Introduction to Stanley–Reisner rings

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The theory of Stanley–Reisner rings originated from two fascinating papers: Stanley's paper on Upper Bound Theorem (apprev. UBT) [**St75**] and Reisner's one [**Re76**] on a topological criterion of the Cohen–Macaulay-ness of a Stanley–Reisner ring. Applying Reisner's result [**Re76**], Stanley [**St75**] showed that McMullen's Upper Bound Theorem [**Mc70**] for the boundary complex of a simplicial polytope can be deduced from the Cohen–Macaulay-ness (and Dehn–Sommerville equation) of the corresponding Stanley–Reisner ring. As a result, he succeeded to generalize UBT to a simplicial sphere. Since then, through other important works—Hochster's beautiful formulas on local cohomologies and Tor modules, Stanley's striking application [**St80**] of toric variety to solve g-conjecture, and connections with graphs, posets, Gröbner basis and so on—, the theory of Stanley–Reisner rings has developed in interactions with combinatorics, topology, commutative algebra, and computational algebra. Stanley–Reisner rings are also known to be related with toric topology, and studies in this direction have attracted attentions [**BP**].

The aim of this talk is to introduce the theory with its historical background in view, based on [**BH, St**], of combinatorics and commutative algebra. Concretely, we will mention main notions and theorems such as (sequentially) Cohen-Macaulay-ness, Gorenstein*-ness, shellability, Alexander duality, Upper Bound Theorem, g-Theorem, and Hochster's formulas. Other important ones and recent developments will be also introduced as long as we have some time.

References

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