INTRODUCTION TO STANLEY–REISNER RINGS

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Abstract. This is the lecture note for the author's talk in "(Non)-Commutative algebra and Topology" at Shinshu University. This note (talk) is based on books on combinatorial commutative algebra such as [10, 37, 52].

Contents

0. Notations and conventions

In this note, unless otherwise said, all the rings are assumed to be commutative with the identity element and the symbols below has the fixed meaning.

- $n :=$ a fixed positive integer.
- $[n] := \{1, \dots, n\}$.
- $\Delta :=$ a simplicial complex on $[n]$.
- $|\Delta| :=$ a geometric realization of Δ.
- $\mathscr{F}(\Delta) :=$ the set of the facets of Δ.
- $\widetilde{H}_i(\Delta; \Bbbk)$ (resp. $\widetilde{H}^i(\Delta; \Bbbk) :=$) the i-th reduced homology (resp. cohomology) of Δ with coefficients in $\Bbbk$.
- $\widetilde{\chi}(\Delta) :=$ the reduced Euler characteristic of Δ.

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- $\partial P :=$ the boundary of a polytope P.
- $\Bbbk :=$ a field.
- $S := \Bbbk[x_1, \dots, x_n]$.
- $\mathfrak{m} := (x_1, \dots, x_n)$.
- $d := \dim \Delta + 1$.
- $\underline{\boldsymbol{e}}_i := i$-th unit vector.
- $\underline{\boldsymbol{e}}_F := \sum_{i \in F} \underline{\boldsymbol{e}}_i$ for $F \subseteq [n]$.
- For $\underline{\boldsymbol{a}} \in \mathbb{Z}^n$, $\operatorname{supp}(\underline{\boldsymbol{a}}) := \{i \in [n] \mid a_i > 0\}$.
- $\mathbb{Z}_{\geq 0} := \{r \in \mathbb{Z} \mid r \geq 0\}$.
- $\preceq$ denotes the partial order on $\mathbb{Z}^n$ defined by
$$\underline{\boldsymbol{a}} \preceq \underline{\boldsymbol{b}} \quad \Longleftrightarrow \quad a_i \leq b_i \quad \text{for all } i.$$
- For $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}} \in \mathbb{Z}^n$, set $[\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}] := \{\underline{\boldsymbol{c}} \in \mathbb{Z}^n \mid \underline{\boldsymbol{a}} \preceq \underline{\boldsymbol{c}} \preceq \underline{\boldsymbol{b}}\}$.
- $M(g) :=$ the module whose degree is shifted by g.
- $C^{\bullet}[r] :=$ the complex translated by r.
- $\operatorname{mod}_G S :=$ the category consisting of finitely generated G-graded S-modules and degree-preserving S-homomorphisms.

1. Face enumeration and Stanley–Reisner rings

The theory of Stanley-Reisner rings has its origin at three famous theorems, called Upper Bound Theorem, g-Theorem, and Hochster's formula.

1.1. Simplicial complexes and f-vectors (historical background).

Throughout this note, n denotes a fixed positive number n, and set $[n] := \{1, \dots, n\}$.

Definition 1.1. Let $2^{[n]}$ be the power set of $[n]$.

(1) A subset $\Delta \subseteq 2^{[n]}$ is called a (finite abstract) *simplicial complex*[1] (on $[n]$) if
$$F \subseteq G \subseteq [n]\,,\ G \in \Delta \quad \Longrightarrow \quad F \in \Delta$$
(2) An element F of Δ is called a *face* of Δ, and its *dimension*, $\dim F$, is defined by
$$\dim F := \#F - 1.$$
(3) The maximal faces of Δ (w.r.t. $\subseteq$) are called *facets*. The set of all the facets is denoted by $\mathscr{F}(\Delta)$
(4) The *dimension* $\dim \Delta$ of Δ is then defined by
$$\dim \Delta := \max \{\dim F \mid F \in \Delta\}\,.$$
(5) For a simplicial complex Δ, a geometric simplicial complex $|\Delta|$ can be constructed up to homeomorphism, which is called a *geometric realization* of Δ.
(6) For $F_1, \dots, F_r \subseteq 2^{[n]}$, set
$$\langle F_1, \dots, F_r \rangle = \{G \subseteq [n] \mid G \subseteq F_i \text{ for some } i\}$$

Remark 1.2. Whenever $\Delta \neq \varnothing$, the complex Δ has a unique face $\varnothing \in 2^{[n]}$ of dimension -1.

Henceforth Δ denotes a simplicial complex on $[n]$ with $\Delta \neq \varnothing$ of dimension $d-1$, unless otherwise said.

[1] All the simplicial complex Δ may admit *ghost vertices*, that is, there may exist $k \in [n]$ such that $\{k\} \notin \Delta$.

Definition 1.3. The vector $f(\Delta) := (f_{-1}(\Delta), f_0(\Delta), \dots, f_{d-1}(\Delta))$, where

$$f_i(\Delta) := \#\{F \in \Delta \mid \dim F = i\},$$

is called the f-vector of Δ.

Remark 1.4.

$$f_{-1}(\Delta) \neq 0 \quad \Longleftrightarrow \quad f_{-1}(\Delta) = 1 \quad \Longleftrightarrow \quad \Delta \neq \varnothing.$$

Problem. Characterize vectors of integers which appear as f-vectors of some special class of Δ. For example, how about those for *simplicial spheres*, i.e., simplicial complexes Δ such that $|\Delta| \approx \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ denotes a $(d-1)$-dimensional sphere.

In the study of f-vectors, *h-vectors* are frequently considered instead of f-vectors themselves.

Definition 1.5. The vector $h(\Delta) := (h_0(\Delta), \dots, h_d(\Delta))$ such that

$$\sum_{i=0}^{d} h_i(\Delta) t^i = \sum_{i=0}^{d} f_{i-1}(\Delta) t^i (1-t)^{d-i}$$

is called the *h-vector* of Δ.

Unless there's no fear of confusion, $h_i(\Delta)$ and $f_i(\Delta)$ are simply denoted by h_i and f_i.

Lemma 1.6. *For Δ of dim $d-1$,*

$$h_i = \sum_{k=0}^{i} (-1)^{i-k} \binom{d-k}{i-k} f_{k-1}, \quad f_i = \sum_{k=0}^{i+1} \binom{d-k}{i+1-k} h_k.$$

In particular,

$$h_0 = f_{-1} = 1, \quad h_1 = f_0 - d, \quad h_d = (-1)^{d-1}\widetilde{\chi}(\Delta), \quad \sum_{i=0}^{d} h_i = f_{d-1},$$

where $\widetilde{\chi}(\Delta) := \sum_{i=-1}^{d-1} (-1)^i f_i = \chi(\Delta) - 1$.

As Lemma 1.6 shows,

$$\boxed{\text{knowing } h(\Delta) \text{ is equivalent to knowing } f(\Delta).}$$

h-vectors can be easily computed by the following triangle like Pascal's.

$$\begin{array}{ccccccccccc}
 & & & & 1 & & f_0 & & & & \\
 & & & 1 & & f_0 - 1 & & f_1 & & & \\
 & & \iddots & & x & & y & & \ddots & & \\
 & \iddots & & & & y - x & & & & f_{d-1} & \\
1 & h_1 & h_2 & \cdots & & \cdots & & \cdots & h_d & &
\end{array}$$

Example 1.7. (1) $\Delta := \langle \{i, i+1 \mid i = 1, \dots, n-1\} \cup \{\{n, 1\}\} \rangle$. The geometric realization $|\Delta|$ is then an n-gon as in Figure 1.

$$f(\Delta) = (1, n, n).$$

Moreover

$$\begin{array}{cccccc} & & 1 & & n & \\ & 1 & & n-1 & & n \\ 1 & & n-2 & & 1 & \end{array}$$

Thus $h(\Delta) = (1, n-2, 1)$.

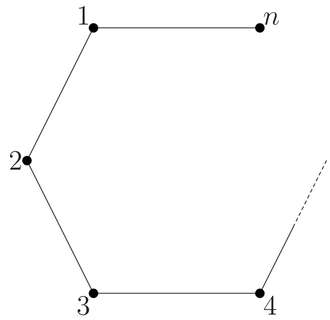

FIGURE 1. n-gon

(2) $\Delta := \langle \{1,2,3\}, \{3,4,5\} \rangle$. $|\Delta|$ is as in Figure 2.

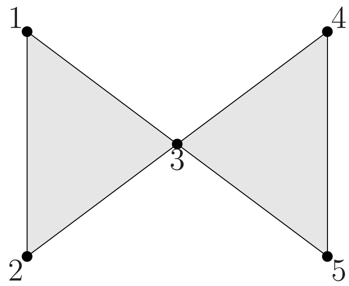

FIGURE 2.

$$f(\Delta) = (1, 5, 6, 2), \qquad h(\Delta) = (1, 2, -1, 0).$$

(3) $\Delta := \langle \{1,2,3\}, \{1,3,4\}, \{1,3,5\} \rangle$. $|\Delta| =$ Figure 3.

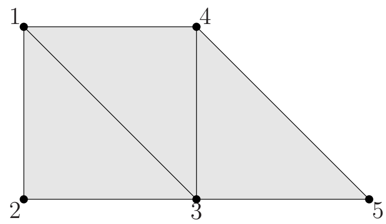

FIGURE 3.

$$f(\Delta) = (1, 5, 7, 3), \qquad h(\Delta) = (1, 2, 0, 0).$$

(4) $\Delta := \langle \{1,2,3\}, \{1,3,4\}, \{1,4,5\}, \{1,2,5\}, \{2,3,6\}, \{3,4,6\}, \{2,5,6\}, \{4,5,6\} \rangle$. $|\Delta| =$ Figure 4.

$$f(\Delta) = (1, 6, 12, 8), \qquad h(\Delta) = (1, 3, 3, 1).$$

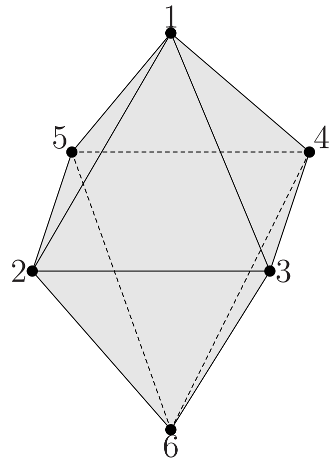

FIGURE 4. octahedron

(5) $\Delta := \langle \{1, 2, 4\}, \{2, 4, 5\}, \{2, 3, 5\}, \{3, 5, 6\}, \{1, 3, 6\}, \{1, 4, 6\}\rangle$. $|\Delta| =$ Figure 5.

$$f(\Delta) = (1, 6, 12, 6), \qquad h(\Delta) = (1, 3, 3, -1).$$

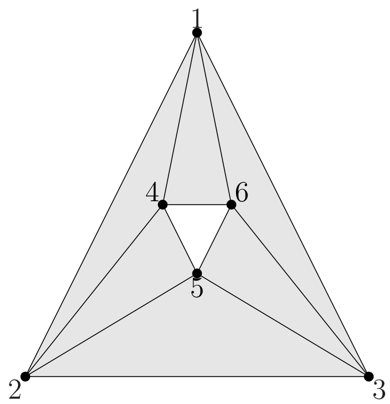

FIGURE 5. cylinder

h-vectors first appear in Sommerville's paper [47], where he did not give a name to them, to describe a relation among components of f-vectors of boundary of polytopes, which is the generalization of [14].

Recall that a convex polytope is called a *simplicial polytope* if each faces are simplex. Let $\mathscr{P}(n, d)$ be the set of simplicial polytopes of dimension d with n vertices. Clearly, for $P \in \mathscr{P}(n, d)$,

$$\partial P \approx \mathbb{S}^{d-1},$$

where ∂P denotes the boundary complex of P. For $\Delta = \partial P$ with $P \in \mathscr{P}(n, d)$ (more precisely when Δ is the boundary complex of P), Dehn and Sommerville discovered the following beautiful equations.

Theorem 1.8 (Dehn–Sommerville equation for simplicial polytopes [14, 47])**.** *For $\Delta = \partial P$ with $P \in \mathscr{P}(n, d)$ (hence $\dim \Delta = d - 1$),*

$$h_i = h_{d-i}$$

for all i.

In the theorem, $h_0 = h_d$ implies $\widetilde{\chi}(\Delta) = (-1)^{d-1}$, which is just Euler's polyhedron formula. Actually the equations hold for simplicial spheres (more generally Eulerian complex), which will be proved later.

Let $C(n,d)$ be a polytope given as the convex hull of n $(\geq d+1)$ distinct points on the curve $\left\{(t,t^2,\dots,t^d) \in \mathbb{R}^d \mid t \in \mathbb{R}\right\}$. Such polytopes are called *cyclic polytopes*. It is well-known that $C(n,d) \in \mathscr{P}(n,d)$ and its combinatorial data (e.g. the f-vectors and the *face poset* of its boundary complex (See Subsection 3.1 for the definition)) does not depend on the choice of n-distinct points.

In 1957, T. S. Motzkin [38] claimed that $\partial C(n,d)$ has a maximal f-vector, i.e.

$$f_i(\partial P) \leq f_i(\partial C(n,d)) \tag{*}$$

holds for all i and $P \in \mathscr{P}(n,d)$.

Later on, McMullen proved the claim above by showing

Theorem 1.9 (Upper Bound Theorem (apprev. UBT) McMullen 1970 [34])**.** *For $\Delta = \partial P$ with $P \in \mathscr{P}(n,d)$ and any i,*

$$h_i \leq \binom{n-d+i-1}{i}.$$

Indeed, Dehn-Sommerville equation and Upper Bound Theorem imply Motzkin's inequality, since

$$h_i(\partial C(n,d)) = \binom{n-d+i-1}{i}$$

for all i with $0 \leq i \leq \lfloor d/2 \rfloor$ (see [10, 52]).

Remark 1.10. One can also naturally define the notion of f-vector for convex polytopes that is not necessarily simplicial. Clearly the inequality (*) still holds for such polytopes.

It is natural to ask whether UBT holds for simplicial spheres or not, but unfortunately McMullen's proof cannot be applied to them; his proof depends on the shellability of convex polytopes (see Definition 2.1 and Theorem 2.4) and there is a simplicial sphere that is not shellable (e.g. triangulations of non-PL spheres such as double suspension of Poincaré homology 3-sphere).

Stanley showed UBT for simplicial spheres proving that the corresponding Stanley–Reisner ring is Cohen–Macaulay.

1.2. **Stanley–Reisner rings and h-vectors.** From now to the end, $\Bbbk$ denotes a field, let $S := \Bbbk[x_1,\dots,x_n]$ be a polynomial ring over $\Bbbk$, and set $\mathfrak{m} := (x_1,\dots,x_n)$. The following symbols will be used.

- Alphabets $\underline{\boldsymbol{a}} := (a_1,\dots,a_n), \underline{\boldsymbol{b}} := (b_1,\dots,b_n),\dots$ in the bold font with an underline denotes elements of $\mathbb{Z}^n$.
- $\underline{\boldsymbol{e}}_i$ denotes the i-th unit vector, and for $F \subseteq [n]$, set $\underline{\boldsymbol{e}}_F := \sum_{i\in F} \underline{\boldsymbol{e}}_i$.
- $\preceq$ denotes the partial order on $\mathbb{Z}^n$ defined by
$$\underline{\boldsymbol{a}} \preceq \underline{\boldsymbol{b}} \iff a_i \leq b_i \quad \text{for all } i.$$
- For $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}} \in \mathbb{Z}^n$, set $[\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}] := \{\underline{\boldsymbol{c}} \in \mathbb{Z}^n \mid \underline{\boldsymbol{a}} \preceq \underline{\boldsymbol{c}} \preceq \underline{\boldsymbol{b}}\}$.
- For $\underline{\boldsymbol{a}} \in \mathbb{Z}^n$, set $|\underline{\boldsymbol{a}}| := \sum_{i=1}^n a_i$ and $x^{\underline{\boldsymbol{a}}} := \prod_{i=1}^n x_i^{a_i}$.
- For $F \subseteq [n]$, set $x_F := x^{\underline{\boldsymbol{e}}_F} = \prod_{i\in F} x_i$.
- S has a natural structure of a $\mathbb{Z}^n$-graded ring with the grading given by
$$S_{\underline{\boldsymbol{a}}} := \begin{cases} \Bbbk \cdot x^{\underline{\boldsymbol{a}}} & \text{for } \underline{\boldsymbol{a}} \in \mathbb{Z}^n_{\geq 0} \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{Z}_{\geq 0}$ is the set of non-negative integers. Note that

- $S_{\underline{\mathbf{0}}} = \Bbbk$, where $\underline{\mathbf{0}} = (0, \dots, 0)$.
- $S_{\underline{\boldsymbol{e}}_i} = \Bbbk \cdot x_i$.

• Also, S has a natural structure of $\mathbb{Z}$-graded ring:

$$S_i := \bigoplus_{|\underline{\boldsymbol{a}}|=i} S_{\underline{\boldsymbol{a}}}.$$

See Appendix A for basics of graded rings and modules. As is stated below Definition A.1, any $\mathbb{Z}^n$-graded S-module is tacitly regarded as a $\mathbb{Z}$-graded one with the natural $\mathbb{Z}$-grading.

For $G = \mathbb{Z}^n$ or $\mathbb{Z}$, let $\operatorname{mod}_G S$ be the category consisting of finitely generated graded S-modules and of degree-preserving S-homomorphisms.

Remark 1.11. The ideal $\mathfrak{m}$ is clearly a maximal ideal of S, and moreover it is also the greatest ideal among all the ideal belonging to $\operatorname{mod}_G S$. In other words, S is *graded local*. By this structure, S is equipped with many properties that a usual noetherian local ring has. See [10, 21, 22]

Definition 1.12. For $M \in \operatorname{mod}_{\mathbb{Z}} S$,

$$H_M(t) := \sum_{i=-\infty}^{\infty} (\dim_{\Bbbk} M_i) t^i \in \mathbb{Z}[\![t, t^{-1}]\!]$$

is called the *Hilbert Series* of M.

The following is a classical result in commutative algebra. See [10, 31] for details.

Proposition 1.13. *For $M \in \operatorname{mod}_{\mathbb{Z}} S$ with $d := \dim M$, where $\dim M$ denotes the* Krull dimension *of M, there exists a polynomial $Q_M(t) \in \mathbb{Z}[t, t^{-1}]$ with $Q_M(1) \neq 0$ such that*

$$H_M(t) = \frac{Q_M(t)}{(1-t)^d}.$$

Definition 1.14. For not necessarily empty Δ on $[n]$, set

$$I_\Delta := (x_F \mid F \subseteq [n]\,,\ F \notin \Delta) \subseteq S, \quad \Bbbk[\Delta] := S/I_\Delta.$$

The ideal I_Δ is called the *Stanley–Reisner ideal* of Δ, and $\Bbbk[\Delta]$ the *Stanley–Reisner ring* of Δ.

Clearly $I_\Delta, \Bbbk[\Delta] \in \operatorname{mod}_{\mathbb{Z}^n} S$. Note that I_Δ is set to be (0) when $\Delta = 2^{[n]}$, and $I_\Delta = S$ when $\Delta = \varnothing$.

Recall that an ideal of S is called *monomial* if it is generated by monomials in S. A monomial of the form x_F with $F \subseteq [n]$ is called a *squarefree monomial*, and an ideal of S is said to be *squarefree* if it is generated by squarefree monomial ideals. Clearly I_Δ is squarefree for any Δ. Moreover there is the following one-to-one corresponding:

$$\left\{\begin{matrix}\text{simplicial complexes} \\ \text{on } [n]\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{squarefree monomial} \\ \text{ideals of } S\end{matrix}\right\} = \left\{\begin{matrix}\text{radical monomial} \\ \text{ideals of } S\end{matrix}\right\}$$

For $\underline{\boldsymbol{a}} \in \mathbb{Z}^n$, define

$$\operatorname{supp}(\underline{\boldsymbol{a}}) := \{i \in [n] \mid a_i > 0\}\,,$$

and call it the *support* of $\underline{\boldsymbol{a}}$. For $F \subseteq [n]$, set

$$\mathfrak{p}_F := (x_i \mid i \in [n] \setminus F), \quad \Bbbk[F] := S/\mathfrak{p}_F.$$

Lemma 1.15. *Let Δ_1, Δ_2 be simplicial complexes on $[n]$.*

(1) $I_{\Delta_1 \cap \Delta_2} = I_{\Delta_1} + I_{\Delta_2}$ *and* $I_{\Delta_1 \cup \Delta_2} = I_{\Delta_1} \cap I_{\Delta_2}$.

(2) *$I_\Delta = \bigcap_{F \in \mathscr{F}(\Delta)} \mathfrak{p}_F$, and hence*

$$\dim \Bbbk[\Delta] = \max\{\#F \mid F \in \mathscr{F}(\Delta)\} = \dim \Delta + 1.$$

(3) *There is one-to-one corresponding*

$$\mathscr{F}(\Delta) \ni F \longleftrightarrow \mathfrak{p}_F \in \operatorname{Ass}(\Bbbk[\Delta]),$$

where $\operatorname{Ass}(-)$ *denotes the set of associated prime ideals (see Definition A.2 for the definition).*

(4) *As $\mathbb{Z}^n$-graded $\Bbbk$-vector spaces,*

$$\Bbbk[\Delta] = \bigoplus_{F \in \Delta} x_F \cdot \Bbbk[F].$$

In particular, setting $d := \dim \Bbbk[\Delta] = \dim \Delta + 1$, *one obtains*

$$\begin{aligned} H_{\Bbbk[\Delta]}(t) &= \sum_{F \in \Delta} \frac{t^{\#F}}{(1-t)^{\#F}} \\ &= \frac{1}{(1-t)^d} \sum_{i=0}^{d} f_{i-1}(\Delta) t^i (1-t)^{d-i} = \frac{1}{(1-t)^d} \sum_{i=0}^{d} h_i(\Delta) t^i. \end{aligned}$$

Proof. (1): an easy exercise. (2): an easy consequence of the fact that $\Delta = \langle F \mid F \in \mathscr{F}(\Delta)\rangle$. (3): use Proposition A.3 in conjunction with the fact that in the present case, the monomial m in Proposition A.3 can be chosen to be x_G for some $G \in \Delta$. (4): follows from the fact that

$$0 \neq x^{\underline{\boldsymbol{a}}} \in \Bbbk[\Delta] \quad \Longleftrightarrow \quad \operatorname{supp}(\underline{\boldsymbol{a}}) \in \Delta.$$

□

Thus,

$$\boxed{\text{we can apply some results in commutative algebra, to the study of } h\text{-vectors!}}$$

1.3. **Cohen–Macaulay-ness and Gorenstein-ness.**

Definition 1.16. Let $M \in \operatorname{mod}_{\mathbb{Z}} S$, and $\underline{\boldsymbol{\theta}} := \theta_1, \dots, \theta_d$ of homogeneous elements (w.r.t. $\mathbb{Z}$-grading) in S.

(1) The sequence $\underline{\boldsymbol{\theta}}$ is called a *homogeneous system of parameters* (abbrev. *h.s.o.p*) if $d = \dim M$ and $l(M/(\underline{\boldsymbol{\theta}})M) < \infty$, or equivalently $(M/(\underline{\boldsymbol{\theta}})M)_i = 0$ for all $i \gg 0$.

(2) $\underline{\boldsymbol{\theta}}$ is a (homogeneous) *regular sequence* on M in $\mathfrak{m}$ if
   (a) each θ_i is in $\mathfrak{m}$,
   (b) $M/(\underline{\boldsymbol{\theta}})M \neq 0$, and
   (c) each θ_i is a non-zero divisor on $M/(\theta_1, \dots, \theta_{i-1})M$, where $(\theta_1, \dots, \theta_{i-1})$ is set to be (0) when $i = 1$.

(3) The maximal length of homogeneous regular sequence on M in $\mathfrak{m}$ is called the *depth* of S, and denoted by $\operatorname{depth} M$.

Lemma 1.17.

(1) *Every $M \in \operatorname{mod}_{\mathbb{Z}} S$ admits a h.s.o.p., and it is always algebraically independent over the field $\Bbbk$.*

(2) *Whenever $\#\Bbbk = \infty$, one always can be chosen a h.s.o.p. from S_1. Such a h.s.o.p. is called a* linear s.o.p. *(abbrev. l.s.o.p.).*

(3) $\operatorname{depth} M \leq \dim M$ (*whenever* $M \neq 0$).

Sketch of Proof. The first two assertions follow from a classical result, so-called "Prime Avoidance", in commutative ring theory. The last one is an immediate consequence of the fact that any homogeneous regular sequence is a part of some h.s.o.p. (see [10, 31] for details). □

Definition 1.18. An S-module $M \in \operatorname{mod}_{\mathbb{Z}} S$ is said to be *Cohen-Macaulay* if it satisfies the following conditions equivalent to each other:

(1) Any h.s.o.p $\underline{\boldsymbol{\theta}} = \theta_1, \dots, \theta_d$ is a *regular sequence* on M.
(2) There exists a h.s.o.p which is regular on M.
(3) $\operatorname{depth} M = \dim M$.

Remark 1.19. Formally, CM-ness and depth is defined for finitely generated modules over a noetherian local ring, and a finitely generated module M over a noetherian ring A is said to be CM if the localization $M_\mathfrak{p}$ at $\mathfrak{p}$ is a CM $R_\mathfrak{p}$-module for any $\mathfrak{p} \in \operatorname{Spec}(A)$. It is not trivial but well-known that the conditions in Definition 1.18 is equivalent to say that $M_\mathfrak{p}$ is CM for all $\mathfrak{p} \in \operatorname{Spec}(S)$ and also that $M_\mathfrak{p}$ is CM for all maximal ideal $\mathfrak{p}$ of S. Moreover it follows that $\operatorname{depth} M = \operatorname{depth}_{S_\mathfrak{m}} M_\mathfrak{m}$. See [10, 21, 22] for details.

In commutative algebra, especially in the theory of Cohen-Macaulay modules or Gorenstein modules (Definition 1.24), canonical modules play a very important role to give a duality.

Definition 1.20. Let I be an ideal of S with $I \in \operatorname{mod}_G S$, where $G = \mathbb{Z}^n$ or $\mathbb{Z}$, and set $R := S/I$. Assume R is Cohen-Macaulay of dimension d. An R-module $M \in \operatorname{mod}_G R$ is said to be a G-graded *canonical module* of R if

$$\operatorname{Ext}^i_R(\Bbbk, M) \cong \begin{cases} 0 & \text{for } i \neq d, \\ \Bbbk & \text{for } i = d, \end{cases}$$

in $\operatorname{mod}_G R$.

Remark 1.21. A G-graded canonical module of R is necessarily CM as an S-module and it is a maximal CM as an R-module (i.e., a CM R-module of dimension $\dim R$). Moreover it is indecomposable.

The notion of a canonical module also defined for a (not necessarily graded) Cohen-Macaulay ring. If $M \in \operatorname{mod}_G R$ is a canonical module in the sense of Definition 1.20, then the shifted modules $M(g)$ where $g \in G$ (see Definition A.1) are canonical in the non-graded sense, while not in the sense of Definition 1.20. It is also noteworthy that with the hypothesis of Definition 1.20, R necessarily admits a G-graded canonical module (see Proposition 1.23).

If $R = S/I$ with $I \in \operatorname{mod}_G R$ is CM of dimension d and M is a Cohen-Macaulay R-module of dimension t, then there exists a natural isomorphism

$$M \longrightarrow \operatorname{Ext}^{d-t}_R(\operatorname{Ext}^{d-t}_R(M, \omega_R), \omega_R)$$

(see [10, 31]).

Set $\underline{\mathbf{1}} := (1, \dots, 1) \in \mathbb{Z}^n$ and $\omega_S := x^{\mathbf{1}} \cdot S = S(-\underline{\mathbf{1}})$. It is well-known that ω_S is a $\mathbb{Z}^n$-graded canonical module of S.

Theorem 1.22 (Grothendieck vanishing and non-vanishing (cf. [10, 31]))**.** *For* $M \in \operatorname{mod}_G S$, *it follows that*

$$\operatorname{Ext}^{n-i}_S(M, \omega_S) \begin{cases} = 0 & \text{if } i < \operatorname{depth}_S M \text{ or } i > \dim_S M \\ \neq 0 & \text{if } i = \operatorname{depth}_S M,\ \dim_S M \end{cases}$$

The bottom non-vanishing Ext module is very important particularly in the CM case.

Proposition 1.23 (cf. [10, 21, 22]). *Assume R is CM of dimension d. Then $\omega_R := \operatorname{Ext}_S^{n-d}(R, \omega_S)$ is a G-graded canonical module of R.*

Definition 1.24. Let I be an ideal of S with $I \in \operatorname{mod}_{\mathbb{Z}} S$, and set $R := S/I$.

(1) The ring R is said to be *Gorenstein* (abbrev. *Gor*) if R is CM and $R \cong \omega_R(-a(R))$ for some $a(R) \in \mathbb{Z}$. The integer $a(R)$ is called the *a-invariant* (or Gorenstein parameter) of R.

(2) Δ is said to be *CM* (resp. *Gor*) (over $\Bbbk$) if so is $\Bbbk[\Delta]$.

(3) Finally, Δ is said to be *Gorenstein** (abbrev. *Gor**) if Δ is Gor and $a(\Bbbk[\Delta]) = 0$.

Remark 1.25. It is well-know that CM-ness, Gor-ness, and Gor*-ness does not depend on an extension of the base field $\Bbbk$ (under the assumption Definition 1.18 and 1.24). There several characterization for CM-ness and Gor-ness. See [10, 31] for details.

Definition 1.26.

(1) For Δ and $F \in \Delta$, the *link* $\operatorname{lk}_\Delta F$ of Δ with respect to F is defined to be the complex

$$\operatorname{lk}_\Delta F := \{G \in \Delta \mid F \cap G = \varnothing,\ F \cup G \in \Delta\}.$$

Note that $\operatorname{lk}_\Delta \varnothing = \Delta$.

(2) Δ is said to be *pure* if $\dim F = \dim \Delta$ for all $F \in \mathscr{F}(\Delta)$.

(3) Δ is said to be *Eulerian* if it is pure and $\widetilde{\chi}(\operatorname{lk}_\Delta F) = (-1)^{\dim \operatorname{lk}_\Delta F}$ for all $F \in \Delta$.

Proposition 1.27. *Set $d := \dim \Delta + 1$ and $R := \Bbbk[\Delta]$. The following hold.*

(1) $(-1)^d H_R(t^{-1}) = \sum_{F \in \Delta} (-1)^{d - \dim F} \widetilde{\chi}(\operatorname{lk}_\Delta F) \left(\dfrac{t}{1-t}\right)^{\#F}$. *In particular, if Δ is Eulerian, then*

$$(-1)^d H_R(t^{-1}) = H_R(t).$$

(2) *Assume Δ is CM over $\Bbbk$. Then*

(a) *Δ is pure.*

(b) *For every l.s.o.p. $\underline{\boldsymbol{\theta}}$,*

$$(1-t)^d H_R(t) = H_{R/\underline{\boldsymbol{\theta}}R}(t).$$

(c) $H_{\omega_R}(t) = (-1)^d H_R(t^{-1})$.

Sketch of Proof. (1) follows from a simple calculation in conjunction with the fact that $H_R(t) = \sum_{F \in \Delta} (t/(1-t))^{\#F}$ and that $t/(1-t)$ is transformed into $-(1 + t/(1-t))$ by the substitution $t \to t^{-1}$.

(2) (a) It is well known that all the associated prime ideals of CM module have the same codimension, which in conjunction with Lemma 1.15 implies that all $F \in \mathscr{F}(\Delta)$ have the same cardinality.

(b) Set $\underline{\boldsymbol{\theta}}_i := \theta_1, \dots, \theta_i$ for all i with $1 \le i \le d = \dim \Bbbk[\Delta]$ and $\underline{\boldsymbol{\theta}}_0 := 0$. By the definition of CM-ness,

$$0 \longrightarrow \left(M/(\underline{\boldsymbol{\theta}}_{i-1})M\right)(-1) \xrightarrow{\theta_i} M/(\underline{\boldsymbol{\theta}}_{i-1})M \longrightarrow M/(\underline{\boldsymbol{\theta}}_i)M \longrightarrow 0$$

is exact, which implies $(1-t)H_{M/(\underline{\boldsymbol{\theta}}_{i-1})M}(t) = H_{M/(\underline{\boldsymbol{\theta}}_i)M}(t)$.

(c) Needs some arguments. See [10, 31]. □

Corollary 1.28. *Set $d := \dim \Delta + 1$.*

(1) (*R. P. Stanley 1975* [48]) *For a CM* Δ,
$$0 \le h_i \le \binom{n-d+i-1}{i}$$
for all i*. In particular, UBT holds for CM* Δ*.*

(2) *Dehn–Sommerville equation* ($h_i = h_{d-i}$) *holds for any Eulerian* Δ*.*

Proof. (1) Choose a l.s.o.p. $\boldsymbol{\theta}$ of $\Bbbk[\Delta]$, set $R := \Bbbk[\Delta]/(\boldsymbol{\theta})$, and let $y_1, \dots, y_{n-d}$ be a basis of R_1 (recall that $\boldsymbol{\theta}$ is algebraically (hence linearly) independent over $\Bbbk$). Clearly, the following inequalities hold
$$0 \le \dim_\Bbbk R_i \le \dim_\Bbbk (\Bbbk[y_1, \dots, y_{n-d}])_i = \binom{n-d-i+1}{i}.$$
(2) Immediate consequence of Proposition 1.27 and $(-1)^d H_R(t^{-1}) = \dfrac{\sum_{i=0}^d h_{d-i}}{(1-t)^d}$. □

So the natural question is which complexes are CM or Eulerian? As for CM-ness the problem is immediately settled by Hochster's formula.

Theorem 1.29 (Hochster's formula for Ext)**.** *For* Δ*,* $i \in \mathbb{Z}$*, and* $\underline{\boldsymbol{a}} \in \mathbb{Z}^n$ *with* $F := \operatorname{supp}(\underline{\boldsymbol{a}})$*, it follows that*
$$\operatorname{Ext}_S^{n-i}(\Bbbk[\Delta], \omega_S)_{\underline{\boldsymbol{a}}} \cong \begin{cases} H_{i-\#F-1}(\operatorname{lk}_\Delta F; \Bbbk) & \text{if } a_i \ge 0 \text{ for all } i, \\ 0 & o.w. \end{cases}$$

Sketch of Proof. Set $d := \dim \Delta + 1$. It is known [10] that $\operatorname{Ext}_S^{n-i}(\Bbbk[\Delta], \omega_S)$ is isomorphic (in $\operatorname{mod}_{\mathbb{Z}^n} S$) to the i-th homology of the following complex
$$0 \longrightarrow D_d \longrightarrow \cdots \longrightarrow D_1 \longrightarrow D_0 \longrightarrow 0$$
such that

- $D_i = \bigoplus_{F \in \Delta,\ \#F = i} \Bbbk[F]$ and
- $D_i \to D_{i-1}$ is just the sum of the natural map $\Bbbk[F] \to \Bbbk[F']$ with a suitable sign ± 1, where $F' \subseteq F$.

It is easy to verify that $(D_\bullet)_{\underline{\boldsymbol{a}}} \cong \widetilde{C}_\bullet(\operatorname{lk}_\Delta \operatorname{supp}(\underline{\boldsymbol{a}}); \Bbbk)[-\#F-1]$, where $[-]$ means a homological translation, if $\underline{\boldsymbol{a}} \notin \mathbb{Z}^n_{\ge 0}$ and otherwise $(D_\bullet)_{\underline{\boldsymbol{a}}} = 0$. □

Remark 1.30. Hochster's formula is usually (and quite often) introduced as the formula for the *local cohomologies*. By the graded local duality, the above formula is equivalent to the usual one. See [10, 52] for the usual formula.

Corollary 1.31.

(1) (Reisner 1976 [41]) Δ *is CM over* $\Bbbk$ *if and only if* $\widetilde{H}_i(\operatorname{lk}_\Delta F; \Bbbk) = 0$ *for all* $F \in \Delta$ *and* $i < \dim \operatorname{lk}_\Delta F$ ($= \dim \Delta - \#F$*, actually*)*.*

(2) Δ *is Gor* over* $\Bbbk$ *if and only if*
$$H_i(\operatorname{lk}_\Delta F; \Bbbk) \cong \begin{cases} \Bbbk & \text{if } i = \dim \operatorname{lk}_\Delta F, \\ 0 & \text{otherwise} \end{cases}$$
for all $F \in \Delta$*.*

Proof. (1) Immediate consequence of Hochster's formula.

(2) This follows from Hochster's formula and the fact that $\Bbbk[\Delta] \cong M$ in $\operatorname{mod}_{\mathbb{Z}^n} \Bbbk[\Delta]$ for any indecomposable $M \in \operatorname{mod}_{\mathbb{Z}^n} \Bbbk[\Delta]$ with $\Bbbk[\Delta] \cong M$ as $\mathbb{Z}^n$-graded $\Bbbk$-vector spaces (See [10]). □

Note that homologies of links are isomorphic to local homologies.

Proposition 1.32 (Munkres [39, Lemma 3.3])**.** *Set* $X := |\Delta|$*. For* $p \in X$*, a simplex* σ *of* X *with* $p \in \operatorname{relint}\sigma$*, and a face* $F \in \Delta$ *corresponding to* σ*,*

$$H_i(X, X \setminus \{p\}\,; \Bbbk) \cong \widetilde{H}_{i-\#F-1}(\operatorname{lk}_\Delta F; \Bbbk)$$

for all i*.*

Consequently, the following hold.

Corollary 1.33.

(1) (Munkres and Stanley) *The following hold.*
   (a) Δ *is CM over if and only if* $H_i(X; \Bbbk) \cong H_i(X, X \setminus \{p\}\,; \Bbbk) = 0$ *for all* $p \in X$ *and* $i < \dim X$*.*
   (b) Δ *is Gor* over* $\Bbbk$ *if and only if*

$$H_i(X; \Bbbk) \cong H_i(X, X \setminus \{p\}) \cong \begin{cases} \Bbbk & \text{if } i = \dim X, \\ 0 & \text{otherwise.} \end{cases}$$

   (c) *In particular, CM-ness and Gor*-ness of* Δ *depends only on* X *and* $\Bbbk$*.*

(2) Δ *is Gor* if and only if* Δ *is CM and Eulerian.*

(3) *A simplicial sphere is Gor*. In particular, Dehn–Sommerville equation and UBT holds for simplicial spheres.*

(4) *If* $|\Delta|$ *is a homology manifold (with or without boundary) and its homology vanishes expect for the top homology, then* Δ *is CM.*

Remark 1.34. CM-ness depends on the characteristic of $\Bbbk$. For example, let Δ be a triangulation of $\mathbb{R}P^2$. Since $\mathbb{R}P^2$ is a manifold, it follows that Δ is CM over $\Bbbk$ when $\operatorname{char}\Bbbk \neq 2$ and not CM when $\operatorname{char}\Bbbk = 2$.

1.4. **More about h-vectors.**

Lemma 1.35. *Let* r *be a positive integer. For any* $k \in \mathbb{Z}$ *with* $k > 0$*, there exist unique integers* $k(r) > k(d-1) > \cdots > k(1) \geq 0$ *such that*

$$k = \sum_{i=1}^{r} \binom{k(i)}{i}. \tag{1.1}$$

Proof. An easy exercise. □

The representation in (1.1) is called the *r-th Macaulay representation* (abbrev. M-rep.). For a positive integer k with the M-rep.

$$k = \sum_{i=1}^{r} \binom{k(i)}{i},$$

set

$$k^{\langle r\rangle} = \sum_{i=1}^{r} \binom{k(i)+1}{i+1}.$$

Theorem 1.36 (Macaulay and Stanley)**.** *For* $(h_0, \dots, h_d) \in \mathbb{Z}^{d+1}$*, the following are equivalent.*

(1) *There exists an ideal* I *of* S *with* $I \in \operatorname{mod}_{\mathbb{Z}} S$ *such that* $H_{S/I}(t) = \sum_{i=0}^{d} h_i t^i$ *(and hence* S/I *is finite-dimensional over* $\Bbbk$*).*

(2) *There exists an ideal I of S with $I \in \operatorname{mod}_{\mathbb{Z}} S$ such that S/I is Cohen-Macaulay of dimension d and*
$$H_{S/I}(t) = \frac{\sum_{i=0}^{d} h_i t^i}{(1-t)^d}.$$
(3) $h_0 = 1$ *and* $0 \le h_{i+1} \le h_i^{\langle i \rangle}$ *for all* $i \ge 1$.

Definition 1.37. A vector satisfying the conditions in Theorem 1.36 called an M-vector.

In 1971, McMullen [35] conjectured that for an vector $h := (h_0, \dots, h_d) \in \mathbb{Z}^{d+1}$ the following are equivalent:

(1) There exists $P \in \bigcup_n \mathcal{P}(n,d)$ with $h(\partial P) = h$.
(2) Set $g_i := h_i - h_{i-1}$. It follows that
   (a) $h_0 = 1$,
   (b) $h_i = h_{d-i}$ for all i, and
   (c) $(g_0, \dots, g_{\lfloor d/2 \rfloor})$ is an M-vector.

This conjecture, called *g-conjecture*, has already been proved to be true by Billera, Lee, and Stanley.

Theorem 1.38 (g-theorem)**.** *McMullen's conjecture is true.*

Billera–Lee [3, 4] proved (2) $\Rightarrow$ (1), and Stanley [50] proved the inverse making use of the theory of toric variety and the hard Lefschetz theorem (See also [19, 52]). Now the following conjecture naturally comes up.

Conjecture 1.39 ((new) g-conjecture)**.** *The same characterization of h-vectors holds for simplicial spheres.*

This conjecture is still open. Clearly, the only problem is whether the h-vector of any simplicial sphere satisfies the above three conditions (only the condition (c) remains to be proven) or not.

More generally, it is natural to try to find the characterization of h-vectors for (homological or topological) manifolds (with or without boundary). A manifold is not necessarily CM, while it satisfies parts of the conditions for CM-ness.

Definition 1.40.

(1) $\Bbbk[\Delta]$ with $\dim \Bbbk[\Delta] = d$ is said to be *Buchsbaum* (abbrev. Bbm) if $\operatorname{Ext}_S^{n-i}(\Bbbk[\Delta], \omega_S)_{\underline{\boldsymbol{a}}} = 0$ for all $i \ne d$ and $\underline{\boldsymbol{a}} \ne \underline{\boldsymbol{0}}$.
(2) Δ is said to be *Bbm* over $\Bbbk$ if $\Bbbk[\Delta]$ is Bbm.

Remark 1.41. In general, Bbm-ness is more subtle and the definition above is valid almost only for Stanley-Reisner rings.

By Hochster's formula for Ext and Munkres' isomorphism, the following is clear.

Theorem 1.42 (Miyazaki and Schenzel)**.** *Let* $\dim \Delta = d-1$ *and set* $X := |\Delta|$. *The following are equivalent:*

(1) Δ *is Bbm over* $\Bbbk$.
(2) $\widetilde{H}_i(\operatorname{lk}_\Delta F; \Bbbk) = 0$ *for all* $F \ne \varnothing$ *and* $i < d - \#F - 1$.
(3) $\widetilde{H}_i(X, X \setminus \{p\}) = 0$ *for all* $p \in Z$ *and* $i < \dim X$.

In particular, if X is a homology manifold (with or without ∂), then Δ is Bbm.

Now we obtain the following implications.

$$\begin{tikzcd} \text{simplicial sphere} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \begin{array}{l}\text{homology mfd}\\ \text{whose homologies are vanished}\\ \text{expect for the top one}\end{array} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{homology mfd} \arrow[d, Rightarrow] \\ \text{Gor*} \arrow[r, Rightarrow] & \text{CM} \arrow[r, Rightarrow] & \text{Bbm} \end{tikzcd}$$

As a corollary of the characterization of h-vectors of Hilbert series of CM graded $\Bbbk$-algebra due to Macaulay and Stanley, the following holds.

Theorem 1.43 (Stanley). *For $h = (h_0, \dots, h_d) \in \mathbb{Z}^{n+1}$, the following are equivalent.*

(1) *$h = h(\Delta)$ for some CM Δ of dimension d.*
(2) *h is an M-vector.*

As for Bbm complexes, it is still an open problem to characterize their h-vectors. (Note that components of h-vector of a Bbm complex could be negative (e.g. the cylinder above)).

Problem 1.44 (Hibi [26]). Give a (combinatorial) characterization of h-vectors of Bbm Δ.

There are some results on h-vectors of Bbm Δ. See [44, 53] for details.

2. Algebraic properties of Stanley–Reisner rings

2.1. **Shellability and CM-ness.** As is stated in the previous section, McMullen proved UBT by the shellability of simplicial polytopes. Let us recall the definition of shellability. Recall that $\mathscr{F}(\Delta)$ denotes the set of all the facets of Δ.

Definition 2.1. A simplicial complex Δ is said to be *shellable* if there is a linear ordering, called *shelling*, $F_1, \dots, F_r$ on $\mathscr{F}(\Delta)$ with the following equivalent conditions:

(1) $\langle F_i \rangle \cap \langle F_1, \dots, F_{i-1} \rangle$ is pure (actually pure shellable) of dimension $\dim F_i - 1$ for each i.
(2) $\langle F_i \rangle \setminus \langle F_1, \dots, F_{i-1} \rangle$ has a unique minimal face G_i for each i. Hence $\Delta = \coprod_{i=1}^{r} [G_i, F_i]$, where $G_1 := \varnothing$ and $[G_i, F_i] := \{G \subseteq [n] \mid G_i \subseteq G \subseteq F_i\}$.

Remark 2.2. Each $\langle F_i \rangle \cap \langle F_1, \dots, F_{i-1} \rangle$ is homeomorphic to a sphere or a ball. Moreover it is well-known [6] that any shellable complex is contractible or has a homotopy type of wedge of spheres (of various dimension).

Example 2.3. In Example 1.7, the simplicial complexes in Figures 2 and 5 are not shellable, and the other complexes are pure and shellable. The complex in the following Figure 6 is not pure but shellable.

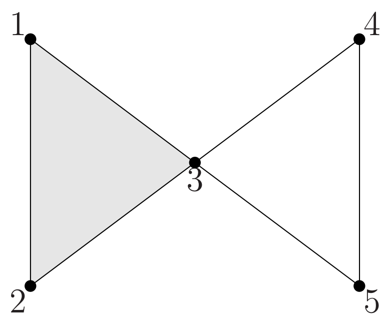

Figure 6.

The ordering $\{1,2,3\},\{3,4\},\{4,5\},\{3,5\}$ is a shelling of this complex, and one obtains the decomposition

$$[\varnothing,\{1,2,3\}]\coprod[\{4\},\{3,4\}]\coprod[\{5\},\{4,5\}]\coprod[\{3,5\},\{3,5\}].$$

McMullen used the following fact to prove UBT for simplicial polytopes.

Theorem 2.4 (Bruggesser–Mani [7])**.** *Every boundary complex of a simplicial polytope is pure and shellable.*

Recall that, on the other hand, Stanley's proof of UBT uses CM-ness the boundary complex. So it is natural to expect that there is a relation between pure-shellability and CM-ness.

Note that for two simplicial complex Δ,Δ',

$$\Delta\subseteq\Delta' \iff I_\Delta\supseteq I_{\Delta'}.$$

Proposition 2.5. *A pure shellable Δ is CM over any field $\Bbbk$.*

Sketch of Proof. Let $F_1,\dots,F_r$ be a shelling of Δ and set $\Delta_i:=\langle F_1,\dots,F_i\rangle$. Since there exists the following exact sequence

$$0\longrightarrow\Bbbk[\Delta_i]\longrightarrow\Bbbk[\Delta_{i-1}]\oplus\Bbbk[\Delta_i]\longrightarrow\Bbbk[\Delta_{i-1}\cap\Delta_i]\longrightarrow 0$$

induced from the assertion (1) of Lemma 1.15 and since each $\Bbbk[\Delta_{i-1}\cap\Delta_i]$ is CM of dimension $d-1$ (which is easy to verify), the desired assertion follows from the induction on r and the long exact sequence of Ext modules induced from the short exact sequence above. □

In view of $\Bbbk[\Delta]$, a shelling can be considered as a filtration of $\Bbbk[\Delta]$. There is another proof in this point of view. (Compare this with Theorem 2.7).

Another Proof. Let $F_1,\dots,F_r$ be a shelling of Δ, $G_1:=\varnothing$, and $G_2,\dots,G_r$ unique minimal faces. Set $\Delta_{r-i}:=\langle F_1,\dots,F_i\rangle$ for $i=1,\dots,r$. The filtration

$$\varnothing=\Delta_r\subset\Delta_{r-1}\subset\cdots\subset\Delta_1\subset\Delta_0=\Delta$$

induces the one

$$I_\Delta=I_{\Delta_0}\subset I_{\Delta_1}\subset I_{\Delta_2}\subset\cdots\subset I_{\Delta_{r-1}}\subset I_{\Delta_r}=S,$$

and hence

$$0\subset I_{\Delta_1}/I_\Delta\subset I_{\Delta_2}/I_\Delta\subset\cdots\subset I_{\Delta_{r-1}}/I_\Delta\subset\Bbbk[\Delta]. \tag{2.1}$$

By the definition of shelling, $I_{\Delta_{i+1}}/I_{\Delta_i}\cong(S/\mathfrak{p}_{F_{i+1}})(-\underline{\boldsymbol{e}}_{G_{i+1}})$ for $i=0,\dots,r-1$. Now use the long exact sequence of Ext and the fact that each $S/\mathfrak{p}_{F_{i+1}}$ is CM of dim $\dim\Bbbk[\Delta]$ (since Δ is pure). □

Remark 2.6. The inverse of Proposition 2.5 is false. For example, every triangulation of a ball or a sphere is CM over any $\Bbbk$, while there exists a unshellable triangulation of a ball and a sphere (see the end of Subsection 1.1 and [43, 57]).

As for shellability, Dress gives the following algebraic criterion.

Theorem 2.7 (Dress [15])**.** *A simplicial complex Δ is shellable if and only if $M:=\Bbbk[\Delta]$ is* clean*, i.e., there is a filtration*

$$0=\Phi_0(M)\subsetneq\Phi_1(M)\subsetneq\cdots\subsetneq\Phi_r(M)=M$$

such that $\Phi_i(M)/\Phi_{i-1}\cong S/\mathfrak{p}$ for some minimal prime ideal $\mathfrak{p}$ of M.

Now it is natural to quest a suitable notion to fit the "???" below.

$$\begin{tikzcd}
\Delta : \text{pure shellable} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \Delta : \text{CM} \arrow[d, Rightarrow] \\
\Delta : \text{shellable} \arrow[r, Rightarrow] & \Delta : ???
\end{tikzcd}$$

The required notion is a sequential Cohen–Macaulay-ness.

Definition 2.8 (cf. [45, 52])**.** Let $M \in \operatorname{mod}_{\mathbb{Z}^n} S$.

(1) M is said to be *sequentially Cohen–Macaulay* (abbrev. SCM) if there exists a filtration, called a *Cohen–Macaulay filtration*

$$0 = \Phi_0(M) \subsetneq \Phi_1(M) \subsetneq \cdots \subsetneq \Phi_r(M) = M$$

with $\Phi_i(M) \in \operatorname{mod}_{\mathbb{Z}^n} S$ such that

(a) $\dim(\Phi_i(M)/\Phi_{i-1}(M)) < \dim(\Phi_{i+1}(M)/\Phi_i(M))$ for each i, and
(b) Each quotient $\Phi_i(M)/\Phi_{i-1}(M)$ is CM.

(2) A simplicial complex Δ is said to be *SCM* over $\Bbbk$ if so is $\Bbbk[\Delta]$.

Remark 2.9. A Cohen-Macaulay filtration, if exists, is unique, and moreover

$$\dim(\Phi_i(M)) = \dim\left(\Phi_i(M)/\Phi_{i-1}(M)\right) \tag{2.2}$$

for all i. In [45], Schenzel considers SCM modules over a (not necessarily graded) noetherian ring. There, he calls SCM modules *Cohen-Macaulay filtered modules* and studies algebraic aspects of SCM modules in view of commutative ring theory.

Proposition 2.10. *For Δ,*

(1) *Δ is CM over $\Bbbk$ if and only if it is pure and SCM over $\Bbbk$.*
(2) *A shellable Δ is SCM over any field $\Bbbk$.*

Proof. (1) The implication $\Rightarrow$ is clear. For the inverse, note that if follows from Lemma 1.15 that all the associated prime ideals of $\Bbbk[\Delta]$ has the same codimension, which implies that the length of Cohen-Macaulay filtration of $\Bbbk[\Delta]$ must be less than or equal to 1.

(2) As is well-known [6], each shelling $F_1, \dots, F_r$ can be arranged so that $\dim F_1 \geq \dim F_2 \geq \cdots \geq F_r$. The filtration as in (2.1) induced from the arranged shelling is then a CM filtration. ◻

Definition 2.11. For Δ and $i \in \mathbb{Z}$, set

$$\Delta^{[i]} := \langle F \mid F \in \Delta,\ \dim F = i \rangle, \quad \Delta^{\langle i \rangle} := \bigcup_{k \geq i} \Delta^{[k]}$$

$\Delta^{[i]}$ is called the *pure i-skeleton* of Δ.

Example 2.12. Let Δ be the simplicial complex in Figure 6. The complexes $\Delta^{[2]}$ and $\Delta^{[1]}$ are as follows.

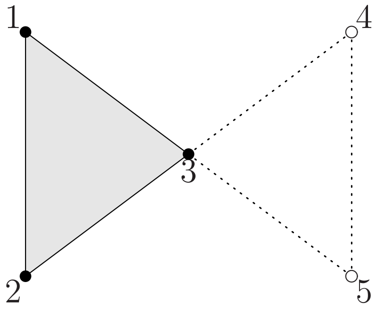

FIGURE 7. $\Delta^{[2]}$

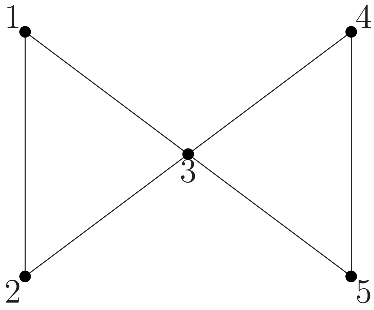

FIGURE 8. $\Delta^{[1]}$

Set $d = \dim \Delta + 1$, and

$$\Phi_j(\Bbbk[\Delta]) := I_{\Delta^{\langle j \rangle}} / I_\Delta$$

for $j = -1, 0, \dots, d$. One obtains the following filtration.

$$0 = \Phi_{-1}(\Bbbk[\Delta]) \subseteq \Phi_0(\Bbbk[\Delta]) \subseteq \cdots \subseteq \Phi_{d-1}(\Bbbk[\Delta]) \subseteq \Phi_d(\Bbbk[\Delta]) = \Bbbk[\Delta] \tag{2.3}$$

Theorem 2.13. *For Δ above, the following are equivalent.*

(1) *Δ is SCM over $\Bbbk$.*
(2) (Stanley) *Each $\Phi_i(\Bbbk[\Delta])/\Phi_{i-1}(\Bbbk[\Delta])$ is equal to 0 or CM over $\Bbbk$.*
(3) (Duval) *Each $\Delta^{[i]}$ is empty or CM over $\Bbbk$, for all i.*
(4) (Peskine and Stanley) *Each $Ext_S^{n-i}(\Bbbk[\Delta], \omega_S)$ is 0 or CM of dimension i.*

Sketch of Proof. See [16] for the equivalence (2) $\Leftrightarrow$ (3).

The implication (2) $\Rightarrow$ (4) is an easy consequence of the following: the first is the fact that $\dim(\Phi_i(\Bbbk[\Delta])/\Phi_{i-1}(\Bbbk[\Delta])) = i$ whenever $\Phi_i(\Bbbk[\Delta]) \neq \Phi_{i-1}(\Bbbk[\Delta])$, and the second is the long exact sequences of Ext modules induced from each short exact sequences given by the filtration (2.3).

(4) $\Rightarrow$ (1): Set $D_S^\bullet := \omega_S[n]$, where $[n]$ denotes the translation of complex. The assertion can be shown by the spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^{n+p}(\operatorname{Ext}^{n-q}(\Bbbk[\Delta], \omega_S), \omega_S) \Rightarrow H^{p+q}\left(\mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}(\Bbbk[\Delta], D_S^\bullet), D_S^\bullet)\right)$$

and the fact that $\mathbf{R}\operatorname{Hom}(\mathbf{R}\operatorname{Hom}(\Bbbk[\Delta], D_S^\bullet), D_S^\bullet)$ is quasi-isomorphic to the complex where the 0-th component is $\Bbbk[\Delta]$ and the others are 0.

(1) $\Rightarrow$ (2): Set $R := \Bbbk[\Delta]$ and let

$$0 = \Psi_0(R) \subsetneq \Psi_1(R) \subsetneq \cdots \subsetneq \Psi_r(R) = R$$

be a CM filtration. For $i = 1, \dots, r$, set $d_i := \dim \Psi_i(R)/\Psi_{i-1}(R) = \dim \Psi_i(R)$. The assertion immediately holds when one shows that the filtration coincides with the filtration given by skipping all the $\Phi_i(R)$ with $\Phi_i(R) = \Phi_{i-1}(R)$ in (2.3). The last assertion follows from the following facts:

- $\dim \Phi_i(R) \leq i$.
- $\dim M = \dim \Phi_{i+1}(R)$ for any $M \in \operatorname{mod}_{\mathbb{Z}^n} S$ with $\Phi_i(R) \subsetneq M \subseteq \Phi_{i+1}(R)$.
- each $\Psi_i(R)$ is the largest $\mathbb{Z}^n$-graded submodule of R with dimension less than or equal to
$$d_i := \dim \Psi_i(R)/\Psi_{i-1}(R) = \dim \Psi_i(R),$$
and is equal to $\Phi_{d_i}(R)$.

$\square$

2.2. **Alexander duality.**

Definition 2.14. For Δ on $[n]$, define

$$\Delta^{\vee} := \{F \subseteq [n] \mid F^c := [n] \setminus F \notin \Delta\}.$$

The simplicial complex $\Delta^{\vee}$ is called the *Alexander dual* of Δ with respect to $[n]$. Clearly, $I_{\Delta^{\vee}} := (x_{F^c} \mid F \in \mathscr{F}(\Delta))$.

Theorem 2.15 ((Combinatorial) Alexander duality (cf. [17]))**.** *For all i,*

$$\widetilde{H}^{i-2}(\Delta^{\vee}; \Bbbk) \cong \widetilde{H}_{n-i-1}(\Delta; \Bbbk).$$

Sketch of Proof. Let $\Gamma := 2^{[n]}$ be the $(n-1)$-simplex. From the one-to-one corresponding $\Delta \ni F \mapsto F^c \in \Gamma \setminus \Delta^{\vee}$ (and suitable choice of orientations), we get the isomorphism of complexes

$$\widetilde{C}_{\bullet}(\Delta; \Bbbk) \xrightarrow{\cong} \left(\widetilde{C}^{\bullet}(\Gamma; \Bbbk)/\widetilde{C}^{\bullet}(\Delta^{\vee}; \Bbbk)\right)[n-2].$$

Since Γ is contractible, it follows that

$$H_{n-i-1}(\Delta; \Bbbk) \cong H^{i-3}(\Gamma, \Delta^{\vee}; \Bbbk) \cong H^{i-2}(\Delta^{\vee}; \Bbbk).$$

□

Corollary 2.16 (cf. [17])**.** *For $F \subseteq [n]$ and i,*

$$\widetilde{H}^{i-2}(\operatorname{lk}_{\Delta^{\vee}} F; \Bbbk) \cong \widetilde{H}_{n-\#F-i-1}(\Delta_{F^c}; \Bbbk).$$

Sketch of Proof. Note that $\operatorname{lk}_{\Delta^{\vee}} F = (\Delta_{F^c})^{\vee}$ as simplicial complexes on F^c, where $\vee$ in the right hand denotes the Alexander dual with respect to F^c. Applying Theorem 2.15, one obtains the desired isomorphism. □

Definition 2.17. For $i \in \mathbb{Z}$ and $g \in G$, the non-negative integer

$$\beta_{i,g} := \beta_{i,g}(M) := \dim_{\Bbbk} \operatorname{Tor}_i^S(M, \Bbbk)_g$$

is called the (i, g)-th graded *Betti number* of M.

Remark 2.18. As is well-known, S has a finite global dimension n, and hence $\beta_{i,g} = 0$ for all $i > n$. Moreover each $\beta_{i,g}$ can be characterized as follows: there exists a *minimal graded free resolution* of M such that

$$P_{\bullet} : \cdots \longrightarrow \bigoplus_{g \in G} S(-g)^{\beta_{i,g}} \longrightarrow \cdots \bigoplus_{g} S(-g)^{\beta_{1,g}} \longrightarrow \bigoplus_{g} S(-g)^{\beta_{0,g}} \longrightarrow 0,$$

with $H_i(P_{\bullet}) = 0$ for $i \neq 0$ and $H_0(P_{\bullet}) \cong M$.

Definition 2.19. Let $M \in \operatorname{mod}_{\mathbb{Z}} S$.

(1) The integer $\operatorname{indeg} M := \inf \{i \in \mathbb{Z} \mid M_i \neq 0\}$ is called the *initial degree* of M.
(2) Set $\operatorname{reg}_S M := \sup \{r \in \mathbb{Z} \mid \beta_{i,i+r}(M) \neq 0 \ {}^{\exists} i\}$. (Note that $\operatorname{indeg} M \leq \operatorname{reg}_S M < \infty$ whenever $M \neq 0$). The integer $\operatorname{reg}_S M$ is called the *Castelnuovo-Mumford regularity* of M.
(3) M is said to have a *q-linear resolution* if $\operatorname{indeg} M = \operatorname{reg}_S M$, or equivalently M has a graded free resolution of the following form

$$\cdots \longrightarrow S(-q-2)^{\beta_{2,q+2}} \longrightarrow S(-q-1)^{\beta_{1,q+1}} \longrightarrow S(-q)^{\beta_{0,q}} \longrightarrow 0.$$

Betti numbers are important in commutative ring theory. For example, the following hold.

Proposition 2.20 (cf. [10, 31])**.** *For $M \in \operatorname{mod}_{\mathbb{Z}} S$ with $M \neq 0$, it follows that*

(1) $\operatorname{projdim}_S M := \max \{i \in \mathbb{Z} \mid {}^{\exists} j \ s.t. \ \beta_{i,j} \neq 0\}$ $(\leq n = \operatorname{gldim} S)$.

(2) (Auslander–Buchsbaum formula) $\operatorname{projdim}_S M = n - \operatorname{depth} M$. *In particular, M is CM if and only if* $\operatorname{projdim}_S M = n - \dim M$ *(recall that* $\dim S = n$*), or equivalently* $\beta_{i,j} = 0$ *for all* $i > n - \dim M$.

Let $\underline{\boldsymbol{\theta}} := \theta_1, \dots, \theta_d$ be the sequence of the variables of S, and consider the exterior algebra

$$K_\bullet(\underline{\boldsymbol{\theta}}; S) := S \langle y_1, \dots, y_d \mid d(y_i) = \theta_i \rangle , \tag{2.4}$$

where y_i is a S-free basis of homological degree 1.

Definition 2.21. With the above notation,

(1) $K_\bullet(\underline{\boldsymbol{\theta}}; S)$ is called the *Koszul complex* of S with respect to $\underline{\boldsymbol{\theta}}$.
(2) For an S-module M, the *Koszul complex* of M with respect to $\underline{\boldsymbol{\theta}}$ is defined to be

$$K_\bullet(\underline{\boldsymbol{\theta}}; M) := K_\bullet(\underline{\boldsymbol{\theta}}; S) \otimes_S M.$$

The following is well-known.

Proposition 2.22 (cf. [10, 31])**.** *Let $\underline{\boldsymbol{x}} := x_1, \dots, x_n$ be the sequence of variables of S. The Koszul complex $K_\bullet(\underline{\boldsymbol{x}}; S)$ then gives a $\mathbb{Z}^n$-graded minimal S-free resolution of $\Bbbk$, where each degree (in the sense of $\mathbb{Z}^n$-grading) of y_i in (2.4) is set to be $\underline{\boldsymbol{e}}_i$.*

Theorem 2.23 (Hochster (cf. [10, 52])**.** *For $\underline{\boldsymbol{a}} \in \mathbb{Z}^n$ with $F := \operatorname{supp}(\underline{\boldsymbol{a}})$, it follows that*

$$\beta_{i,\underline{\boldsymbol{a}}}(\Bbbk[\Delta]) = \begin{cases} \dim_\Bbbk \widetilde{H}^{\#F-i-1}(\Delta_F; \Bbbk) & \text{if } \underline{\boldsymbol{a}} \in [\underline{\mathbf{0}}, \underline{\mathbf{1}}] := \{\underline{\boldsymbol{b}} \in \mathbb{Z}^n \mid \underline{\mathbf{0}} \preceq \underline{\boldsymbol{b}} \preceq \underline{\mathbf{1}}\} , \\ 0 & o.w. \end{cases}$$

Proof. For simplicity, set $R := \Bbbk[\Delta]$. By Proposition 2.22, the Koszul complex $K_\bullet := K_\bullet(\underline{\boldsymbol{x}}; S)$ is a minimal $\mathbb{Z}^n$-graded free resolution of $\Bbbk$. Let $V := \bigoplus_{i=1}^n \Bbbk \cdot v_i$ be the $\mathbb{Z}^n$-graded $\Bbbk$-vector space with basis v_i of degree $\underline{\boldsymbol{e}}_i$, and $\bigwedge V$ be the exterior algebra of V over $\Bbbk$. Regarding $\bigwedge V$ as just a $\mathbb{Z}^n$-graded $\Bbbk$-vector space, not a differential graded algebra, one can describe $K_\bullet$ as follows. For $\sigma = \{i_1, \dots, i_s\} \subseteq [n]$ with $i_1 < \cdots < i_s$, let v_σ denote $v_{i_1} \wedge \cdots \wedge v_{i_s}$. Clearly $K_i = S \otimes_\Bbbk \bigwedge^i V$ and the differential maps of $K_\bullet$ is described (with a suitable choice of sign $\pm$) as

$$K_s \ni z \otimes v_\sigma \mapsto \sum_{j=1}^s \pm (z x_i) \otimes v_{\sigma \setminus \{i_j\}} \in K_{s-1},$$

where $\sigma = \{i_1, \dots, i_s\}$ with $1 \le i_1 < \cdots < i_s \le n$. It thus follows that

$$\operatorname{Tor}_i^S(R, \Bbbk) \cong H^{-i}(R \otimes_S K_\bullet) \cong H_i(R \otimes_\Bbbk \bigwedge V)$$

in $\operatorname{mod}_{\mathbb{Z}^n} S$.

It is an easy exercise to verify that for $\underline{\boldsymbol{a}} \in [\underline{\mathbf{0}}, \underline{\mathbf{1}}]$ with $\operatorname{supp}(\underline{\boldsymbol{a}}) = F$, the assignment

$$\widetilde{C}^i(\Delta_F; \Bbbk) = \operatorname{Hom}_\Bbbk(\widetilde{C}_i(\Delta_F; \Bbbk), \Bbbk) \ni y_G^* \mapsto \pm x_G \otimes v_{F \setminus G} \in (K_{\#F-i-1})_{\underline{\boldsymbol{a}}},$$

where y_G denotes the basis of $\widetilde{C}_i(\Delta_F; \Bbbk)$ corresponding to $G \in \Delta_F$ and y_G^* its dual basis, gives rise to the isomorphism $\widetilde{C}^\bullet(\Delta_F; \Bbbk)[\#F - 1] \to (K_\bullet)_{\underline{\boldsymbol{a}}}$.

If $\underline{\boldsymbol{a}} \notin \mathbb{Z}^n_{\ge 0}$, then $(K_\bullet)_{\underline{\boldsymbol{a}}} = 0$. In the remaining case where $\underline{\boldsymbol{a}} \succeq \underline{\mathbf{0}}$ and $\underline{\boldsymbol{a}} \not\preceq \underline{\mathbf{1}}$, it then follows that $\underline{\boldsymbol{a}}' := \underline{\boldsymbol{e}}_{\operatorname{supp}(\underline{\boldsymbol{a}})} \prec \underline{\boldsymbol{a}}$. Setting $u := x^{\underline{\boldsymbol{a}} - \underline{\boldsymbol{a}}'} \in \mathfrak{m}$, one can easily verify that the multiplication map $u \cdot : (K_\bullet)_{\underline{\boldsymbol{a}}'} \to (K_\bullet)_{\underline{\boldsymbol{a}}}$ is isomorphism of complexes. Therefore $H_\bullet(K_\bullet)_{\underline{\boldsymbol{a}}} = 0$ since $H_\bullet(K_\bullet)$ is annihilated by $\mathfrak{m}$. $\square$

Corollary 2.24 (Eagon–Reinser [17])**.** *For Δ with $\dim \Delta = d - 1$, Δ is CM over $\Bbbk$ if and only if $I_{\Delta^\vee}$ has a $(n-d)$-linear resolution.*

Proof. Note that $\operatorname{indeg} I_{\Delta^\vee} = n-d$. On the other hand, from Hochster's formulas (Theorems 1.29 and 2.23), Alexander duality, and $\beta_{i,\underline{\mathbf{a}}}(I_{\Delta^\vee}) = \beta_{i-1,\underline{\mathbf{a}}}(\Bbbk[\Delta^\vee])$ for all i, it follows that

$$\Delta \text{ is CM over } \Bbbk \iff \operatorname{reg}_S I_{\Delta^\vee} \le n-d.$$

□

Now it is natural to ask about the blank below.

$$\begin{array}{ccc} \Delta: \text{ CM} & \Longleftrightarrow & I_{\Delta^\vee}: \text{ has a lin. res.} \\ \Downarrow & & \Downarrow \\ \Delta: \text{ SCM} & \Longleftrightarrow & I_{\Delta^\vee}: \text{ ???.} \end{array}$$

Definition 2.25. Let I be an ideal of S with $I \in \operatorname{mod}_{\mathbb{Z}} S$.

(1) For $i \in \mathbb{Z}$, set $I_{\langle i \rangle} = (f \mid f \in I_i)$.
(2) I is said to be *component-wise linear* if each $I_{\langle i \rangle}$ has a i-linear resolution.

Theorem 2.26 (Herzog–Hibi [24])**.** *$\Bbbk[\Delta]$ is SCM if and only if $I_{\Delta^\vee}$ is component-wise linear.*

Sketch of Proof. For a simplicial complex Γ and $i \in \mathbb{Z}$ with $0 \le k \le n$, set

$$(I_\Gamma)_{[k]} := (x_F \mid F \notin \Gamma,\ \#F = k)\,.$$

It is known [24, Proposition 1.5] that I_Γ is component-wise linear if and only if $(I_\Gamma)_{[k]}$ has a linear resolution for all k with $0 \le k \le n$.

The assertion is now an immediate consequence from the fact above, Eagon-Reiner's formula, Duval's criterion for SCM-ness, and the following fact

$$(I_{\Delta^\vee})_{[k]} = I_{(\Delta^{[n-k-1]})^\vee}$$

for $k = 1, \dots, n$, which can be easily verified. □

We will complete this section to mention the result on componentwise linear ideals by Herzog, Reiner and Welker. First let us recall the definition of Golod rings. See [2, 23] for details.

Let I be an ideal of S with $I \in \operatorname{mod}_{\mathbb{Z}} S$, and set $R := S/I$. Note that R is graded local with the unique maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$, where the maximal ideal $\mathfrak{m}$ of S is identified its image by the natural surjective homomorphism $S \twoheadrightarrow R$. Clearly $\Bbbk \cong R/\mathfrak{m} \cong R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$.

Definition 2.27. Let $M \in \operatorname{mod}_{\mathbb{Z}} R$.

(1) The formal power series

$$P_M^R(t) := \sum_{i=0}^{\infty} \dim_\Bbbk(\operatorname{Tor}_i^R(\Bbbk, M))t^i \in \mathbb{Z}[\![t]\!]$$

is called the *Poincaré series* of M.
(2) The *Poincaré series* of R is, by definition, $P_\Bbbk^R(t)$.

It is well-known that the following inequality holds.

Proposition 2.28 (Serre (cf. [2, 23]))**.** *Let R, M be as above. There is the following inequality*

$$P_M^R(t) \leqslant \frac{P_M^S(t)}{1 - t(P_R^S(t) - 1)}, \tag{2.5}$$

where $\leqslant$ means that each coefficients of the formal power series P_M^R are greater than or equal to those of the rational function in the left hand.

Remark 2.29. (1) In [2, 23], the same inequality as (2.5) is proved under the condition that S is a regular local ring, R a noetherian local ring, and $S \twoheadrightarrow R$ a homomorphism of local rings. The inequality (2.5) is an immediate consequence of this equality, since

$$\operatorname{Tor}^S_\bullet(M, \Bbbk) \cong \operatorname{Tor}^{S_\mathfrak{m}}_\bullet(M_\mathfrak{m}, S_\mathfrak{m}/\mathfrak{m}S_\mathfrak{m}), \quad \operatorname{Tor}^R_\bullet(M, \Bbbk) \cong \operatorname{Tor}^{R_\mathfrak{m}}_\bullet(M_\mathfrak{m}, R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m})$$

for any $M \in \operatorname{mod}_\mathbb{Z} R$.

(2) Since S is regular, both of $P^S_M(t)$ and $P^S_R(t)$ is just a polynomial of degree less than or equal to n.

Definition 2.30. With the notation above, the $\Bbbk$-algebra R is said to be *Golod* if the local ring $R_\mathfrak{m}$ is Gold, or equivalently the equality holds in (2.5) for the residue field $\Bbbk$, that is,

$$P^R_\Bbbk(t) = \frac{P^S_\Bbbk(t)}{1 - t(P^S_R(t) - 1)}. \tag{2.6}$$

By Proposition 2.22, one can easily verify that

$$P^S_\Bbbk(t) = \sum_{i=0}^{n} \dim_\Bbbk H_i(K_\bullet(\underline{\boldsymbol{x}}; \Bbbk))t^i = (1 + t)^n.$$

It thus follows that R is Golod if and only if

$$P^R_\Bbbk(t) = \frac{(1 + t)^n}{1 - t(P^S_R(t) - 1)}.$$

Clearly a Golod ring gives a typical example of those whose Poincaré series is rational. Whether a Poincaré series is always rational or not was a problem posed by Serre and Kaplansky ([46]). This problem has already been resolved negatively in Anick's famous paper [1].

The name "Golod" comes from Golod's characterization for rings satisfying (2.6). Recall that Koszul complex $K_\bullet(\underline{\boldsymbol{x}}; R)$ has a structure of DGA, and hence one can define *Massey products* (also called *Massey operation*) on $H_\bullet(K_\bullet(\underline{\boldsymbol{x}}; R))$. See [23, 28, 32, 33] for the definition of Massey products.

Theorem 2.31 (Golod [20] (cf. [2, 23])). *With the above notation, R is Golod if and only if all the* Massey products (Massey operation) *on $H_\bullet(K_\bullet(\underline{\boldsymbol{x}}; R)) = \operatorname{Tor}^S_\bullet(\Bbbk, R)$ vanishes.*

Herzog, Reiner, and Welker proved the following making use of this characterization.

Theorem 2.32 (Herzog–Reiner–Welker, [25]). *For a componentwise linear ideal I, the $\mathbb{Z}$-graded $\Bbbk$-algebra S/I is Golod.*

3. Further developments

3.1. **Simplicial posets.** In this subsection, all the posets (i.e. *partially ordered sets*) are assumed to be finite. For a simplicial complex Δ, Δ itself together with the inclusion $\subseteq$ forms the poset, called the *face poset*. Let P be a poset with the order $\leq$. Recall that

(1) P is called a *Boolean lattice* (or *Boolean algebra*) if it is isomorphic to a face poset of a simplex.
(2) P is said to be *simplicial* if it satisfies the following conditions:
  (a) P has the least element $\hat{0}$.
  (b) each intervals $[\hat{0}, p] := \{p' \in P \mid \hat{0} \leq p' \leq p\}$ are a Boolean lattice for all $p \in P$.

Clearly, a face poset of a simplicial complex is simplicial. More generally it is well-known [5] that every simplicial poset appears as the face poset of a regular CW complex X_P, i.e., the set of the closed cells of X with the order given by inclusion (while the face poset of an arbitrary regular CW complex is not necessarily simplicial).

The notion of f-vectors and h-vectors can be generalized to simplicial posets as follows. Let P be a simplicial poset. For $p \in P$, the interval $[\hat{0}, p]$ is then isomorphic to the face poset of a $(r-1)$-simplex. The integer r is called the *rank* of p, denoted by $\operatorname{rk} p$; hence $\operatorname{rk} p$ is just the maximal length of the chains in P that ends at p. The *rank* of P, denoted by $\operatorname{rk} P$, is defined to be $\max\{\operatorname{rk} p \mid p \in P\}$. Set $d := \operatorname{rk} P$. The vector $f(P) := (f_{-1}, \dots, f_{d-1})$, where

$$f_i := \#\{p \in P \mid \operatorname{rk} p = i + 1\},$$

is called the *f-vector* of P, and the *h-vector* of P is then defined to be $h(P) := (h_0, \dots, h_d)$, where

$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i}.$$

A simplicial poset is called a *simplicial cell sphere* if (the underlying space of) its associated regular CW complex X_P is homeomorphic to a sphere. For a simplicial cell sphere, the following characterization of h-vectors are well-known.

Theorem 3.1 (Masuda [29] and Stanley [51])**.** *A vector $(h_0, \dots, h_d) \in \mathbb{Z}^{d+1}$ is the h-vector of a simplicial cell sphere if and only if it satisfies the following conditions:*

(1) *$h_0 = 1$ and $h_i = h_{d-i}$ for all i,*
(2) *$h_i \geq 0$ for all i, and*
(3) *if $h_i = 0$ for some i with $1 \leq i \leq d-1$, then the sum $\sum_{i=0}^{d} h_i$ is even.*

The "if" part is proved by Stanley [51] and he conjectured that the "only if" part also holds. Later on, Masuda [29] proved it making use of the theory of toric topology [12, 30].

3.2. Squarefree modules and Alexander duality.

3.2.1. *Squarefree modules and Alexander duality.* In [54], Yanagawa introduced the notion, called a *squarefree module*, as a generalization of Stanley-Reisner rings, and extended the theory of Stanley-Reisner rings to squarefree modules. The full subcategory $\operatorname{Sq} S$ of $\operatorname{mod}_{\mathbb{Z}^n} S$ consisting of squarefree S-modules is abelian and contains, as objects, basic modules related with Stanley–Reisner rings such as $\Bbbk[\Delta]$, I_Δ, $I_\Delta / I_{\Delta'}$ where $\Delta \subseteq \Delta'$, and ω_S. In [36], Miller generalized squarefree modules to the notion, called *positively $\underline{\boldsymbol{a}}$-determined S-modules.* As the class of squarefree modules contains Stanley-Reisner rings (hence residue ring by squarefree monomial ideals), that of positively $\underline{\boldsymbol{a}}$-determined modules does residue rings of S by monomial ideals generated by monomials of degree $\preceq \underline{\boldsymbol{a}}$. On other hand, Römer [42] defined the notion of squarefree modules over the exterior algebra $E := \bigwedge S_1$ over $\Bbbk$ with respect to S_1.

By the notions above, Miller [36] and Römer [42] independently succeeded to discover the duality $\mathscr{A}$ on $\operatorname{Sq} S$, called *the Alexander duality functor*, which plays the role of Alexander duality. See their paper in loc. cit. for the precise definitions by them (or see below). It is noteworthy that $\mathscr{A}$ sends $\Bbbk[\Delta]$ to $I_{\Delta^\vee}$, not to $\Bbbk[\Delta^\vee]$. Many of facts on Alexander duality still hold for $\mathcal{A}$. For example, Theorems 2.24 and 2.26 hold for squarefree modules and $\mathscr{A}$.

Here we shall give another description of the Alexander duality functor following [56]. Let A be the *incidence algebra* over $\Bbbk$ associated with the subposet $P := [\underline{\boldsymbol{0}}, \underline{\boldsymbol{1}}]$ of $\mathbb{Z}^n$ (See the paper in loc. sit. for the definition). The algebra A is a finite-dimensional associative $\Bbbk$-algebra, and the category $\operatorname{mod} A$ of finitely generated left A-modules is equivalent to $\operatorname{Sq} S$ (hence $\operatorname{Sq} S$ has enough injectives, enough projectives, and finite global dimension). Through this equivalence, $\mathscr{A}$ coincides with $\operatorname{Hom}_A(-, \Bbbk)$ (precisely, the composition of $\operatorname{Hom}_A(-, \Bbbk)$ and the functor induced from the ring isomorphism $A \to A^{\mathrm{op}}$).

3.2.2. *Relation with Koszul duality.* As is shown in [55, 56], the category $D^b(\operatorname{Sq} S)$ can be regarded as a triangulated full subcategory of $D^b(\operatorname{mod}_{\mathbb{Z}^n} S)$, and $\mathbf{R}\operatorname{Hom}_S(-,\omega_S)$ induces the functor $\mathscr{D} : D^b(\operatorname{Sq} S)^{\mathrm{op}} \to D^b(\operatorname{Sq} S)$. Let $\mathscr{A} : D^b(\operatorname{Sq} S)^{\mathrm{op}} \to D^b(\operatorname{Sq} S)$ denote the functor induced from the Alexander duality functor by abuse of notation. The composition $\mathscr{D} \circ \mathscr{A}^{\mathrm{op}}$ and $\mathscr{A} \circ \mathscr{D}^{\mathrm{op}}$ then correspond to the functors giving the Koszul duality $D^b(\operatorname{mod} A) \cong D^b(\operatorname{mod} A^!)$ (Since A and $A^!$ are finite-dimensional over $\Bbbk$, Koszul duality induces this equivalence). See [56] for details.

3.2.3. *Relation with Auslander–Reiten translate and Nakayama functor.* In [8], Brun and Fløystad investigated the composition $\mathscr{A} \circ \mathscr{D}^{\mathrm{op}}$ and inferred that $\mathscr{A} \circ \mathscr{D}^{\mathrm{op}}$ corresponds to the Nakayama functor on $D^b(\operatorname{mod} A)$ and hence to Auslander–Reiten translate on $D^b(\operatorname{mod} A_P)$. (Actually they proved that there is the corresponding similar to above for positively $\underline{\boldsymbol{a}}$-determined modules. See [8] for details).

3.3. **Sheaves associated with squarefree modules.** In [55], Yanagawa introduced a *sheaf M^+ associated with a squarefree modules $M \in \operatorname{Sq} S$.* These sheaves are those on the (geometric) $(n-1)$-simplex X and with values in the category of $\Bbbk$-vector spaces. For example, $\Bbbk[\Delta]^+ = j_* \Bbbk_{|\Delta|}$, where $\Bbbk_{|\Delta|}$ denotes the constant sheaf on $|\Delta|$ with stalk $\Bbbk$ and $j : |\Delta| \to X$ is the natural embedding. In particular, S^+ is just the constant sheaf $\Bbbk_X$. The construction of M^+ gives an exact functor from $\operatorname{Sq} S$ to the category $\operatorname{Sh}(X)$ of $\Bbbk_X$-modules, and through this functor, local duality (for $M \in \operatorname{Sq} S$ with $M_{\underline{\mathbf{0}}} = 0$) corresponds to Poincaré–Verdier duality. Moreover through the functor $D^b(\operatorname{Sq} S) \to D^b(\operatorname{Sh}(X))$ induced from $(-)^+$, the complex $\omega_S[n-1]$ corresponds to a dualizing complex on X. For $M \in \operatorname{Sq} S$ and $\underline{\boldsymbol{a}} \in \mathbb{Z}^n_{\geq 0}$, it follows that

$$\operatorname{Ext}_S^{n-i}(M,\omega_S)_{\underline{\boldsymbol{a}}} \cong \begin{cases} H^{i-1}(X, M^+) & \text{if } \underline{\boldsymbol{a}} = \underline{\mathbf{0}} \\ H_c^{i-1}(U_\sigma, j^* M^+) & \text{otherwise} \end{cases}$$

for all $i \geq 1$, where σ denotes the face of X corresponding to $\operatorname{supp}(\underline{\boldsymbol{a}})$, H_c^{i-1} the $(i-1)$-th cohomology with compact support, U_σ the (open) star of σ, and j the embedding $U_\sigma \to X$. When $M = \Bbbk[\Delta]$, it follows that $H_c^{i-1}(U_\sigma, j^* M^+) \cong H^{i-1}(|\Delta|, |\Delta| \setminus \{p\}; \Bbbk)$ for any p in the interior of σ. The above isomorphism (in conjunction with Proposition 1.32) thus gives a generalization of Theorem 1.29 (except for $\operatorname{Ext}^n(\Bbbk[\Delta],\omega_S)$). See [55] for details.

3.4. **Toric face rings.** A Stanley–Reisner ring is one of main subject in combinatorial commutative algebra; another main subject is an *affine semigroup ring.* For a given affine monoid C, i.e., a finitely generated additive submonoid of $\mathbb{Z}^N$ for some positive integer N, its affine semigroup ring $\Bbbk[C]$ is a $\Bbbk$-algebra with the $\Bbbk$-basis C whose multiplication is induced from the addition in C. For example, the polynomial ring S is an affine semigroup ring associated with $\mathbb{Z}^n_{\geq 0}$. An affine semigroup ring $\Bbbk[C]$ has a connection with the cone generated by C (in $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}C$). See [9, 10, 37] for details.

A *toric face ring* is a common generalization of Stanley-Reisner rings and affine semigroup rings. Roughly speaking, a toric face ring is an $\Bbbk$-algebra given by gluing affine semigroup rings along a given rational pointed fan (see [9, 11, 27] for the definition). For example, a Stanley-Reisner ring $\Bbbk[\Delta]$ can be constructed by gluing each polynomial ring $\Bbbk[F]$ with $F \in \Delta$ along the fan consisting of the cones generated by a face of Δ.

In [11, 27], the following facts on Stanley-Reisner rings can be generalized to toric face rings.

- Hochster's formula for local cohomologies (Remark 1.30) and Tor modules (Theorem 2.23).
- The relation among CM-ness, Gor*-ness, and Eulerian-ness.
- Relation among shellability, CM, and SCM.

With an additional condition, in [40], Bbm-ness, CM-ness, and Gor*-ness for toric face rings are shown to depend only on $\Bbbk$ and the regular CW complex associated with the given fan (as for CM-ness, this assertion also follows from the results in [13]).

Appendix A. A brief review of graded algebras and modules

In this section, we will recall basics on graded modules. See [10, 21, 22] for details.

Definition A.1. Let $G = \mathbb{Z}^n$ or $\mathbb{Z}$, and R be a commutative ring.

(1) The ring R is said to be *G-graded* if
  (a) R has a decomposition $R = \bigoplus_{g \in G} R_g$ as $\mathbb{Z}$-modules such that
  (b) $R_g \cdot R_h \subseteq R_{g+h}$.
(2) An R-module M is said to be *G-graded* if
  (a) M has a decomposition $M = \bigoplus_{g \in G} M_g$ as $\mathbb{Z}$-modules such that
  (b) $R_g \cdot M_h \subseteq M_{g+h}$ for all $g, h \in G$.
(3) For a G-graded module M and $g \in G$, $M(g)$ denotes the G-graded module such that $M = M(g)$ as underlying S-modules and the grading of M is given by $M(g)_h := M_{g+h}$.

Any $\mathbb{Z}^n$-graded module M has the natural structure of a $\mathbb{Z}$-graded module by setting

$$M_i := \bigoplus_{|\underline{a}|=i} M_{\underline{a}}.$$

Thus all the $\mathbb{Z}^n$-graded module M is tacitly regarded as a $\mathbb{Z}$-graded one.

Let us recall the definition of associated prime ideals.

Definition A.2. Let R be a noetherian ring and M a finitely generated R-module.

(1) A prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ is said to be an *associated prime ideal* if
$$\mathfrak{p} = \operatorname{Ann}(m) := \{r \in R \mid rm = 0\}$$
for some $m \in M$.
(2) The set of associated prime ideals of M is denoted by $\operatorname{Ass}_R(M)$.

Henceforth let R be a noetherian commutative graded ring, and $\operatorname{Mod}_G R$ denotes the category consisting of G-graded R-modules and of *degree preserving* R-homomorphisms, i.e., R-homomorphisms $f : M \to N$ with M, N G-graded such that $f(M_g) \subseteq N_g$. The full subcategory of $\operatorname{Mod}_G R$ consisting of finitely generated G-graded R-modules is denoted by $\operatorname{mod}_G R$. It is well-known that $\operatorname{Mod}_G R$ has enough injectives and projectives (see [10]).

From an easy observation, the following hold.

Lemma A.3.

(1) *For any $M \in \operatorname{mod}_G R$ and $\mathfrak{p} \in \operatorname{Ass}_R(M)$, it follows that $\mathfrak{p} \in \operatorname{mod}_G R$ and $\mathfrak{p} = \operatorname{Ann}(m)$ for some $m \in M_g$ and $g \in G$.*
(2) *For each $M \in \operatorname{mod}_G R$, there exists a complex*
$$P_\bullet : \cdots \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$
such that
  (a) *P_i is finite direct sums of finitely many copies of $R(g)$ for various g (hence it is a finitely generated free R-module if one forgets the grading),*
  (b) *each differential map is degree-preserving, and*
  (c) *$H_0(P_\bullet) \cong M$ and $H_i(P_\bullet) = 0$ for $i \neq 0$.*

(3) *For* $M, N \in \operatorname{mod}_G R$ *(actually* N *is not needed to be finitely generated),*

$$\operatorname{Hom}_S(M, N) = \bigoplus_{g \in G} \operatorname{Hom}_{\operatorname{Mod}_G R}(M, N(g)).$$

(4) *In particular, with the natural structure of an* R*-module,* $\operatorname{Hom}_S(M, N) \in \operatorname{mod}_G R$ *(whenever* $M, N \in \operatorname{mod}_G R$*), where*

$$\operatorname{Hom}_S(M, N)_g := \operatorname{Hom}_{\operatorname{Mod}_G R}(M, N(g)).$$

(5) *For* $M, N \in \operatorname{Mod}_G R$*,* $M \otimes_R N \in \operatorname{Mod}_G R$*, where*

$$(M \otimes_R N)_g := \sum_{\substack{y \in M_h,\ z \in N_{h'} \\ h + h' = g}} \mathbb{Z} \cdot (y \otimes z) \subset M \otimes_R N.$$

Corollary A.4. *With the grading induced from results in Lemma A.3, it follows that* $\operatorname{Ext}^i_R(M, N) \in \operatorname{mod}_G R$ *and* $\operatorname{Tor}^R_i(M, N) \in \operatorname{mod}_G R$ *for* $M, N \in \operatorname{mod}_G R$*.*

Proof. If one define ${}^*\operatorname{Hom}_S(M, N) := \bigoplus_{g \in G} \operatorname{Hom}_{\operatorname{Mod}_G R}(M, N(g))$ for $M, N \in \operatorname{Mod}_G R$, then one gets the bifunctor

$${}^*\operatorname{Hom}_S(-, -) : (\operatorname{Mod}_G R)^{\mathrm{op}} \times \operatorname{Mod}_G R \to \operatorname{Mod}_G R.$$

As the usual Hom functor, ${}^*\operatorname{Hom}_S(M, -)$ and ${}^*\operatorname{Hom}_S(-, N)$ are left exact. One can thus define ${}^*\operatorname{Ext}_S(-, -)$. Lemma A.3 tells us that ${}^*\operatorname{Ext}_S(M, N) \cong \operatorname{Ext}_S(M, N)$ for $M \in \operatorname{mod}_G R$ and $N \in \operatorname{Mod}_G R$.

Similarly one can verify the assertion on Tor by Lemma A.3. □

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