Semi-tilting modules and mutation

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In this talk, using the notion of mutation, we will provide a partial answer to the Wakamatsu tilting conjecture. Let R be a commutative complete local ring and A a noetherian R-algebra, i.e., A is a ring endowed with a ring homomorphism $\varphi : R \to A$ whose image is contained in the center of A and A is a finitely generated R-module. A module $T \in \text{mod-}A$ is said to be a Wakamatsu tilting module if the following conditions are satisfied: (1) $\text{Ext}^i_A(T,T) = 0$ for $i \neq 0$; (2) A admits a right resolution $A \to T^\bullet$ in mod-A with $T^\bullet \in \mathcal{K}^+(\text{add}(T))$ and $\text{Ext}^j_A(\text{Z}^i(T^\bullet), T) = 0$ for all $i, j \geq 1$ (see [2]). The Wakamatsu tilting conjecture states that proj dim ${}_{\text{End}_A(T)}T =$ proj dim T_A for every Wakamatsu tilting module $T \in \text{mod-}A$ (see [1]). Note that if both proj dim ${}_{\text{End}_A(T)}T$ and proj dim T_A are finite then T is a tilting module and proj dim ${}_{\text{End}_A(T)}T =$ proj dim T_A.

A module $T \in \text{mod-}A$ is said to be a semi-tilting module if the following conditions are satisfied: (1) $\text{Ext}^i_A(T,T) = 0$ for $i \neq 0$; (2) A admits a right resolution $A \to T^\bullet$ in mod-A with $T^\bullet \in \mathcal{K}^{\text{b}}(\text{add}(T))$. Note that ${}_{\text{End}_A(T)}T$ is a Wakamatsu tilting module of finite projective dimension. We will show that for a basic semi-tilting module $T = U \oplus X \in \text{mod-}A$ with X indecomposable, if X is generated by U then there exists a non-split exact sequence $0 \to Y \to E \to X \to 0$ in mod-A with Y indecomposable, $E \in \text{add}(U)$ and $U \oplus Y$ a semi-tilting module. Note that for a basic semi-tilting module $T \in \text{mod-}A$ there always exists a direct summand X of T such that $T \cong U \oplus X$ and X is generated by U unless T is projective. Assume that R is Cohen-Macaulay and A is a maximal Cohen-Macaulay R-module. For a semi-tilting module $T \in \text{mod-}A$ which is a maximal Cohen-Macaulay R-module, we denote by $\mathcal{L}({}^\perp T)$ the full subcategory of mod-A consisting of modules M which are maximal Cohen-Macaulay R-modules and $\text{Ext}^i_A(M,T) = 0$ for $i \neq 0$. We will show that if the number of isomorphism classes of indecomposable modules in $\mathcal{L}({}^\perp T)$ is finite then T is a tilting module, i.e., the Wakamatsu tilting conjecture holds true for such a Wakamatsu tilting module.

References

[1] A. Beligiannis and I. Reiten, Homological and homotopical aspects of torsion theories, Mem. Amer. Math. Soc. **188** (2007), no. 883, viii+207 pp.

[2] T. Wakamatsu, Stable equivalence for self-injective algebras and a generalization of tilting modules, J. Algebra **134** (1990), no. 2, 298–325.