Topology of Polyhedral products and Golod property of Stanley-Reisner ring, I

Kouyemon Iriye (OPU)

19 February 2014; Matsumoto

Introduction

In this series of talks I aim to show ideas of the proofs of the following theorems and talk about their algebraic and topological background.

Theorem (I. & Kishimoto '13 and '14)

Let K be a simplicial complex on the vertex set $[m] = \{1, \dots, m\}$ and $\underline{X} = \{X_i\}_{i=1}^m$ be a set of based CW-complexes. If the Alexander dual of K is sequentially Cohen-Macaulay over $\mathbb{Z}$, there is a homotopy equivalence

$$Z_K(C\underline{X}, \underline{X}) \simeq \bigvee_{I \subset [m]} |\Sigma K_I| \wedge \widehat{X}^I.$$

Corollary

If the Alexander dual of K is sequentially Cohen-Macaulay over $\mathbb{Z}$, the moment-angle complex $Z_K(D^2, S^1)$ is homotopy equivalent to a wedge of spheres with dimension greater than 1.

Corollary

If the Alexander dual of K is sequentially Cohen-Macaulay over $\mathbb{Z}$, the moment-angle complex $Z_K(D^2, S^1)$ is homotopy equivalent to a wedge of spheres with dimension greater than 1.

Corollary (Herzog-Reiner-Welker, '99)

If the Alexander dual of K is sequentially Cohen-Macaulay over $\mathbb{Z}$, K is Golod over any field.

References

1. V. Buchstaber and T. Panov, *Torus actions and their applications in topology and combinatorics,* University Lecture Series, vol. 24(2002), AMS.

2. V. Buchstaber and T. Panov, *Toric topology,* arXive:1210.2368.

3. K. Iriye and D. Kishimoto, *Topology of polyhedral products and Golod property of Stanley-Reisner rings*, arXive:1306.6221.

4. T. Panov, *Cohmology of face rings, and torus actions*, arXive:0506526v3.

5. T. Panov, *Moment-angle manifolds and complexes*, Lecture Notes KAIST '2010, arXive:1008.5047v2.

Plan of talks

- Today. I will explain all technical terms in the main result and topological and homotopy theoretical background.
  - Simplicial complex
  - Polyhedral Products
  - Coordinate subspace arrangements
- Second day. Bridge between algebra and topology
- Third day. Sketch of Proofs.

Simplicial Complex

Definition (2.1)

Let V be a finite set. A subset K of 2^V is called a (abstract) simplicial complex on the vertex set V, if the following two conditions are satisfied:

1. $\{v\} \in K$ for any $v \in V$, and
2. $\sigma \subset \tau \in K$, then $\sigma \in K$. In particular, $\emptyset \in K$.

Simplicial Complex

Definition (2.1)

Let V be a finite set. A subset K of 2^V is called a (abstract) simplicial complex on the vertex set V, if the following two conditions are satisfied:

1. $\{v\} \in K$ for any $v \in V$, and
2. $\sigma \subset \tau \in K$, then $\sigma \in K$. In particular, $\emptyset \in K$.

If $K = 2^V$, we write $K = \Delta^V$. An element of a simplicial complex K is called a face or simplex of K and the dimension of a face σ is defined as $\dim \sigma = |\sigma| - 1$. The dimension of K is the maximum dimension of its faces. A maximal face is called facet and K is called pure if all its facets have the same dimension.

For a simplicial complex K we define its geometrical realization denoted by $|K|$ as follows. Let $V = \{v_1, \cdots, v_m\}$ be the vertex set of K and $\{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$ be the canonical basis of $\mathbb{R}^m$. For a simplex $\sigma = \{v_{i_1}, \cdots, v_{i_k}\}$ of K we define a subspace $|\sigma| \subset \mathbb{R}^m$ as the convex hull of the points $\{\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k}\}$.

For a simplicial complex K we define its geometrical realization denoted by $|K|$ as follows. Let $V = \{v_1, \cdots, v_m\}$ be the vertex set of K and $\{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$ be the canonical basis of $\mathbb{R}^m$. For a simplex $\sigma = \{v_{i_1}, \cdots, v_{i_k}\}$ of K we define a subspace $|\sigma| \subset \mathbb{R}^m$ as the convex hull of the points $\{\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k}\}$. That is,

$$|\sigma| = \{\lambda_{i_1}\mathbf{e}_{i_1} + \cdots + \lambda_{i_k}\mathbf{e}_{i_k} \mid \sum_{j=1}^{k} \lambda_{i_j} = 1,\ \lambda_{i_j} \geq 0 \text{ for } j = 1, \cdots, k\}$$

For a simplicial complex K we define its geometrical realization denoted by $|K|$ as follows. Let $V = \{v_1, \cdots, v_m\}$ be the vertex set of K and $\{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$ be the canonical basis of $\mathbb{R}^m$. For a simplex $\sigma = \{v_{i_1}, \cdots, v_{i_k}\}$ of K we define a subspace $|\sigma| \subset \mathbb{R}^m$ as the convex hull of the points $\{\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_k}\}$. That is,

$$|\sigma| = \{\lambda_{i_1}\mathbf{e}_{i_1} + \cdots + \lambda_{i_k}\mathbf{e}_{i_k} \mid \sum_{j=1}^{k} \lambda_{i_j} = 1,\ \lambda_{i_j} \geq 0 \text{ for } j = 1, \cdots, k\}$$

Then we define

$$|K| = \bigcup_{\sigma \in K} |\sigma| \subset \mathbb{R}^m.$$

All polyhedron are also called geometrical realization of K if they are homeomorphic to $|K|$.

Let K be a simplicial complex on the vertex set V. For a subset $I \subset V$ we define the full subcomplex K_I as

$$K_I = \{\sigma \subset I \mid I \in K\}.$$

Let K be a simplicial complex on the vertex set V. For a subset $I \subset V$ we define the full subcomplex K_I as

$$K_I = \{\sigma \subset I \mid I \in K\}.$$

For a simplex $\sigma \in K$ we define the link $\mathrm{link}_K(\sigma)$ and face-deletion $\mathrm{fdel}_K(\sigma)$ as

$$\begin{aligned} \mathrm{link}_K(\sigma) &= \{\tau \in K \mid \sigma \cap \tau = \emptyset,\ \sigma \cup \tau \in K\}, \\ \mathrm{fdel}_K(\sigma) &= \{\tau \in K \mid \sigma \not\subset \tau\}. \end{aligned}$$

In particular, $\mathrm{link}_K(\sigma) = K$ for $\sigma = \emptyset$.
If σ is a vertex, say $\{v\}$, then $\mathrm{fdel}_K(\{v\})$ is simply written as

$$\mathrm{del}_K(v) \quad \text{or} \quad K - v.$$

Let K be a simplicial complex on the vertex set $[m]$ and $\mathbf{k}$ be a ring, that is, a commutative ring with unit. The Stanley-Reisner ring or face ring of K over $\mathbf{k}$ is

$$\mathbf{k}[K] = \mathbf{k}[v_1, \cdots, v_m]/I(K)$$

where $\mathbf{k}[v_1, \cdots, v_m]$ is the polynomial ring and $I(K)$ is the ideal generated by square free monomials

$$v_{i_1} \cdots v_{i_k}, \quad \{i_1, \cdots, i_k\} \notin K.$$

The Stanley-Reisner ring $\mathbf{k}[K]$ is a graded ring with $\deg v_i = 2$.

Let K be a simplicial complex on the vertex set $[m]$ and $\mathbf{k}$ be a ring, that is, a commutative ring with unit. The Stanley-Reisner ring or face ring of K over $\mathbf{k}$ is

$$\mathbf{k}[K] = \mathbf{k}[v_1, \cdots, v_m]/I(K)$$

where $\mathbf{k}[v_1, \cdots, v_m]$ is the polynomial ring and $I(K)$ is the ideal generated by square free monomials

$$v_{i_1} \cdots v_{i_k}, \quad \{i_1, \cdots, i_k\} \notin K.$$

The Stanley-Reisner ring $\mathbf{k}[K]$ is a graded ring with $\deg v_i = 2$.

Recall that an algebra A over $\mathbf{k}$ is called Cohen-Macaulay if it is a free and finitely generated module over its polynomial subring.

Definition (2.2)

A simplicial set K is called Cohen-Macaulay over $\mathbf{k}$ (simply CM/$\mathbf{k}$) if so is its Stanley-Reisner ring.

Definition (2.2)

A simplicial set K is called Cohen-Macaulay over $\mathbf{k}$ (simply CM/$\mathbf{k}$) if so is its Stanley-Reisner ring.

A simplicial complex K is s-acyclic over $\mathbf{k}$ if

$$\tilde{H}_i(K;\mathbf{k}) = 0 \quad \text{for all} \quad 0 \le i \le s,$$

where $\tilde{H}_i(K;\mathbf{k})$ denotes the reduced i-th homology group of K with coefficient $\mathbf{k}$.

Definition (2.2)

A simplicial set K is called Cohen-Macaulay over $\mathbf{k}$ (simply CM/$\mathbf{k}$) if so is its Stanley-Reisner ring.

A simplicial complex K is *s*-acyclic over $\mathbf{k}$ if

$$\tilde{H}_i(K;\mathbf{k}) = 0 \quad \text{for all} \quad 0 \leq i \leq s,$$

where $\tilde{H}_i(K;\mathbf{k})$ denotes the reduced i-th homology group of K with coefficient $\mathbf{k}$.

Theorem (Reisner '76)

A simplicial complex K is CM/$\mathbf{k}$ if and only if $\mathrm{link}_K(\sigma)$ is $(\dim \mathrm{link}_K(\sigma) - 1)$-acyclic for any face σ of K.
In particular, K itself $\dim K - 1$-acyclic and pure.

Define the pure d-skeleton $K^{[d]}$ of K as the subcomplex of K generated by all d-dimensional faces of K. Stanley extended the concept of Cohen-Macaulayness to nonpure complexes:

Define the pure d-skeleton $K^{[d]}$ of K as the subcomplex of K generated by all d-dimensional faces of K. Stanley extended the concept of Cohen-Macaulayness to nonpure complexes:

Definition (2.3, Stanley '96)

A simplicial complex K is sequentially Cohen-Macauley over $\mathbf{k}$ (SCM/$\mathbf{k}$ for short) if the pure d-skeleton $K^{[d]}$ is CM/$\mathbf{k}$ for every $d \geq 0$.

A class of shellable complexes is the most well-studied class of sequentially Cohen-Macauley complexes.

Definition (2.4, Björner and Wachs, '96)

A simplicial complex K is shellable if there is an ordering $F_1, \cdots, F_k$ of facets of K such that the complex

$$G_i = \langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$$

is pure and $\dim F_i - 1$-dimensional for all $i = 2, \cdots, k$. If G_i is the entire boundary of F_i, then F_i is called a spanning facet. Here $\langle F_1, \ldots, F_{i-1} \rangle$ denotes the subcomplex generated by the facets $F_1, \ldots, F_{i-1}$.

Definition (2.5)

Let K be a simplicial complex on the vertex set $V \subset X$. The Alexander dual of K with respect to X is the simplicial complex

$$K_X^* = \{\sigma \subset X \mid X \setminus \sigma \notin K\}.$$

If there is no reference to any underlying set X, it is assumed that $X = V$.

Definition (2.5)

Let K be a simplicial complex on the vertex set $V \subset X$. The Alexander dual of K with respect to X is the simplicial complex

$$K_X^* = \{\sigma \subset X \mid X \setminus \sigma \notin K\}.$$

If there is no reference to any underlying set X, it is assumed that $X = V$.

It is easy to see that $(K_X^*)_X^* = K$.

For a simplicial complex K on V, $\sigma \subset V$ is called minimal non-face if σ is not a face of K but all its boundary are faces of K.

$$\sigma \text{ is a facet of } K \iff \sigma^c = X \setminus \sigma \text{ is a minimal non-face of } K_X^*$$

Theorem (Combinatorial Alexander Duality)

For a simplicial complex $K \neq \Delta^{[m]}$ *on the vertex set* $V \subset [m]$ *it holds that* $\tilde{H}_i(K^*_{[m]}; \mathbf{k}) \cong \tilde{H}^{m-3-i}(K; \mathbf{k})$. *Here as a convention* $\tilde{H}_{-1}(\{\emptyset\}; \mathbf{k}) = \tilde{H}^{-1}(\{\emptyset\}; \mathbf{k}) = \mathbf{k}$.

Theorem (Combinatorial Alexander Duality)

For a simplicial complex $K \neq \Delta^{[m]}$ on the vertex set $V \subset [m]$ it holds that $\tilde{H}_i(K^*_{[m]}; \mathbf{k}) \cong \tilde{H}^{m-3-i}(K; \mathbf{k})$. Here as a convention $\tilde{H}_{-1}(\{\emptyset\}; \mathbf{k}) = \tilde{H}^{-1}(\{\emptyset\}; \mathbf{k}) = \mathbf{k}$.

There is a simple proof of this theorem in "A. Björner and M. Tancer, *Note: Combinatorial Alexander duality - short and elementary proof*, Discrete & Computational Geometry, Vol. 42(2009), 586–593."

Theorem (Combinatorial Alexander Duality)

For a simplicial complex $K \neq \Delta^{[m]}$ on the vertex set $V \subset [m]$ it holds that $\tilde{H}_i(K^*_{[m]}; \mathbf{k}) \cong \tilde{H}^{m-3-i}(K; \mathbf{k})$. Here as a convention $\tilde{H}_{-1}(\{\emptyset\}; \mathbf{k}) = \tilde{H}^{-1}(\{\emptyset\}; \mathbf{k}) = \mathbf{k}$.

There is a simple proof of this theorem in "A. Björner and M. Tancer, *Note: Combinatorial Alexander duality - short and elementary proof*, Discrete & Computational Geometry, Vol. 42(2009), 586–593."

Let $C_i(K)$ be a free $\mathbf{k}$-module with the free basis $\{\sigma \in K \mid \dim\sigma = i\}$. The *reduced chain complex* of K over $\mathbf{k}$ is the complex $\tilde{C}_*(K)$:

$$\cdots \to C_{i+1}(K) \xrightarrow{\partial_{i+1}} C_i(K) \xrightarrow{\partial_i} C_{i-1}(K) \to \cdots C_0(K) \to C_{-1}(K) \to 0,$$

whose boundary operator ∂_i defined as

$$\partial_i(\sigma) = \sum_{j\in\sigma} sgn(j,\sigma)\sigma \setminus \{j\},$$

where $sgn(j,\sigma) = (-1)^{\sharp\{s\in\sigma \mid s<j\}}$.

The n-th *reduced homology group* of K over $\mathbf{k}$ is defined as

$$\tilde{H}_n(K) = \tilde{H}_n(K; \mathbf{k}) = \mathrm{Ker}\ \partial_n / \mathrm{Im}\ \partial_{n+1}.$$

The n-th *reduced homology group* of K over $\mathbf{k}$ is defined as

$$\tilde{H}_n(K) = \tilde{H}_n(K; \mathbf{k}) = \mathrm{Ker}\ \partial_n / \mathrm{Im}\ \partial_{n+1}.$$

The dual cochain complex $\tilde{C}^*(K)$ is the *reduced cochain complex* of K over $\mathbf{k}$, where

$$C^i(K) = \mathrm{Hom}_{\mathbf{k}}(C_i(K), \mathbf{k})$$

which has the dual free basis $\{\sigma^* \mid \sigma \in K,\ \dim \sigma = i\}$ and whose coboundary operator $\delta^i = (\partial_i)^*$ is given by

$$\delta^i(\sigma^*) = \sum_{j \notin \sigma,\ \sigma \cup \{j\} \in K} sgn(j, \sigma \cup \{j\})(\sigma \cup \{j\})^*.$$

The n-th *reduced cohomology group* of K over $\mathbf{k}$ is defined as

$$\tilde{H}^n(K) = \tilde{H}^n(K; \mathbf{k}) = \mathrm{Ker}\ \delta^n / \mathrm{Im}\ \delta^{n-1}.$$

Lemma (2.6.)

Let K be a simplicial complex on the vertex set $[m]$. Then

$$\tilde{H}_i(K) \cong \tilde{H}_{i+1}(\Delta^{[m]}, K).$$

Lemma (2.6.)

Let K be a simplicial complex on the vertex set $[m]$. Then

$$\tilde{H}_i(K) \cong \tilde{H}_{i+1}(\Delta^{[m]}, K).$$

Proof.

There is the long exact sequence of the pair $(\Delta^{[m]}, K)$:

$$\cdots \to \tilde{H}_{i+1}(\Delta^{[m]}) \to \tilde{H}_{i+1}(\Delta^{[m]}, K) \to \tilde{H}_i(K) \to \tilde{H}_i(\Delta^{[m]}) \to \cdots$$

Since $\tilde{H}_{i+1}(\Delta^{[m]}) = \tilde{H}_i(\Delta^{[m]}) = 0$, the long exact sequence implies that $\tilde{H}_i(K) \cong \tilde{H}_{i+1}(\Delta^{[m]}, K)$. □

Lemma (2.6.)

Let K be a simplicial complex on the vertex set $[m]$. Then

$$\tilde{H}_i(K) \cong \tilde{H}_{i+1}(\Delta^{[m]}, K).$$

Proof.

There is the long exact sequence of the pair $(\Delta^{[m]}, K)$:

$$\cdots \to \tilde{H}_{i+1}(\Delta^{[m]}) \to \tilde{H}_{i+1}(\Delta^{[m]}, K) \to \tilde{H}_i(K) \to \tilde{H}_i(\Delta^{[m]}) \to \cdots$$

Since $\tilde{H}_{i+1}(\Delta^{[m]}) = \tilde{H}_i(\Delta^{[m]}) = 0$, the long exact sequence implies that $\tilde{H}_i(K) \cong \tilde{H}_{i+1}(\Delta^{[m]}, K)$. □

Lemma (2.7.)

Let K be a simplicial complex on the vertex set $[m]$. Then

$$\tilde{H}_{i+1}(\Delta^{[m]}, K) \cong \tilde{H}^{m-i-3}(K^*).$$

Sketch of Proof.

$(i+1)$-st reduced chain complex $C_{i+1}(\Delta^{[m]}, K)$ has a free basis

$$\{\sigma \subset [m] \mid \sigma \notin K,\ \dim \sigma = i+1\} = \{\sigma \subset [m] \mid \sigma \notin K,\ |\sigma| = i+2\}.$$

Sketch of Proof.

$(i+1)$-st reduced chain complex $C_{i+1}(\Delta^{[m]}, K)$ has a free basis

$$\{\sigma \subset [m] \mid \sigma \notin K,\ \dim \sigma = i+1\} = \{\sigma \subset [m] \mid \sigma \notin K,\ |\sigma| = i+2\}.$$

On the other hand $(m-i-3)$-rd reduced cochain complex $C^{m-i-3}(K^*)$ has a free basis

$$\{\sigma^* \mid \sigma \in K^*,\ \dim \sigma = m-i-3\}$$
$$= \{\sigma^* \mid \sigma \subset [m], \sigma^c \notin K,\ |\sigma| = m-i-2\}.$$

Sketch of Proof.

$(i+1)$-st reduced chain complex $C_{i+1}(\Delta^{[m]}, K)$ has a free basis

$$\{\sigma \subset [m] \mid \sigma \notin K,\ \dim\sigma = i+1\} = \{\sigma \subset [m] \mid \sigma \notin K,\ |\sigma| = i+2\}.$$

On the other hand $(m-i-3)$-rd reduced cochain complex $C^{m-i-3}(K^*)$ has a free basis

$$\begin{aligned} &\{\sigma^* \mid \sigma \in K^*,\ \dim\sigma = m-i-3\} \\ &\qquad = \{\sigma^* \mid \sigma \subset [m], \sigma^c \notin K,\ |\sigma| = m-i-2\}. \end{aligned}$$

Therefore, the correspondence $\sigma \mapsto p(\sigma)(\sigma^c)^*$ induces an isomorphism $C_{i+1}(\Delta^{[m]}, K) \to C^{m-i-3}(K^*)$ of modules, where $p(\sigma) = \prod_{i\in\sigma}(-1)^{i-1}$.
Now what we have to do is to check that these isomorphisms commute with (co)boundary maps. $\square$

Sketch of Proof.

$(i+1)$-st reduced chain complex $C_{i+1}(\Delta^{[m]}, K)$ has a free basis

$$\{\sigma \subset [m] \mid \sigma \notin K,\ \dim \sigma = i+1\} = \{\sigma \subset [m] \mid \sigma \notin K,\ |\sigma| = i+2\}.$$

On the other hand $(m-i-3)$-rd reduced cochain complex $C^{m-i-3}(K^*)$ has a free basis

$$\{\sigma^* \mid \sigma \in K^*,\ \dim \sigma = m-i-3\}$$
$$= \{\sigma^* \mid \sigma \subset [m], \sigma^c \notin K,\ |\sigma| = m-i-2\}.$$

Therefore, the correspondence $\sigma \mapsto p(\sigma)(\sigma^c)^*$ induces an isomorphism $C_{i+1}(\Delta^{[m]}, K) \to C^{m-i-3}(K^*)$ of modules, where
$p(\sigma) = \prod_{i \in \sigma} (-1)^{i-1}$.
Now what we have to do is to check that these isomorphisms commute with (co)boundary maps. □

Lemmas 2.6 and 2.7 imply the combinatorial Alexander duality theorem.

Let K_i be a simplicial complex on the vertex set V_i for $i = 1, 2$ with $V_1 \cap V_2 = \emptyset$. Then we define the simplicial join as

$$K_1 * K_2 = \{\sigma_1 \sqcup \sigma_2 \mid \sigma_i \in K_i\}.$$

Let K_i be a simplicial complex on the vertex set V_i for $i = 1, 2$ with $V_1 \cap V_2 = \emptyset$. Then we define the simplicial join as

$$K_1 * K_2 = \{\sigma_1 \sqcup \sigma_2 \mid \sigma_i \in K_i\}.$$

As a special case we have the simplicial cone CK and simplicial suspension ΣK of a simplicial complex K.

$$\begin{aligned} CK &= \Delta^{[1]} * K, \\ \Sigma K &= \partial\Delta^{[2]} * K = CK_+ \cup CK_-, \end{aligned}$$

where $CK_\pm$ is a copy of CK with $CK_+ \cap CK_- = K$.

It is well-know that

$$\begin{aligned}
|K_1 * K_2| &\approx |K_1| * |K_2| \simeq \Sigma|K_1| \wedge |K_2|, \\
|CK| &\approx C|K|, \\
|\Sigma K| &\approx \Sigma|K|.
\end{aligned}$$

It is well-know that

$$\begin{aligned} |K_1 * K_2| &\approx |K_1| * |K_2| \simeq \Sigma|K_1| \wedge |K_2|, \\ |CK| &\approx C|K|, \\ |\Sigma K| &\approx \Sigma|K|. \end{aligned}$$

Here $X_1 * X_2 = \{(x_1, t, x_2) \in X_1 \times [0,1] \times X_2\}/\sim$, where $(x_1, 0, x_2) \sim (x_1', 0, x_2)$ for all $x_1, x_1' \in X_1$ and $(x_1, 1, x_2) \sim (x_1, 1, x_2',)$ for all $x_2, x_2' \in X_2$.

$$\begin{aligned} CX &= X \times [0,1]/X \times \{0\}(\text{or } X \times [0,1]/(X \times \{0\} \cup \{*\} \times [0,1]), \\ \Sigma X &= CX/X \times \{1\}(\text{or } CX/(X \times \{1\} \cup \{*\} \times [0,1]), \\ X \wedge Y &= X \times Y/(X \times \{*\} \cup \{*\} \times Y), \end{aligned}$$

where in the last definition X and Y have the base-point $*$. We will use the following well-known facts freely: $S^n \wedge S^m \approx S^{n+m}$.

Polyhedral Product

- Let K be a simplicial complex on the vertex set $[m] = \{1, \ldots, m\}$.
- Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i \in [m]}$ be a sequence of pairs of spaces.

Polyhedral Product

- Let K be a simplicial complex on the vertex set $[m] = \{1, \ldots, m\}$.
- Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i \in [m]}$ be a sequence of pairs of spaces.

Definition (3.1)

The polyhedral product $Z_K(\underline{X}, \underline{A})$ is defined as

$$Z_K(\underline{X}, \underline{A}) = \bigcup_{\sigma \in K} D(\sigma) \quad (\subset X_1 \times \cdots \times X_m)$$

where $D(\sigma) = Y_1 \times \cdots \times Y_m$ for $Y_i = \begin{cases} X_i & i \in \sigma \\ A_i & i \notin \sigma. \end{cases}$

If $(X_i, A_i) = (X, A)$, then we write simply as $Z_K(X, A)$.

Here are some examples of polyhedral products.

- Let $K = \{\{1\},\{2\},\emptyset\}$ be the discrete simplicial complex on [2] and $(X_i, A_i) = (D^{n_i}, S^{n_i - 1})$ for $i = 1, 2$. Then

$$\begin{array}{rcl} Z_K(\underline{X}, \underline{A}) & = & D^{n_1} \times S^{n_2-1} \cup S^{n_1-1} \times D^{n_2} \\ & = & \partial D^{n_1+n_2} = S^{n_1+n_2-1}. \end{array}$$

Here are some examples of polyhedral products.

- Let $K = \{\{1\},\{2\},\emptyset\}$ be the discrete simplicial complex on [2] and $(X_i, A_i) = (D^{n_i}, S^{n_i-1})$ for $i = 1, 2$. Then

$$\begin{aligned} Z_K(\underline{X},\underline{A}) &= D^{n_1}\times S^{n_2-1}\cup S^{n_1-1}\times D^{n_2} \\ &= \partial D^{n_1+n_2} = S^{n_1+n_2-1}. \end{aligned}$$

- Let $(X, A) = (D^2, S^1)$, then

$$Z_K(D^2, S^1) \subset (D^2)^m \subset (\mathbf{C})^m$$

  is the moment-angle complex introduced by Davis-Januskiewicz '91.

- Let $K = \partial\Delta^{[m]}$ and $(\underline{X}, \underline{A}) = \{(X_i, *)\}_{i=1}^{m}$.

$$Z_K(\underline{X}, *) = * \times X_2 \times \cdots \times X_m \cup \cdots \cup X_1 \times \cdots \times X_{m-1} \times *,$$

which is known as a fat wedge.

- Let $K = \partial\Delta^{[m]}$ and $(\underline{X}, \underline{A}) = \{(X_i, *)\}_{i=1}^{m}$.

$$Z_K(\underline{X}, *) = * \times X_2 \times \cdots \times X_m \cup \cdots \cup X_1 \times \cdots \times X_{m-1} \times *,$$

  which is known as a fat wedge.
- Let $K = \partial\Delta^{[m]}$ and $(\underline{X}, \underline{A}) = \{(CX_i, X_i)\}_{i=1}^{m}$. Then

$$\begin{aligned} Z_K(\underline{X}, \underline{A}) &= X_1 \times CX_2 \times \cdots \times CX_m \cup \dots \\ &\quad \cup CX_1 \times \cdots \times CX_{m-1} \times X_m \\ &\simeq X_1 * \cdots * X_m \simeq \Sigma^{m-1} X_1 \wedge \cdots \wedge X_m \end{aligned}$$

- Let $K = \partial\Delta^{[m]}$ and $(\underline{X}, \underline{A}) = \{(X_i, *)\}_{i=1}^{m}$.

$$Z_K(\underline{X}, *) = * \times X_2 \times \cdots \times X_m \cup \cdots \cup X_1 \times \cdots \times X_{m-1} \times *,$$

which is known as a fat wedge.

- Let $K = \partial\Delta^{[m]}$ and $(\underline{X}, \underline{A}) = \{(CX_i, X_i)\}_{i=1}^{m}$. Then

$$\begin{aligned} Z_K(\underline{X}, \underline{A}) &= X_1 \times CX_2 \times \cdots \times CX_m \cup \dots \\ &\quad \cup CX_1 \times \cdots \times CX_{m-1} \times X_m \\ &\simeq X_1 * \cdots * X_m \simeq \Sigma^{m-1} X_1 \wedge \cdots \wedge X_m \end{aligned}$$

This example is due to Porter '65. Porter defined this special polyhedral product to introduce higher order Whitehead products. Our results can be considered a generalization of his result.

Theorem (Bahri, Bendersky, Cohen and Gitler, '10)

Let $\underline{X} = \{X_i\}_{i=1}^m$ be a collection of based CW-complex. Then there is a homotopy equivalence

$$\Sigma Z_K(C\underline{X}, \underline{X}) \simeq \Sigma \left(\bigvee_{I \subset [m]} \Sigma |K_I| \wedge \hat{X}^I \right),$$

where $\widehat{X}^I = \wedge_{i \in I} X_i$ and $X \vee Y$ is the one-point union of X and Y.

Theorem (Bahri, Bendersky, Cohen and Gitler, '10)

Let $\underline{X} = \{X_i\}_{i=1}^m$ *be a collection of based CW-complex. Then there is a homotopy equivalence*

$$\Sigma Z_K(C\underline{X}, \underline{X}) \simeq \Sigma \left(\bigvee_{I \subset [m]} \Sigma |K_I| \wedge \hat{X}^I \right),$$

where $\widehat{X}^I = \wedge_{i \in I} X_i$ *and* $X \vee Y$ *is the one-point union of* X *and* Y*.*

This theorem is a generalization of the well-known decomposition

$$\Sigma(X_1 \times \cdots \times X_m) \simeq \Sigma \left(\bigvee_{I \subset [m]} \hat{X}^I \right).$$

As a special case we have the following homotopy equivalence

$$\Sigma Z_K(D^2, S^1) \simeq \Sigma \left(\bigvee_{I \subset [m]} \Sigma|K_I| \wedge \overbrace{S^1 \wedge \cdots \wedge S^1}^{|I|} \right) = \bigvee_{I \subset [m]} \Sigma^{|I|+2}|K_I|.$$

Thus by using Bahri, Bendersky, Cohen and Gitler's theorem we obtain the following Hochster's theorem. This is an example of application of topology to algebra.

As a special case we have the following homotopy equivalence

$$\Sigma Z_K(D^2, S^1) \simeq \Sigma \left(\bigvee_{I \subset [m]} \Sigma |K_I| \wedge \overbrace{S^1 \wedge \cdots \wedge S^1}^{|I|} \right) = \bigvee_{I \subset [m]} \Sigma^{|I|+2} |K_I|.$$

Thus by using Bahri, Bendersky, Cohen and Gitler's theorem we obtain the following Hochster's theorem. This is an example of application of topology to algebra.

Theorem (Hochster)

$$\begin{aligned} \mathrm{Tor}_{\mathbb{Z}[v_1,\cdots,v_m]}(\mathbb{Z}[K],\mathbb{Z}) &\cong H^*(Z_K(D^2,S^1);\mathbb{Z}) \\ &\cong \bigoplus_{I \subset [m]} \tilde{H}^*(K_I;\mathbb{Z}) \end{aligned}$$

where $H^{-1}(\emptyset;\mathbb{Z}) = \mathbb{Z}$.

Question

When is the homotopy equivalence given by Bahri, Bendersky, Cohen and Gitler desuspended ?

Question

When is the homotopy equivalence given by Bahri, Bendersky, Cohen and Gitler desuspended ?

We have

$$Z_{\square}(D^2, S^1) = Z_{K_1}(D^2, S^1) \times Z_{K_2}(D^2, S^1) = S^3 \times S^3,$$

where K_1 and K_2 are the discrete simplicial complex with 2-vertexes. This example is a special case of the following example, and shows that there might be a class of simplicial complexes whose moment-angle complex decomposes as a wedge of suspension spaces.

Let K_1 be a simplicial complex on $[m_1]$ and K_2 be a simplicial complex on $[m_1+1, m_1+m_2] = \{m_1+1, \cdots, m_1+m_2\}$. Then

$$\begin{aligned}
Z_{K_1 * K_2}(\underline{X}, \underline{A}) &= \bigcup_{(\sigma_1,\sigma_2)\in K_1 * K_2} D(\sigma_1 \sqcup \sigma_2) \\
&= \bigcup_{(\sigma_1,\sigma_2)\in K_1 * K_2} D(\sigma_1) \times D(\sigma_2) \\
&= (\bigcup_{\sigma_1 \in K_1} D(\sigma_1)) \times (\bigcup_{\sigma_2 \in K_2} D(\sigma_2)) \\
&= Z_{K_1}(\underline{X}, \underline{A}) \times Z_{K_2}(\underline{X}, \underline{A}),
\end{aligned}$$

in the last equation $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^{m_1}$ in the first summand and $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=m_1+1}^{m_1+m_2}$ in the second.

If

$$Z_K(D^2, S^1) \simeq \bigvee_{I \subset [m]} \Sigma |K_I| \wedge \hat{S^1}^I = \bigvee_{I \subset [m]} \Sigma^{|I|+1} |K_I|,$$

$Z_K(D^2, S^1)$ is homotopy equivalent to a suspension space. Then the cohomology ring $H^*(Z_K(D^2, S^1); \mathbf{k})$ has the trivial product.

If

$$Z_K(D^2, S^1) \simeq \bigvee_{I \subset [m]} \Sigma |K_I| \wedge \hat{S^1}^I = \bigvee_{I \subset [m]} \Sigma^{|I|+1} |K_I|,$$

$Z_K(D^2, S^1)$ is homotopy equivalent to a suspension space. Then the cohomology ring $H^*(Z_K(D^2, S^1); \mathbf{k})$ has the trivial product.

Definition (3.2)

A simplicial complex K is called Golod over $\mathbf{k}$ if the cohomology ring $H^*(Z_K(D^2, S^1); \mathbf{k})$ has the trivial product, that is, for $\alpha, \beta \in \tilde{H}^*(Z_K(D^2, S^1); \mathbf{k})$ we have $\alpha\beta = 0$. In particular, if the moment-angle complex $Z_K(D^2, S^1)$ is a suspension, K is Golod.

If

$$Z_K(D^2, S^1) \simeq \bigvee_{I \subset [m]} \Sigma|K_I| \wedge \hat{S^1}^I = \bigvee_{I \subset [m]} \Sigma^{|I|+1}|K_I|,$$

$Z_K(D^2, S^1)$ is homotopy equivalent to a suspension space. Then the cohomology ring $H^*(Z_K(D^2, S^1); \mathbf{k})$ has the trivial product.

Definition (3.2)

A simplicial complex K is called Golod over $\mathbf{k}$ if the cohomology ring $H^*(Z_K(D^2, S^1); \mathbf{k})$ has the trivial product, that is, for $\alpha, \beta \in \tilde{H}^*(Z_K(D^2, S^1); \mathbf{k})$ we have $\alpha\beta = 0$. In particular, if the moment-angle complex $Z_K(D^2, S^1)$ is a suspension, K is Golod.

Golodness condition on a simplicial complex K is a necessary condition for the moment-angle complex $Z_K(D^2, S^1)$ being homotopy equivalent to a wedge of suspension spaces.

Question

For a Golod simplicial complex K, is there a homotopy equivalence $Z_K(D^2, S^1) \simeq \bigvee_{I\subset[m]} \Sigma^{|I|+1}|K_I|$?

Question

For a Golod simplicial complex K, is there a homotopy equivalence $Z_K(D^2, S^1) \simeq \bigvee_{I \subset [m]} \Sigma^{|I|+1}|K_I|$?

Definition (3.3, Jöllenbeck, '06)

A simplicial complex K is said to satisfy the strong gcd-condition if the set of minimal non-faces of K admits a strong gcd-order. A strong gcd-order is a linear order, $M_1, \cdots, M_r$, of the minimal non-faces of K such that whenever $1 \leq i < j \leq r$ and $M_i \cap M_j = \emptyset$, there is a k with $i < k \neq j$ such that $M_k \subset M_i \cup M_j$.

Question

For a Golod simplicial complex K, is there a homotopy equivalence $Z_K(D^2, S^1) \simeq \bigvee_{I \subset [m]} \Sigma^{|I|+1}|K_I|$?

Definition (3.3, Jöllenbeck, '06)

A simplicial complex K is said to satisfy the strong gcd-condition if the set of minimal non-faces of K admits a strong gcd-order. A strong gcd-order is a linear order, $M_1, \cdots, M_r$, of the minimal non-faces of K such that whenever $1 \leq i < j \leq r$ and $M_i \cap M_j = \emptyset$, there is a k with $i < k \neq j$ such that $M_k \subset M_i \cup M_j$.

Theorem (Berglund and Jöllenbeck, '07)

A simplicial complex satisfying the strong gcd-condition is Golod.

Coordinate subspace arrangements

A coordinate subspace in $\mathbb{C}^m$ is

$$L_\omega = \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid z_{i_1} = \cdots = z_{i_k} = 0\}$$

for some subset $\omega = \{i_1, \dots, i_k\} \subset [m]$. Given a simplicial set K, we may define the corresponding coordinate subspace arrangement $\{L_\omega \mid \omega \notin K\}$ and its complement

$$U(K) = \mathbb{C}^m \setminus \bigcup_{\omega \notin K} L_\omega.$$

By definition $Z_K(D^2, S^1) \subset U(K)$.

Coordinate subspace arrangements

A coordinate subspace in $\mathbb{C}^m$ is

$$L_\omega = \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid z_{i_1} = \cdots = z_{i_k} = 0\}$$

for some subset $\omega = \{i_1, \dots, i_k\} \subset [m]$. Given a simplicial set K, we may define the corresponding coordinate subspace arrangement $\{L_\omega \mid \omega \notin K\}$ and its complement

$$U(K) = \mathbb{C}^m \setminus \bigcup_{\omega \notin K} L_\omega.$$

By definition $Z_K(D^2, S^1) \subset U(K)$.

Theorem (Buchstaber-Panov)

There is a T^m-equivariant deformation retraction $U(K) \simeq Z_K(D^2, S^1)$.

Proof.

For $\sigma \subset [m]$ we put

$$U_\sigma = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid z_i \neq 0 \text{ for } i \notin \sigma\}$$

Then

$$U(K) = \bigcup_{\sigma \in K} U_\sigma.$$

There are obvious T^m-equivariant deformation retractions

$$U_\sigma \approx \mathbb{C}^\sigma \times (\mathbb{C} \setminus \{0\})^{[m]\setminus\sigma} \xrightarrow{\simeq} (D^2)^\sigma \times (S^1)^{[m]\setminus\sigma} \approx (D^2, S^1)^\sigma.$$

These patch together to get the required map
$U(K) \to Z_K(D^2, S^1)$. $\square$