Topology of Polyhedral products and Golod property of Stanley-Reisner ring, III

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Plan of talks

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- First day. Main results and topological background.
- Second day. Bridge between algebra and topology.
- Today. Sketch of Proofs.
  - Stratification
  - Splitting of Stratification
  - Generalization

Result and Idea of proof

Our main result is

Theorem (I. & Kishimoto '13 and '14)

If the Alexander dual of K is SCM over $\mathbb{Z}$ and each X_i is a based CW-complex,

$$Z_K(C\underline{X}, \underline{X}) \simeq \bigvee_{\emptyset \neq I \subset [m]} |\Sigma K_I| \wedge \widehat{X}^I$$

where K_I is the full subcomplex of K on I and $\widehat{X}^I = \bigwedge_{i \in I} X_i$.

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We prove the theorem by the following steps.

- We introduce a stratification on real moment-angle complex.

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- We show that the stratification is split.

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We prove the theorem by the following steps.

- We introduce a stratification on real moment-angle complex.
- We show that the stratification is split.
- We generalize the stratification and prove the theorem.

Stratification

Recall that

$$Z_K = Z_K(D^1, S^0) = \bigcup_{\sigma \in K} (D^1, S^0)^\sigma = \bigcup_{\sigma \in K} (D^1)^\sigma \times (S^0)^{[m] \setminus \sigma}$$

where -1 is the base point of $S^0 = \{-1, 1\} \subset [-1, 1] = D^1$.

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Definition (1.1)

For $i = 0, \ldots, m$, we define

$$Z_K^i = \bigcup_{I \subset [m],\ |I| = i} Z_{K_I}$$

where Z_{K_I} lies in $\{(x_1, \ldots, x_m) \in (D^1)^m \mid x_j = -1 \text{ for } j \notin I\}$.

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Then we get a stratification

$$* = Z_K^0 \subset Z_K^1 \subset \cdots \subset Z_K^{m-1} \subset Z_K^m = Z_K.$$

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we will show that Z_K^i is obtained from Z_K^{i-1} by attaching cones along some map $X \to Z_K^{i-1}$.

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Definition (1.2)

For a (continuous) map $f : X \to Y$ the mapping cone of f is the space

$$C_f = Y \cup_f CX = (Y \sqcup CX)/\sim,$$

where $\sim$ is generated by the relation $(1, x) \sim f(x) \in Y$. Here $CX = [0,1] \times X / \{0\} \times X$.
C_f is said a space obtained from Y by attaching a cone along a map $f : X \to Y$.

The following theorem is the key to understand the homotopy type of mapping cones.

Theorem (1.3)

$f \simeq g : X \to Y$ then $C_f \simeq C_g$. In particular, f is null-homotopic, then $C_f \simeq Y \vee \Sigma X$.

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Question (1.4)

How to show that Z_K^i is obtained from Z_K^{i-1} by attaching cones along some map $X \to Z_K^{i-1}$?

Let (X, A), (Y, B) be pairs of spaces and $f : X \to Y$ be a map. If $f(A) \subset B$, then f is written as $f : (X, A) \to (Y, B)$. Moreover f induces a homeomorphism $f|_{X \setminus A} : X \setminus A \to Y \setminus B$ and $f : X \to f(X)$ is a quotient map onto a closed subset $f(X)$ in Y, then f is called a relative homeomorphism.

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If $f : (CX, X) \to (Y, B)$ is a relative homeomorphism and B is closed in Y, then Y is homeomorphic to the mapping cone $C_{f|_X}$.

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Theorem (1.5)

If $f : (CX, X) \to (Y, B)$ is a relative homeomorphism and B is closed in Y, then Y is homeomorphic to the mapping cone $C_{f|_X}$.

Proof.

Define a map $g : B \sqcup CX \to Y$ by defining $g|_B = \text{inclusion} : B \to Y$ and $g|_{CX} = f$. This map induces a map $\tilde{g} : C_{f|_X} = B \cup_{f|_X} CX \to Y$ which is clearly continuous and bijective. By assumption $\tilde{g}$ is a homeomorphism. □

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$$C_{\sigma\subset\tau} = \{(x_1, \dots, x_m) \in (D^1)^m \mid x_i = -1, +1 \text{ for } i \in \sigma,\ i \notin \tau\}$$

which is a $(|\tau| - |\sigma|)$-dimensional face of $(D^1)^m$.

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In particular, vertices of $(D^1)^m$ are
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- A piecewise linear map

  $$i_c : |\mathrm{Sd}\Delta^{m-1}| \to (D^1)^m, \quad \sigma \mapsto C_{\sigma\subset\sigma}$$

  is an embedding, where $\emptyset \neq \sigma \subset [m]$ is a vertex of $\mathrm{Sd}\Delta^{m-1}$.

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So $i_c(|\mathrm{Sd}\Delta^{m-1}|)$ is the union of all proper faces of $(D^1)^m$ having the vertex $(-1,\dots,-1)$.

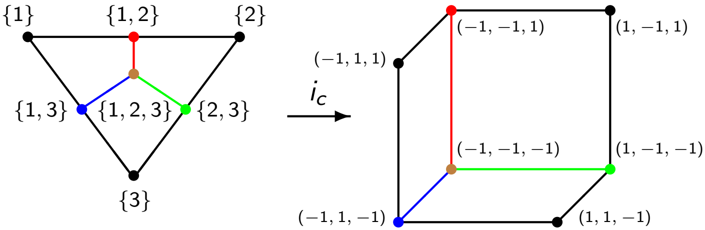

Figure: The embedding $i_c : |\mathrm{Sd}\Delta^2| \to (D^1)^3$

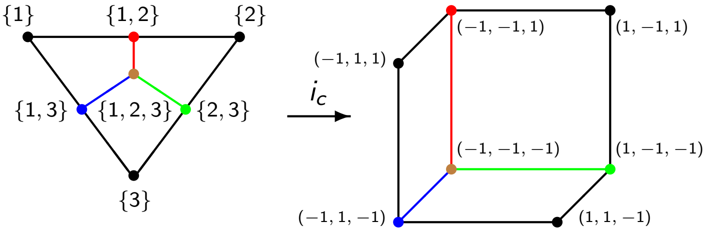

Figure: The embedding $i_c : |\mathrm{Sd}\Delta^2| \to (D^1)^3$

- Define the embeddings

$$i_c : |\mathrm{Sd}K| \to (D^1)^m, \quad \mathrm{C}(i_c) : |\mathrm{C}(\mathrm{Sd}K)| \to (D^1)^m$$

as the restriction of the above embedding and the extension of i_c sending the cone point of $\mathrm{C}(\mathrm{Sd}K)|$ to $(1, \ldots, 1) \in (D^1)^m$, respectively.

By definition, we have

$$Z_K^m = \bigcup_{\rho \in K} (D^1)^\rho \times (S^0)^{[m]\setminus\rho}$$
$$= \bigcup_{\substack{\rho \in K,\\ \sigma \subset [m]\setminus\rho}} (D^1)^\rho \times (-1)^\sigma \times 1^{[m]\setminus(\rho\cup\sigma)} = \bigcup_{\substack{\sigma\subset\tau\subset[m],\\ \tau-\sigma\in K}} C_{\sigma\subset\tau}$$

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$$Z_K^{m-1} = \bigcup_{\substack{\emptyset\neq\sigma\subset\tau\subset[m],\\ \tau-\sigma\in K}} C_{\sigma\subset\tau}.$$

and then

$$Z_K^m - Z_K^{m-1} = \bigcup_{\sigma\subset\tau\in K} C_{\sigma\subset\tau} - \bigcup_{\emptyset\neq\sigma\subset\tau\in K} C_{\sigma\subset\tau}.$$

On the other hand,

$$\mathsf{C}(i_c)(|\mathsf{C}(\mathsf{Sd}K)|) = \bigcup_{\sigma\subset\tau\in K} C_{\sigma\subset\tau}, \quad i_c(|\mathsf{Sd}K|) = \bigcup_{\emptyset\neq\sigma\subset\tau\in K} C_{\sigma\subset\tau}.$$

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Then the map $\mathsf{C}(i_c) : |\mathsf{C}(\mathsf{Sd}K)| \to (D^1)^m$ descends to

$$\mathsf{C}(i_c) : (|\mathsf{C}(\mathsf{Sd}K)|, |\mathsf{Sd}K|) \to (Z_K^m, Z_K^{m-1})$$

which is a relative homeomorphism since

$$Z_K^m - Z_K^{m-1} = \mathsf{C}(i_c)(|\mathsf{C}(\mathsf{Sd}K)|) - i_c(|\mathsf{Sd}K|).$$

More generally, we have:

Proposition (1.6)

The map

$$\mathrm{C}(i_c) : \coprod_{I\subset[m],\ |I|=i} (|\mathrm{C}(\mathrm{Sd}K_I)|, |\mathrm{Sd}K_I|) \to (Z_K^i, Z_K^{i-1})$$

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Corollary (1.7)

Z_K^i is obtained from Z_K^{i-1} by attaching cones along maps $i_c : |\mathrm{Sd}K_I| \to Z_K^{i-1}$ for all $I \subset [m]$ with $|I| = i$.

Triviality

By the above corollary, the proof of the main theorem for a real moment-angle complex is completed by :

Theorem (2.1)

If the Alexander dual of K is SCM over $\mathbb{Z}$, then for any $i = 1, \ldots, m$ and $\emptyset \neq I \subset [m]$ with $|I| = i$, the map $i_c : |\mathrm{Sd}K_I| \to Z_K^{i-1}$ is null homotopic.

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Since the idea is the same, we sketch the proof of this theorem only for shellable complexes, for simplicity.

There are implications of simplicial complexes:

$$\text{shifted} \Rightarrow \text{vertex-decomposable} \Rightarrow \text{shellable} \Rightarrow \text{SCM over } \mathbb{Z}$$

$$\text{pure SCM over } \Bbbk \Leftrightarrow \text{CM over } \Bbbk$$

Definition (2.2)

K is called shellable if there is an ordering of facets $F_1, \ldots, F_k$ (called a shelling ordering) such that the subcomplex

$$\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$$

is pure and $(\dim F_i - 1)$-dimensional for $i = 2, \ldots, k$.

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It is known that:

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If K^* is shellable, so is $(K_I)^*$ for any $\emptyset \neq I \subset [m]$.

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It is known that:

Lemma (2.3)

If K^* is shellable, so is $(K_I)^*$ for any $\emptyset \neq I \subset [m]$.

Then it is sufficient to show that $i_c : |\mathrm{Sd} K| \to Z_K^{m-1}$ is null homotopic. To do this, we try to find a contractible space Δ such that the map $i_c : |\mathrm{Sd} K| \to Z_K^{m-1}$ factors as

$$|\mathrm{Sd} K| \to \Delta \to Z_K^{m-1}.$$

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Lemma (2.4)

The map $i_c : |\mathrm{Sd} K| \to Z_K^{m-1}$ factors as

$$|\mathrm{Sd} K| \xrightarrow{\text{incl}} |\mathrm{Sd} \widehat{K}| \to Z_K^{m-1}.$$

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Proof.

If $\rho \subset [m]$ is a minimal non-face of K, then

$$i_c(\mathrm{Sd}\rho|) = \bigcup_{\emptyset \neq \sigma \subset \rho} C_{\sigma \subset \rho} \subset \bigcup_{\substack{\emptyset \neq \sigma \subset \tau \subset [m] \\ \tau - \sigma \in K}} C_{\sigma \subset \tau} = Z_K^{m-1}.$$

□

Then it is sufficient to show:

Proposition (2.5)

If K^* is shellable, there is a simplicial complex Δ such that

$$K \subset \Delta \subset \widehat{K} \quad \text{and} \quad |\Delta| \simeq *.$$

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We recall the definition of a collapsible complex.

Definition (2.6)

A simplicial complex L is obtained from another simplicial complex K via an elementary collapse if $L = K \setminus \{\sigma, \tau\}$ and σ is a proper face of τ. This means that τ is the only face in K properly containing σ and σ is called free face of K. If L can be obtained from K via a sequence of elementary collapses, then K can be collapsed to L. If K can be collapsed to a 0-simplex $\{\emptyset, \{v\}\}$, then K is collapsible.

We use the following simple lemma to prove Proposition 2.5.

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If K is collapsible, $|K^*|$ is contractible.

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$$\begin{aligned} L^* &= \{\rho \subset [m] \mid \rho^c \notin L\} = \{\rho \subset [m] \mid \rho^c \notin K \setminus \{\sigma, \tau\}\} \\ &= \{\rho \subset [m] \mid \rho^c \notin K \text{ or } \rho^c = \sigma \text{ or } \rho^c = \tau\} \\ &= K^* \cup \{\tau^c, \sigma^c\}. \end{aligned}$$

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If K is collapsible, then there is a sequence of elementary collapses from K to 0-simplex $\{\emptyset, \{1\}\} \subset \Delta^{[m]}$. Then K^* is homotopy equivalent to $\{\emptyset, \{1\}\}^*_{[m]} = \{\sigma \subset [m] \mid \sigma^c \not\in \{\emptyset, \{1\}\}\} = \Delta^{[m]} \setminus \{[m], [2, m]\}$, which is contractible. Therefore, K^* is contractible. $\square$

Proof of Proposition 2.5.

Let $F_1, \ldots, F_k$ be a shelling ordering of K^*, and let $F_{i_1}, \ldots, F_{i_r}$ be all spanning facets, that is, facets satisfying

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$$\Delta = K \cup F_{i_1}^c \cup \cdots \cup F_{i_r}^c$$

where $F_j^c = [m] - F_j$.

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where $F_j^c = [m] - F_j$. Since $F_{i_1}^c, \ldots, F_{i_r}^c$ are minimal non-faces of K, Δ is a simplicial complex satisfying

$$K \subset \Delta \subset \widehat{K}.$$

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Put

$$\Delta = K \cup F_{i_1}^c \cup \cdots \cup F_{i_r}^c$$

where $F_j^c = [m] - F_j$. Since $F_{i_1}^c, \dots, F_{i_r}^c$ are minimal non-faces of K, Δ is a simplicial complex satisfying

$$K \subset \Delta \subset \widehat{K}.$$

On the other hand,

$$\Delta^* = K^* - \{F_{i_1}, \dots, F_{i_r}\}$$

which is collapsible by definition, implying that $|\Delta|$ is contractible by Lemma 2.7. $\square$

Remark

The proof implies that

$$|\Sigma K| \simeq |\Delta|/|K| = \bigvee_{s=1}^{r} S^{m-|F_{i_s}|-1}.$$

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To see this we need the following theorem.

Theorem (2.8)

In the following homotopy commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & Y \end{array} \quad \left(\begin{array}{ccc} |K| & \longrightarrow & |\Delta| \\ \| & & \downarrow \\ |K| & \longrightarrow & C|K| \end{array}\right)$$

the vertical maps induce a map between mapping cones $C_f \to C_g$. Moreover, the vertical maps are homotopy equivalent, then the map $C_f \to C_g$ is a homotopy equivalent.

Generalization

Define $Z^i_K(C\underline{X},\underline{X}) \subset Z_K(C\underline{X},\underline{X})$ similarly to $Z^i_K \subset Z_K$, that is,

$$Z^i_K(C\underline{X},\underline{X}) = \bigcup_{I\subset[m],|I|=i} Z_{K_I}(C\underline{X},\underline{X})$$

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Then there is a stratification

$$* = Z_K^0(C\underline{X}, \underline{X}) \subset Z_K^1(C\underline{X}, \underline{X}) \subset \cdots \subset Z_K^m(C\underline{X}, \underline{X}) = Z_K(C\underline{X}, \underline{X}).$$

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The composite

$$\begin{aligned} |\mathrm{C}(\mathrm{Sd}K)| \times X_1 \times \cdots \times X_m &\xrightarrow{i_c \times 1} (D^1)^m \times X_1 \times \cdots \times X_m \\ &\xrightarrow{\text{perm}} (D^1 \times X_1) \times \cdots \times (D^1 \times X_m) \\ &\xrightarrow{\text{proj}} CX_1 \times \cdots \times CX_m \end{aligned}$$

descends to a relative homeomorphism

$$(|\mathrm{C}(\mathrm{Sd}K)|, |\mathrm{Sd}K)|) \times (X, F) \to (Z_K^m(C\underline{X}, \underline{X}), Z_K^{m-1}(C\underline{X}, \underline{X}))$$

where $X = X_1 \times \cdots \times X_m$ and F is the fat wedge of $X_1, \ldots, X_m$.

We can get an analogous relative homemorphism for the pair

$$(Z_K^i(C\underline{X}, \underline{X}), Z_K^{i-1}(C\underline{X}, \underline{X})) \quad (i = 1, \ldots, m).$$

Then we obtain that $Z_K^i(C\underline{X}, \underline{X})$ is constructed from $Z_K^{i-1}(C\underline{X}, \underline{X})$ by attaching certain spaces, where the attaching maps are explicitly described.

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Theorem (2.9)

If the attaching maps $|\mathrm{Sd} K_I| \to Z_K^{|I|-1}$ are null-homotopic for all $I \subset [m]$, then we have the following decomposition for every collection of based CW-complexes $\underline{X} = \{X_i\}_{i=1}^m$,

$$Z_K(C\underline{X}, \underline{X}) \simeq \bigvee_{\emptyset \neq I \subset [m]} |\Sigma K_I| \wedge \widehat{X}^I.$$

First we consider the case when all CW-complexes have a disjoint base point, that is, $X_i = X_i' \sqcup \{*_i\}$

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$$j : (|\mathrm{Sd}K| \times X) \cup (\mathrm{C}|\mathrm{Sd}K| \times F) \to Z_K^{m-1}(C\underline{X}, \underline{X}),$$

where $X = X_1 \times \cdots \times X_m$ and

$$F = \{*_1\} \times X_2 \times \cdots \times X_m \cup X_1 \times \{*_2\} \times X_3 \times \cdots \times X_m \\ \cup \cdots \cup X_1 \times \cdots \times X_{m-1} \times \{*_m\}$$

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is the fat wedge. Then it is easy to see that

$$(|\mathrm{Sd}K| \times X) \cup (\mathrm{C}|\mathrm{Sd}K| \times F) = (|\mathrm{Sd}K| \times X') \sqcup (\mathrm{C}|\mathrm{Sd}K| \times F)$$

where $X' = X_1' \times \cdots \times X_m'$.

Deforming $\mathsf{C}|\mathrm{Sd}K|$ to its cone point the restriction of j to $\mathsf{C}|\mathrm{Sd}K| \times F$ is naturally homotopic to the composite

$$\mathsf{C}|\mathrm{Sd}K| \times F \to F \to Z_K^{m-1}(C\underline{X}, \underline{X}),$$

where the first map is the projection and the second map in the inclusion.

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$$\mathsf{C}|\mathrm{Sd}K| \times F \to F \to Z_K^{m-1}(C\underline{X}, \underline{X}),$$

where the first map is the projection and the second map in the inclusion.
$Z_K^{m-1}(C\underline{X}, \underline{X})$ has the following subcomplex

$$\begin{aligned}&\{*_1\} \times (CX_2 \times X_3 \times \cdots \times X_m \cup \cdots \cup X_2 \times \cdots \times X_{m-1} \times CX_m)\\ &\cup \{*_2\} \times (CX_1 \times X_3 \times \cdots \times X_m \cup \cdots \cup X_1 \times X_3 \times \cdots \times X_{m-1} \times CX_m)\\ &\cup \cdots \cup \{*_m\} \times (CX_1 \times X_2 \times \cdots \times X_{m-1} \cup \cdots \cup X_1 \times X_2 \times \cdots \times X_{m-2} \times CX_{m-1}),\end{aligned}$$

so we can deform CX_i to its cone point sequentially for $i = 1$ to m. Thus we deform F to the point in $Z_K^{m-1}(C\underline{X}, \underline{X})$.

On the other hand on $|\mathrm{Sd}K| \times X$, j factors as

$$|\mathrm{Sd}K| \times X \to Z_K^{m-1} \times X \to Z_K^{m-1}(C\underline{X}, \underline{X}).$$

By assumption $|\mathrm{Sd}K| \to Z_K^{m-1}$ is null-homotopic, j is deformed to a map

$$|\mathrm{Sd}K| \times X \to \{*\} \times X \to Z_K^{m-1}(C\underline{X}, \underline{X}).$$

Since $\{*\} \times X$ is mapped to the base-point in $Z_K^{m-1}(C\underline{X}, \underline{X})$, we proved that j is null-homotopic.

We use the following lemma to prove Theorem in the general case.

Lemma (2.10)

Suppose that there is a commutative diagram

$$\begin{tikzcd} A_1 \arrow[d, "\alpha"] & B_1 \arrow[l] \arrow[r, "\theta_1"'] \arrow[d, "\beta"] & C_1 \arrow[d, "\gamma"] \\ A_2 & B_2 \arrow[l] \arrow[r, "\theta_2"'] & C_2 \end{tikzcd}$$

in which θ_1, θ_2 are cofibrations and α, β, γ are homotopy equivalences. Then the induced map between pushouts $A_1 \cup_{B_1} C_1 \to A_2 \cup_{B_2} C_2$ is a homotopy equivalence.

We recall a class of simplicial complexes which satisfy the strong gcd-condition.

Definition (Jöllenbeck, '06)

A simplicial complex K is said to satisfy the strong gcd-condition if the set of minimal non-faces of K admits a strong gcd-order. A strong gcd-order is a linear order, $M_1, \cdots, M_r$, of the minimal non-faces of K such that whenever $1 \leq i < j \leq r$ and $M_i \cap M_j = \emptyset$, there is a k with $i < k \neq j$ such that $M_k \subset M_i \cup M_j$.

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Question

Let K be a simplicial complex which satisfies the strong gcd-condition. Can we find a contractible subcomplex of $Z_K^{m-1}(D^1, S^0)$ which contains $i_c(|\mathrm{Sd} K|)$?