomology, the map above induces a multiplication $H^*(X) \otimes H^*(X) \to H^*(X)$ which does not depend on a choice of cellular approximation.

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$$\tilde{\Delta}(re^{i\theta}) = \begin{cases} ((1-r) + re^{2i\theta}, 1) & \text{for } 0 \le \theta \le \pi, \\ (1, (1-r) + re^{2i\theta}) & \text{for } \pi \le \theta \le 2\pi. \end{cases}$$

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Taking an m-fold product, we obtain a cellular approximation $\tilde{\Delta} : (D^2)^m \to (D^2)^m \times (D^2)^m$ which restricts to a cellular approximation for the diagonal map of Z_K for arbitrary K.

$$\begin{array}{ccc} Z_K & \xrightarrow{\tilde{\Delta}} & Z_K \times Z_K \\ \downarrow & & \downarrow \\ (D^2)^m & \xrightarrow{\tilde{\Delta}} & (D^2)^m \times (D^2)^m \end{array}$$

Lemma (2.3)

The cellular cochain algebra $C^*(Z_K)$ defined by the diagonal approximation $\tilde{\Delta} : Z_K \to Z_K \times Z_K$ is multiplicatively isomorphic to $R^*(K)$. Therefore, we get an isomorphism of cohomology algebras:

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Proof of Lemma 2.3.

We first consider the case $m = 1$ and $K = \Delta^{[1]}$, that is, $Z_K = D^2$. The cellular cochain algebra of D^2 is additively generated by the cochains $1 \in C^0(D^2)$, $T^* \in C^1(D^2)$ and $D^* \in C^2(D^2)$ dual to the corresponding cells. The multiplication defined in $C^*(D^2)$ is trivial. To check this it suffices to show that $T^* \cdot T^* = 0$ by degree reason:

$$T^* \cdot T^*(D^2) = (T^* \otimes T^*)(\tilde{\Delta}(D^2)) = (T^* \otimes T^*)(D^2 \times 1 + 1 \times D^2) = 0$$

$\square$

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Since $Z_K \subset (D^2)^m$ is a cell subcomplex and the cellular approximation $\tilde{\Delta} : (D^2)^m \to (D^2)^m \times (D^2)^m$ induces a cellular approximation of Z_K, we obtain a multiplicative map $q : C^*((D^2)^m) \to C^*(Z_K)$. Now consider the commutative diagram

$$\begin{array}{ccc} R^*(\Delta^{[m]}) & \xrightarrow{f} & C^*((D^2)^m) \\ \big\downarrow p & & \big\downarrow q \\ R^*(K) & \xrightarrow{g} & C^*(Z_K). \end{array}$$

$\square$

Proof of Lemma 2.3.

Now consider the commutative diagram

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Here the maps p, q and f are multiplicative, while g is an additive isomorphism. Since p is onto, g is also a multiplicative isomorphism. □

Hochster's theorem

Since in $C^*(Z_K)$ we have

$$\delta T(\sigma,\tau)^* = \sum_{j\in\tau,\ \sigma\cup\{j\}\in K} sgn(j,\tau)T(\sigma\cup\{j\},\tau\setminus\{j\})^*,$$

$C^*(Z_K)$ is a direct sum of smaller subcomplexes as

$$C^*(Z_K) = \bigoplus_{\tau\in[m]} C^{*,2\tau}(Z_K)$$

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where $C^{*,2\tau}(Z_K)$ is the subcomplex generated by the cochains $T(\sigma,\tau\setminus\sigma)^*$ with $\sigma\subset\tau$ and $\sigma\in K$. Since

$$\mathrm{bideg}\, T(\sigma,\tau)^* = (-|\tau|, 2|\sigma|+2|\tau|),$$

we have

$$H^{-i,2j}(Z_K) = \bigoplus_{\tau\in[m],\ |\tau|=j} H^{-i,2\tau}(Z_K)$$

where $H^{-i,2\tau}(Z_K) = H^{-i}[C^{*,2\tau}(Z_K)]$.

Recall the join of two simplicial complexes. Given two simplicial complexes K_1 and K_2 with disjoint vertex sets V_1 and V_2 respectively, their join $K_1 * K_2$ is defined as

$$K_1 * K_2 = \{\sigma_1 \sqcup \sigma_2 \subset V_1 \sqcup V_2 \mid \sigma_i \in K_i \text{ for } i = 1, 2\}.$$

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Now we introduce a multiplication in the sum

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For $I, J \subset [m]$ and $\alpha \in \tilde{H}^*(K_I)$ and $\beta \in \tilde{H}^*(K_J)$ we define the product $\alpha \cdot \beta$ as follows.

Assume that $I \cap J = \emptyset$. Then we have an inclusion of subcomplexes

$$i : K_{I \sqcup J} \hookrightarrow K_I * K_J, \quad \sigma \mapsto (\sigma \cap I) \sqcup (\sigma \cap J)$$

and an injection of reduced simplicial cochains

$$f : \tilde{C}^p(K_I) \otimes \tilde{C}^q(K_I) \to C^{p+q+1}(K_I * K_J).$$

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Now we define

$$\alpha \cdot \beta = \begin{cases} 0, & I \cap J \neq \emptyset, \\ i^* f(\alpha \otimes \beta) \in H^{p+q+1}(K_{I \sqcup J}), & I \cap J = \emptyset. \end{cases}$$

Theorem (Baskakov '02)

There are isomorphisms

$$\tilde{H}^p(K_I) \xrightarrow{\cong} H^{p+1-|I|,2I}(Z_K)$$

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Proof of Baskakov's theorem.

Define a map of cochain complexes

$$\gamma : \tilde{C}^p(K_I) \to C^{p+1-|I|,2I}(Z_K), \quad \sigma^* \mapsto \varepsilon(\sigma)T(\sigma, I \setminus \sigma)^*,$$

where $\varepsilon(\sigma) = (-1)^{\sharp\{(s,t)\in(I\setminus\sigma)\times\sigma \ \mid\ s>t\}}$. Check that
$\mathrm{bideg}\, T(\sigma, I \setminus \sigma)^* = (-|I \setminus \sigma|, 2|\sigma| + 2|I \setminus \sigma|) = (p + 1 - |I|, 2|I|)$. $\square$

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On the other hand

$$\begin{aligned} d(u_{I\setminus\sigma} v_\sigma) &= \sum_{j\in I\setminus\sigma,\ \sigma\cup\{j\}\in K} sgn(j, I\setminus\sigma) u_{I\setminus(\sigma\cup\{j\})} v_{\sigma\cup\{j\}}, \\ &= \sum_{j\notin\sigma,\ \sigma\cup\{j\}\in K_I} sgn(j, I\setminus\sigma) u_{I\setminus(\sigma\cup\{j\})} v_{\sigma\cup\{j\}}. \end{aligned}$$

$\square$

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Now we have to do is to check that

$$\sharp\{s \in I \setminus \sigma \mid s < j\} + \sharp\{(s,t) \in (I \setminus (\sigma \cup \{j\})) \times (\sigma \cup \{j\}) \mid s > t\}$$

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In $R^*(K)$ we have

$$u_{I\setminus\sigma} v_\sigma u_{J\setminus\tau} v_\tau = \begin{cases} 0 & I \cap J \neq \emptyset, \\ u_{(I\sqcup J)\setminus(\sigma\sqcup\tau)} v_{\sigma\cup\tau} & I \cap J = \emptyset, \end{cases}$$

since $u_i u_i = u_i v_i = v_i v_i = 0$. Moreover, if $\sigma \cup \tau \notin K$, then $v_{\sigma\cup\tau} = 0$ by definition. This multiplicative structure coincides with the definition. □

Corollary

A simplicial complex K is Golod over $\mathbf{k}$ if and only if for every pair $I, J \subset [m]$ such that $I \cap J = \emptyset$ the composite of maps

$$\tilde{H}^p(K_I;\mathbf{k}) \otimes \tilde{H}^q(K_J;\mathbf{k}) \to \tilde{H}^{p+q+1}(K_I * K_J;\mathbf{k}) \to \tilde{H}^{p+q+1}(K_{I \sqcup J};\mathbf{k})$$

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is trivial.

Corollary

If a simplicial complex K is Golod over $\mathbf{k}$, then so is its full subcomplex K_I for every $I \subset [m]$.

Theorem

Let K be a simplicial complex. Then the cohomology ring of the real moment-angle complex is given as

$$H^p(Z_K(D^1, S^0)) \cong \bigoplus_{I \subset [m]} \tilde{H}^{p-1}(K_I).$$

The multiplication is given by the following formula under the identification above.

$$\begin{aligned} \tilde{H}^{p-1}(K_I) \otimes \tilde{H}^{q-1}(K_J) \to \tilde{H}^{p-1}(K_{I\cap J^c}) \otimes \tilde{H}^{q-1}(K_J) \\ \cong \tilde{H}^{p+q-1}(K_{I\cap J^c} * K_J) \to \tilde{H}^{p+q-1}(K_{I\cup J}) \end{aligned}$$

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Corollary

A simplicial complex K is Golod over $\mathbf{k}$ *if and only if the cohomology ring of the real moment-angle complex $H^*(Z_K(D^1, S^0); \mathbf{k})$ has the trivial multiplication.*