Discrete Morse theory and combinatorial commutative algebra I

(非) 可換代数とトポロジー

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   - Forman's discrete Morse theory
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   - Graded CW complex and cellular resolutions
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   - Batzies-Welker's resolution for monomial ideals with linear quotients
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5. Algebraic discrete Morse theory
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Introduction

Graded free resolutions

Let $\Bbbk$ be a field and $S := \Bbbk[x_1, \ldots, x_n]$ a polynomial ring over $\Bbbk$.
For $\underline{\mathbf{a}} \in \mathbb{Z}^n_{\geq 0}$, we set $x^{\underline{\mathbf{a}}} := \prod_{i=1}^n x_i^{a_i}$. Recall that

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- S has a structure of $\mathbb{Z}^n$-graded $\Bbbk$-algebra as follows;
  - for $\underline{\mathbf{a}} := (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$, the degree $\deg(x^{\underline{\mathbf{a}}})$ is $\underline{\mathbf{a}}$;
  - $S = \bigoplus_{\underline{\mathbf{a}} \in \mathbb{Z}^n_{\geq 0}} S_{\underline{\mathbf{a}}}$ as $\Bbbk$-vector spaces, where $S_{\underline{\mathbf{a}}} := \Bbbk \cdot x^{\underline{\mathbf{a}}}$;
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For example, an ideal of S generated by some monomials $x^{\underline{\mathbf{a}}}$ is $\mathbb{Z}^n$-graded.
Recall that $\mathbb{Z}^n$ can be regarded as a poset by

$$\underline{\mathbf{a}} \geq \underline{\mathbf{b}} \Longleftrightarrow a_i \geq b_i \quad \forall i.$$

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Graded free resolutions

Let M be a $\mathbb{Z}^n$-graded S-module. A complex of S-modules

$$\mathcal{F} : \cdots \xrightarrow{\partial_3} \mathcal{F}_2 \xrightarrow{\partial_2} \mathcal{F}_1 \xrightarrow{\partial_1} \mathcal{F}_0 \longrightarrow 0$$

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  for some $\lambda_\tau \in \mathbb{k}$.

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$\mathcal{F}$ is said to be minimal if $\deg(\sigma) > \deg(\tau)$ for each σ, τ in $(*)$.

Introduction
Graded free resolutions

- A min. $\mathbb{Z}^n$-gr. free res. is very important in combinatorial commutative algebra and the related field.
- In general, it is hard to compute a minimal $\mathbb{Z}^n$-graded free resolution.

Forman's Morse theory for CW complex

Forman's discrete Morse theory

Example of CW complexes (Definition?)

Cell decomposition of B^2.

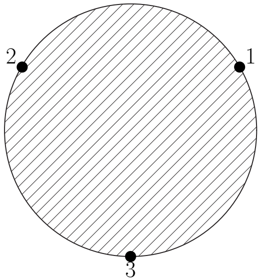

Figure: regular CW

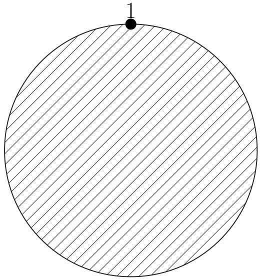

Figure: non-regular CW

Forman's Morse theory for CW complex

Forman's discrete Morse theory

For a CW complex X,

- set $X^{(i)} := \{\text{all the } i\text{-cells}\}$, $X^{(*)} := \bigcup_i X^{(i)}$, and
- $X^i := \bigcup_{j \leq i} X^{(j)}$, which is called i-skeleton of X.

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Recall that we can construct a cellular chain complex $\mathcal{C}(X; \mathbb{Z})$ of X (over $\mathbb{Z}$) as follows;

- $\mathcal{C}_p(X; \mathbb{Z}) = H_i(X^p, X^{p-1}; \mathbb{Z}) \cong \bigoplus_{\sigma \in X^{(p)}} \mathbb{Z} \cdot e_\sigma$;

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- $\partial_p : \mathcal{C}_p(X;\mathbb{Z}) \to \mathcal{C}_{p-1}(X;\mathbb{Z})$ is given by

$$H_p(X^p, X^{p-1};\mathbb{Z}) \xrightarrow{\alpha_p} H_{p-1}(X^{p-1};\mathbb{Z}) \xrightarrow{\beta_{p-1}} H_{p-1}(X^{p-1}, X^{p-2};\mathbb{Z}),$$

where α_p denotes the connecting map and β_{p-1} is the natural map.

Forman's Morse theory for CW complex

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Definition

For $\sigma \in X^{(p)}$ and $\tau \in X^{(p-1)}$, let $[\sigma : \tau]$ denote the coefficients of $\partial_p(e_\sigma)$ in e_τ. Thus $\partial_p(e_\sigma) = \sum_{\tau \in X^{(p-1)}} [\sigma : \tau] \cdot e_\tau$
$[\sigma : \tau] \in \mathbb{Z}$ is called a incidence number of X.

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X: CW cpx. The set of cells $X^{(*)}$ can be ordered as follows;

$$\sigma \geq \tau \Longleftrightarrow \bar{\sigma} \supseteq \tau.$$

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X: CW cpx. The set of cells $X^{(*)}$ can be ordered as follows;

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Let G_X be a directed graph associated with X such that

- the vertices are the cells of X, and
- the edges are $\{ \sigma \to \tau \mid \sigma \geq \tau,\ \dim \sigma = \dim \tau + 1, [\sigma : \tau] \neq 0 \}$.

Forman's Morse theory for CW complex

Forman's discrete Morse theory

Let A a set of some edges in G_X. Then we set G_X^A to be the directed graph whose

- vertices are those of G_X;

A vertex σ which does not appear in any edge in A is called a critical cell.

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Let A a set of some edges in G_X. Then we set G_X^A to be the directed graph whose

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Forman's Morse theory for CW complex

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Theorem (Forman, 1998)

Let X be a fin. reg. CW cpx., and A an acyc. matching of X. Then there exists a (not necessarily reg.) CW cpx X_A such that

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Theorem (Forman, 1998)

Let X be a fin. reg. CW cpx., and A an acyc. matching of X. Then there exists a (not necessarily reg.) CW cpx X_A such that

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Remark

Forman uses so-called Morse function $X^{(*)} \to \mathbb{R}$. The explanation here with an acyclic matching is due to Chari.

Discrete Morse theory for cellular resolutions

Graded CW complex and cellular resolutions

$S := \mathbb{k}[x_1, \ldots, x_n]$, X: CW cpx. Recall that $X^{(*)}$ can be ordered as follows

$$\sigma \geq \tau \iff \bar{\sigma} \supseteq \tau$$

Let $\mathrm{gr} : X^{(*)} \to \mathbb{Z}^n$ be a map.

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Definition

The pair (X, gr) is called a $\mathbb{Z}^n$-graded CW complex if gr is **order-preserving**, i.e., $\mathrm{gr}(\sigma) \geq \mathrm{gr}(\tau)$ if $\sigma \geq \tau$.

Discrete Morse theory for cellular resolutions

Graded CW complex and cellular resolutions

Example

$S := \Bbbk[x_1, \ldots, x_n]$: polynomial ring over a field $\Bbbk$

- $\mathcal{M} := \{\mathrm{m}_1, \ldots, \mathrm{m}_r\}$: a set of monomials of S, X: $(r-1)$-simplex. Labeling each vertices by $\mathrm{m}_1, \ldots, \mathrm{m}_r$, X can be regarded as $2^{\mathcal{M}}$. Hence $X^{(i)} := \{\ \sigma \subseteq \mathcal{M} \mid \#\sigma = i+1\ \}$, and the order on $X = 2^{\mathcal{M}}$ is the one defined by inclusion.
  Define $\mathrm{gr} : 2^{\mathcal{M}} \to \mathbb{Z}^n$ by
  $$\mathrm{gr}(\sigma) := \deg(\mathrm{lcm}(\sigma))$$
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  Define $\mathrm{gr} : 2^{\mathcal{M}} \to \mathbb{Z}^n$ by
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  Then gr is degree-preserving, and (X, gr) is $\mathbb{Z}^n$-graded.
- Clearly, $x^{\mathrm{gr}(\sigma)} = \mathrm{lcm}(\sigma)$.

Discrete Morse theory for cellular resolutions

Graded CW complex and cellular resolutions

Recall that $S := \Bbbk[x_1, \ldots, x_n]$. (X, gr): $\mathbb{Z}^n$-gr. CW.
Consider the chain complex

$$0 \longrightarrow \mathcal{F}^X_{\dim X} \longrightarrow \cdots \longrightarrow \mathcal{F}^X_1 \longrightarrow \mathcal{F}^X_0 \longrightarrow 0$$

such that

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Discrete Morse theory for cellular resolutions

Graded CW complex and cellular resolutions

Definition

For a $\mathbb{Z}^n$-gr. CW (X, gr), the chain complex $\mathcal{F}^X$, constructed above, is called the cellular resolution (of $\mathrm{Coker}(\mathcal{F}_1^X \to \mathcal{F}_0^X)$) supported by X if $\mathcal{F}^X$ is acyclic.

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Proposition

A cellular resolution $\mathcal{F}^X$ is minimal if and only if for $\sigma, \tau \in X^{(*)}$ with $\sigma \geq \tau$ and $\dim \sigma = \dim \tau + 1$, either $\mathrm{gr}(\sigma) \neq \mathrm{gr}(\tau)$ or $[\sigma : \tau] = 0$.

Discrete Morse theory for cellular resolutions

Graded CW complex and cellular resolutions

Example (Taylor resolution)

$J := (\mathrm{m}_1, \cdots, \mathrm{m}_r)$: monomial ideal, $G(J) := \{\mathrm{m}_1, \ldots, \mathrm{m}_r\}$
X: $(r-1)$-simplex, identified with $2^{G(J)}$.
Define $\mathrm{gr}(\sigma) := \deg(\mathrm{lcm}(\sigma))$, and

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It is well known that $\mathcal{F}^X$ gives a $\mathbb{Z}^n$-gr. free res. of J.
$\mathcal{F}^X$ is called the Taylor resolution of J.

Discrete Morse theory for cellular resolutions

Graded CW complex and cellular resolutions

Remark

- Taylor resolutions are cellular, but not minimal in general.

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Graded CW complex and cellular resolutions

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- Bar resolutions are also cellular and not minimal in general.

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For $\underline{\mathbf{a}} \in \mathbb{Z}^n$ and $\mathbb{Z}^n$-gr. CW, set $X_{\leq \underline{\mathbf{a}}}$ to be the subcomplex of X defined by

$$X^{(*)}_{\leq \underline{\mathbf{a}}} := \left\{ \sigma \in X^{(*)} \mid \operatorname{gr} \sigma \leq \underline{\mathbf{a}} \right\}.$$

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- Taylor resolutions are cellular, but not minimal in general.
- Bar resolutions are also cellular and not minimal in general.

For $\underline{\mathbf{a}} \in \mathbb{Z}^n$ and $\mathbb{Z}^n$-gr. CW, set $X_{\leq \underline{\mathbf{a}}}$ to be the subcomplex of X defined by

$$X^{(*)}_{\leq \underline{\mathbf{a}}} := \left\{ \sigma \in X^{(*)} \mid \operatorname{gr} \sigma \leq \underline{\mathbf{a}} \right\}.$$

Proposition

A $\mathbb{Z}^n$-gr. CW (X, gr) supports a free resolution of some graded S-module if and only if $X_{\leq \underline{\mathbf{a}}}$ is either empty or acyclic over $\Bbbk$ for all $\underline{\mathbf{a}} \in \mathbb{Z}^n$.

Discrete Morse theory for cellular resolutions

Graded CW complex and cellular resolutions

Proof.

For $\underline{\mathbf{a}} \in \mathbb{Z}^n$,

$$\left(\mathcal{F}_i^X\right)_{\underline{\mathbf{a}}} = \bigoplus_{\sigma \in X_{\leq \underline{\mathbf{a}}}^{(i)}} \Bbbk \cdot x^{\underline{\mathbf{a}} - \mathrm{gr}\,\sigma} \cdot e_\sigma .$$

Easy observation implies

$$\mathcal{F}_{\underline{\mathbf{a}}}^X \cong \mathcal{C}(X_{\leq \underline{\mathbf{a}}}; \Bbbk).$$

□

Discrete Morse theory for cellular resolutions

Batzies-Welker's theory

Batzies-Welker's idea

- Let A be an acyclic mathching of G_X.

fin. gr. CW X ⇜⇝ cel. res. $\mathcal{F}^X$

Discrete Morse Theory ⇓ ⇓

gr. CW X_A with $X_A \simeq X$ ⇜⇝ cel. res. $\mathcal{F}^{X_A}$ with $\mathcal{F}^{X_A} \simeq \mathcal{F}^X$

Here $\simeq$ denotes a homotopy equivalent.

Discrete Morse theory for cellular resolutions

Batzies-Welker's theory

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- Let A be an acyclic mathching of G_X.

$$\begin{array}{ccc}
\text{fin. gr. CW } X & \leftrightsquigarrow & \text{cel. res. } \mathcal{F}^X \\
\Big\Downarrow \text{Discrete Morse Theory} & & \Big\Downarrow \\
\text{gr. CW } X_A \text{ with } X_A \simeq X & \leftrightsquigarrow & \text{cel. res. } \mathcal{F}^{X_A} \text{ with } \mathcal{F}^{X_A} \simeq \mathcal{F}^X
\end{array}$$

Here $\simeq$ denotes a homotopy equivalent.

- $\operatorname{rank} \mathcal{F}_i^{X_A} = \# X_A^{(i)} \leq \# X^{(i)} = \operatorname{rank} \mathcal{F}_i^X$ for all i

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- $\operatorname{rank} \mathcal{F}_i^{X_A} = \# X_A^{(i)} \leq \# X^{(i)} = \operatorname{rank} \mathcal{F}_i^X$ for all i

$\rightsquigarrow$ We have a "smaller" (minimal in some cases) resolution $\mathcal{F}^{X_A}$.

Discrete Morse theory for cellular resolutions

Batzies-Welker's theory

Definition

(X, gr): $\mathbb{Z}^n$-gr. CW, G_X: associated graph.
An acyclic matching A of G_X is called homogeneous if $\mathrm{gr}\,\sigma = \mathrm{gr}\,\tau$ whenever $\sigma \to \tau \in A$.

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Proposition

(X, gr): fin. $\mathbb{Z}^n$-gr. reg. CW, A: its homogeneous acyclic matching. Then

(1) X_A has natural $\mathbb{Z}^n$-grading $\mathrm{gr}_A : X_A^{(*)} \to \mathbb{Z}^n$ induced by gr (i.e., for $\sigma_A \in X_A^{(*)}$ corresponding to a critical $\sigma \in X^{(*)}$, $\mathrm{gr}_A(c_A) = \mathrm{gr}(c)$).

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(2) With the above grading, $X_{\leq \underline{\mathbf{a}}} \simeq (X_A)_{\leq \underline{\mathbf{a}}}$ for any $\underline{\mathbf{a}} \in \mathbb{Z}^n$.

Discrete Morse theory for cellular resolutions

Batzies-Welker's theory

Theorem (Batzies-Welker, 2002)

(X, gr): fin. $\mathbb{Z}^n$-gr. reg. CW, A: homog. acyc. matching.
Assume $\mathcal{F}^X$ is a cellular resolution of a $\mathbb{Z}^n$-graded S-module M. Then $\mathcal{F}^{X_A}$ is also a cellular resolution of M.

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Proof.

By the hypothesis, $H^i(\mathcal{F}^X) = 0$ for $i \neq 0$ and $H^0(\mathcal{F}^X) \cong M$. Since for any integer i and any $\underline{\mathbf{a}} \in \mathbb{Z}^n$,

$$H^i(\mathcal{F}^X)_{\underline{\mathbf{a}}} \cong H^i(X_{\leq \underline{\mathbf{a}}}; \Bbbk) \cong H^i((X_A)_{\leq \underline{\mathbf{a}}}; \Bbbk) \cong H^i(\mathcal{F}^{X_A})_{\underline{\mathbf{a}}},$$

it follows that $H^i(\mathcal{F}^{X_A}) = 0$ for $i \neq 0$ and $H^0(\mathcal{F}^{X_A}) \cong M$. □

Discrete Morse theory for cellular resolutions

Batzies-Welker's theory

Remark

- Batzies-Welker showed the same assertion in more general situation where S is an affine semigroup ring $\Bbbk[\Lambda]$ and (X, gr) is a compactly $(\mathbb{Z}^n, \Lambda)$-graded CW complex, which is not necessarily finite.

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Discrete Morse theory for cellular resolutions

Description of differential map (but hard to compute)

(X, gr): fin. $\mathbb{Z}^n$-gr. CW, A: its homog. acyc. matching, G_X^A: associated graph.

- A directed path $\sigma_0 \to \sigma_1 \to \cdots \to \sigma_r$ in G_X^A is called gradient path.

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- A directed path $\sigma_0 \to \sigma_1 \to \cdots \to \sigma_r$ in G_X^A is called gradient path.
- For an edge $\sigma \to \tau$ in G_X^A, set

$$w(\sigma \to \tau) := \begin{cases} -[\tau : \sigma] & \text{if } \tau \to \sigma \in A; \\ [\sigma : \tau] & \text{otherwise,} \end{cases}$$

and for a grad. path $\mathcal{P} : \sigma_0 \to \sigma_1 \to \cdots \to \sigma_r$, set

$$w(\mathcal{P}) := \prod_{i=0}^{r-1} w(\sigma_i \to \sigma_{i+1}).$$

Discrete Morse theory for cellular resolutions

Description of differential map (but hard to compute)

With the above notation, for $\sigma, \tau \in G_X^A$, set

$$\mathrm{Path}_{G_X^A}(\sigma, \tau) := \{\text{grad. path from } \sigma \text{ to } \tau\}.$$

$$\partial_i^A(e_{\sigma_A}) = \sum_{\sigma'_A \in X_A^{(i-1)}} \left(\sum_{\mathcal{P} \in \mathrm{Path}_{G_X^A}(\sigma, \sigma')} w(\mathcal{P}) \right) x^{\mathrm{gr}(\sigma) - \mathrm{gr}(\sigma')} \cdot e_{\sigma'_A}$$

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The differential map ∂^A of $\mathcal{F}^{X_A}$ is given as follows; let $\sigma \in X^{(i)}$ be an A-critical cell, and σ_A the corresponding cell of X_A; then

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Application to monomial ideals with linear quotients

Batzies-Welker's resolution for monomial ideals with linear quotients

I: mon. ideal, $G(I) := \{\mathrm{m}_1, \dots, \mathrm{m}_r\}$: min. mon. generators.

Definition

I is said to have linear quotients if there exists a total order $\sqsubseteq$ on $G(I)$ satisfying for $\mathrm{m}, \mathrm{m}' \in G(I)$ with $\mathrm{m}' \sqsubset \mathrm{m}$, $\exists \mathrm{m}'' \in G(I)$ such that

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- $x_i \mathrm{m} = \mathrm{lcm}(\mathrm{m}'', \mathrm{m})$ for some i, and
- $\mathrm{lcm}(\mathrm{m}'', \mathrm{m}) | \mathrm{lcm}(\mathrm{m}', \mathrm{m})$.

Remark

It is well-known that I has linear quotients if and only if $\exists \sqsubseteq$: total order on $G(I)$ such that $(\mathrm{m}_1, \ldots, \mathrm{m}_k) : \mathrm{m}_{k+1}$ is generated by some variables of S for each k.

Application to monomial ideals with linear quotients

Batzies-Welker's resolution for monomial ideals with linear quotients

$X(= 2^{G(I)})$: $(t-1)$-simplex. Recall that the Tylor resolution $\mathcal{F}^X$

- $\mathcal{F}_i^X := \bigoplus_{\sigma \in X^{(i)}} S \cdot e_\sigma$
- $\mathcal{F}_i^X \ni e_\sigma \mapsto \sum_{\mathrm{m} \in \sigma} \pm \frac{\mathrm{lcm}(\sigma)}{\mathrm{lcm}(\sigma \setminus \mathrm{m})} \cdot e_{\sigma \setminus \mathrm{m}} \in \mathcal{F}_{i-1}^X$

gives a not nessarily min. $\mathbb{Z}^n$-gr. free resolution of I.

Let G_X be the graph associated with X.

Application to monomial ideals with linear quotients

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gives a not nessarily min. $\mathbb{Z}^n$-gr. free resolution of I.

Let G_X be the graph associated with X.

Theorem (Batzies-Welker, 2002)

With the above notation, there exists an acycling matching A of G_X such that $\mathcal{F}^{X^A}$ gives a min. $\mathbb{Z}^n$-gr. free res. of the monomial ideal I with linear quotients.

Application to monomial ideals with linear quotients

How to construct an acyclic matching

Construction of the acyclic matching

For $\mathrm{m} \in G(I)$, set

- $J_{\mathrm{m}} := \{ \ i \mid \exists \mathrm{n}_i^{\mathrm{m}} \in G(I) \text{ s.t. } \mathrm{n}_i^{\mathrm{m}} \sqsubset \mathrm{m}, \mathrm{lcm}(\mathrm{n}_i^{\mathrm{m}}, \mathrm{m}) = x_i \mathrm{m} \ \}$,

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- $\prec_p \big|_{G(I) \setminus N_{\mathrm{m}}} := \sqsubset \big|_{G(I) \setminus N_{\mathrm{m}}}$.

Application to monomial ideals with linear quotients

How to construct an acyclic matching

For $\sigma := \left\{\mathrm{m}_0 \prec_{f(\sigma)} \cdots \prec_{f(\sigma)} \mathrm{m}_i\right\}$, define

- $v(\sigma) := \sup\left\{ k \geq 0 \mid \exists \mathrm{m} \in G(I) \text{ s.t. } \begin{array}{l} \mathrm{m} \prec_{f(\sigma)} \mathrm{m}_{i-k} \text{ and} \\ \mathrm{m} \mid \operatorname{lcm}(\mathrm{m}_{i-k}, \ldots, \mathrm{m}_i) \end{array} \right\}.$

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If $v(\sigma) \neq -\infty$, then set

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Let G_X be the directed graph associated with $X = 2^{G(I)}$. Now for each $p \in P$, define the subset A_p, A of the edge sets of G_X as follows.

Application to monomial ideals with linear quotients

How to construct an acyclic matching

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- $v(\sigma) := \sup \left\{ k \geq 0 \mid \exists \mathrm{m} \in G(I) \text{ s.t. } \begin{array}{l} \mathrm{m} \prec_{f(\sigma)} \mathrm{m}_{i-k} \text{ and} \\ \mathrm{m} \mid \mathrm{lcm}(\mathrm{m}_{i-k}, \ldots, \mathrm{m}_i) \end{array} \right\}.$

If $v(\sigma) \neq -\infty$, then set

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Application to monomial ideals with linear quotients

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Application to monomial ideals with linear quotients

How to construct an acyclic matching

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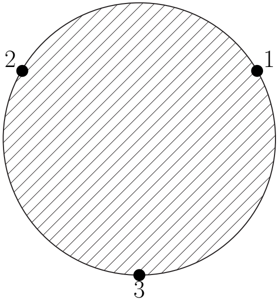
Application to monomial ideals with linear quotients

How to construct an acyclic matching

Example of discrete Morse theory

- Let X be the reg. CW cpx as in the right.



Application to monomial ideals with linear quotients

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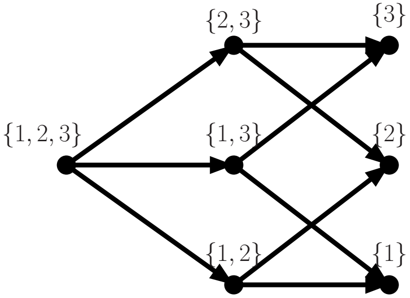

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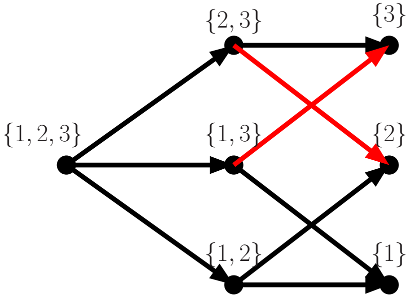

Application to monomial ideals with linear quotients

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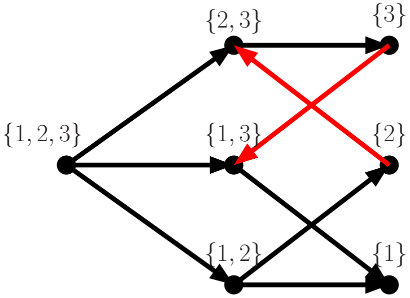

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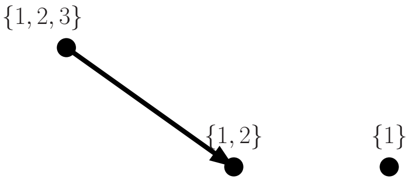

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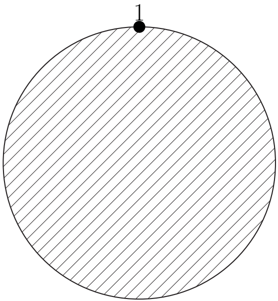
- Consequently, we get the non-reg. CW cpx.



Algebraic discrete Morse theory

Application of algebraic aspects of Forman's theory

Let

- R be (not necessarily commutative) ring, $Z(R)$ the center of R, $R^{\times}$ the grp. of units, and let

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For $\sigma \in X^{(i)}$ and $\tau \in X^{(i-1)}$, define $[\sigma : \tau] \in R$ to satisfy

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With the cpx C, we associate the graph G_X whose

- vertices are $X := \bigcup_{i \geq 0} X^{(i)}$ and
- edges are $\{ \sigma \to \tau \mid \sigma \in X^{(i)}, \tau \in X^{(i-1)}, [\sigma : \tau] \neq 0 \}$.

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For an edge $\sigma \to \tau$ in G_X^A and a path $\mathcal{P} : \sigma_1 \to \cdots \to \sigma_r$, define

- $w(\sigma \to \tau) := \begin{cases} -\frac{1}{[\tau:\sigma]} & \text{if } \tau \to \sigma \in A \text{ (then } [\tau : \sigma] \in Z(R) \cap R^\times) \\ [\sigma : \tau] & \text{otherwise} \end{cases}$
- $w(\mathcal{P}) = \prod_{i=1}^{r-1} w(\sigma_i \to \sigma_{i-1})$

Algebraic discrete Morse theory

Application of algebraic aspects of Forman's theory

For an acyc. matching A, set $X_A^{(i)} := \{ \sigma \in X^{(i)} \mid \sigma \text{ is critical } \}$, and define the complex

$$C^A : \cdots \longrightarrow C_2^A \xrightarrow{\partial_2^A} C_1^A \xrightarrow{\partial_1^A} C_0^A \xrightarrow{\partial_0^A} 0$$

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Theorem (Jòllenberg-Welker, 2005, Sköldberg, 2006)

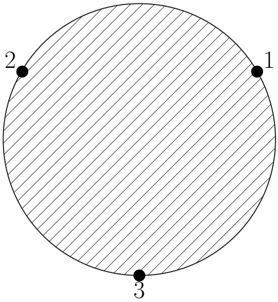
C^A is indeed a complex, and is homotopy equivalent to C.

Algebraic discrete Morse theory

Application of algebraic aspects of Forman's theory

Example of algebraic discrete Morse theory

Let X be the reg. CW cpx as in the right. We regard X as a 2-simplex, and hence as $2^{\{1,2,3\}}$.



Algebraic discrete Morse theory

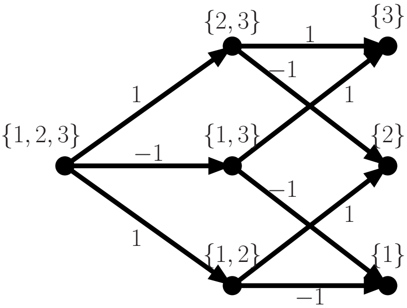
Application of algebraic aspects of Forman's theory

Example of algebraic discrete Morse theory

Let C be the cellular chain complex of X as follows;

- $C_i = \bigoplus_{\sigma \in X^{(i)}} \mathbb{Z} \cdot e_\sigma$;
- $\partial(e_\sigma) := \sum_{i \in \sigma} (-1)^{\varepsilon(i;\sigma)} e_{\sigma \setminus \{i\}}$,

where $\varepsilon(i;\sigma) := \#\{\, j \in \sigma \mid j < i \,\}$.
Then G_X is as in the right.



Algebraic discrete Morse theory

Application of algebraic aspects of Forman's theory

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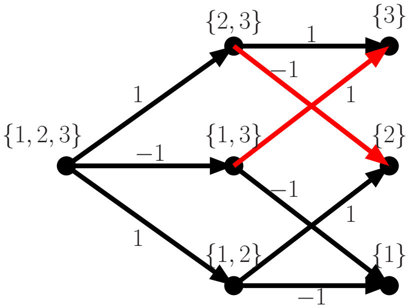
Choose the red arrows as an acyclic matching.



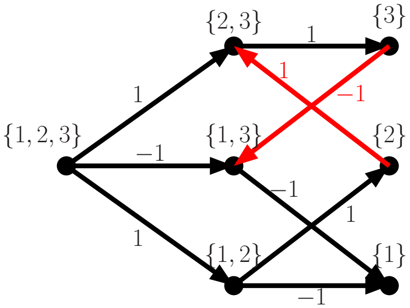
Algebraic discrete Morse theory

Application of algebraic aspects of Forman's theory

Example of algebraic discrete Morse theory

Reverse the red arrows, and change the weights of

- $e_{\{2\}} \to e_{\{2,3\}}$ to $-(1/(-1)) = 1$,
- $e_{\{3\}} \to e_{\{1,3\}}$ to $-(1/1) = -1$.



Algebraic discrete Morse theory

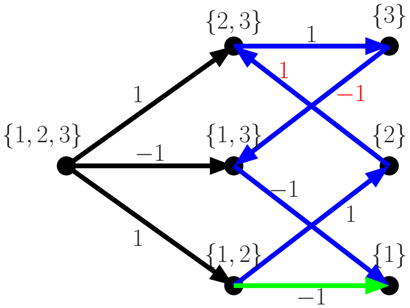
Application of algebraic aspects of Forman's theory

Example of algebraic discrete Morse theory

Now let us compute the differential.
It is easy to check that
$\mathrm{Path}(e_{\{1,2,3\}}, e_{\{1,2\}}) = \{e_{\{1,2,3\}} \to e_{\{1,2\}}\}$.
$\mathrm{Path}(e_{\{1,2\}}, e_{\{1\}}) = \{\mathcal{P}, \mathcal{P}'\}$,
where $\mathcal{P}$ is the green path and $\mathcal{P}'$ is the blue one.
Easy computation shows

$$w(\mathcal{P}) + w(\mathcal{P}') = -1 + 1 = 0.$$

So, $e_{\{1,2\}}$ is mapped to 0 by the differential map.



Algebraic discrete Morse theory

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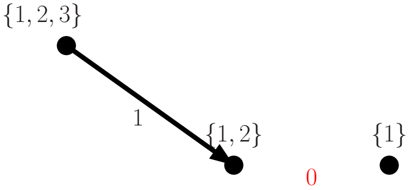
Example of algebraic discrete Morse theory

Thus we obtain the complex C^A. This is just a cellular chain complex of ...

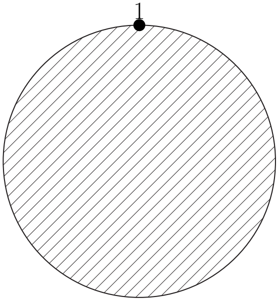


Algebraic discrete Morse theory

Application of algebraic aspects of Forman's theory

Example of algebraic discrete Morse theory

the non-regular CW complex in the right.



Appendix

Definition of CW complex

For a non-negative integer r, B^r denotes a r-dimensional closed ball.

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Definition

X: top. sp. A subset σ of X is said to be an (open) r-cell if there exists a continuous map $f : B^r \to X$ such that

$$f|_{B^r \setminus \partial B^r} : B^r \setminus \partial B^r \xrightarrow{\cong} \sigma.$$

In this case, the continuous map f is called the charcteristic map of σ.

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Definition of CW complex

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A Housdorff top. sp. X together with a set of cells $X^{(*)}$ is said to be a CW complex if

(1) $X = \bigcup_{\sigma \in X^{(*)}} \sigma$ and $\sigma \cap \tau = \emptyset$ for all $\sigma, \tau \in X^{(*)}$ with $\sigma \neq \tau$.

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(2) For any r-cell $\sigma \in X^{(*)}$ and its char. map f_σ, $f_\sigma(\partial B^r)$ non-trivially intersects only finitely many s-cells with $s < r$.

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A CW complex X is said to be regular if for each cell σ, $\bar{\sigma}$ is homeomorphic to a closed ball.

Back to the main