Higher homotopy associativity of power maps on p-regular H-spaces

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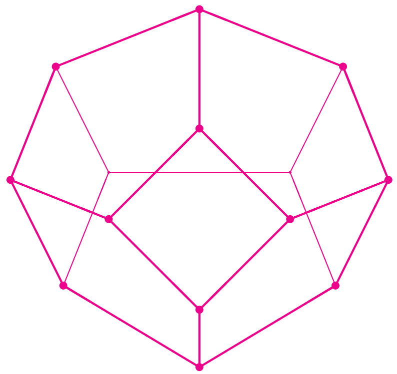

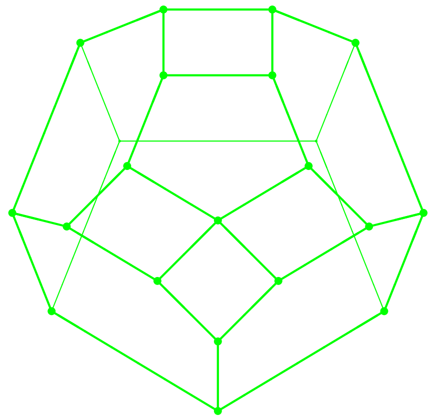

Algebraic and Geometric Models for Spaces and Related Topics 2014

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All spaces are assumed to be pointed, arcwise connected and of the homotopy type of CW-complexes.

Let (X, μ) be a homotopy associative H-space. From the above assumption, (X, μ) is a group-like space. The power maps $\{\Phi_\lambda^X : X \to X\}_{\lambda \in \mathbb{Z}}$ are defined as follows:

- $\Phi_0^X(x) = x_0$
- $\Phi_\lambda^X(x) = \mu(\Phi_{\lambda-1}^X(x), x)$ for $\lambda > 0$
- $\Phi_\lambda^X(x) = \iota(\Phi_{-\lambda}^X(x))$ for $\lambda < 0$,

where $x_0 \in X$ and $\iota\colon X \to X$ denote the homotopy unit and the homotopy inverse on (X, μ), respectively.

• (X,μ) is homotopy commutative $\overset{\text{iff}}{\Longleftrightarrow}$ $\{\Phi_\lambda^X\}_{\lambda\in\mathbb{Z}}$ are H-maps

• If X is a double loop space, then $\{\Phi_\lambda^X\}_{\lambda\in\mathbb{Z}}$ are loop maps

Theorem. [Sullivan 1974]

Let p be an odd prime and $t \geq 1$. Then $S^{2t-1}_{(p)}$ is a loop space $\overset{\text{iff}}{\Longleftrightarrow}$ $t|(p-1)$.

We denote the loop space $S^{2t-1}_{(p)}$ by W_t.

Theorem 1. [Arkowitz-Ewing-Schiffman 1975]

Let p be an odd prime. The power map $\Phi_\lambda^{W_{p-1}}$ on W_{p-1} is an H-map $\overset{\text{iff}}{\Longleftrightarrow} \lambda(\lambda - 1) \equiv 0 \bmod p$.

Remark.

- When $t \neq p-1$, all the power maps $\{\Phi_\lambda^{W_t}\}_{\lambda \in \mathbb{Z}}$ on W_t are H-maps since the multiplication on W_t is homotopy commutative.
- Theorem 1 is generalized to the case of several p-localized finite loop spaces by [McGibbon 1980] and [Theriault 2013].

Theorem 2. [Lin 2012]

Let p be an odd prime and $t \geq 1$ with $t|(p-1)$. The power map $\Phi_\lambda^{W_t}$ on W_t is a loop map $\overset{\text{iff}}{\Longleftrightarrow} \lambda = \alpha^t$ for some p-adic integer $\alpha \in \mathbb{Z}_p^\wedge$.

Remark 3.

- When $\lambda \not\equiv 0 \bmod p$, Theorem 2 is proved by [Rector 1971] and [Arkowitz-Ewing-Schiffman 1975].
- Theorem 2 can also be derived from [Adams-Wojtkowiak 1989] and [Wojtkowiak 1990].

Corollary 4.

Let p and t be as in Theorem 2. Put $m = (p-1)/t$. Assume $\lambda \neq 0$ and write $\lambda = p^a b$ with $a \geq 0$ and $b \not\equiv 0 \bmod p$. The power map $\Phi_\lambda^{W_t}$ on W_t is a loop map $\overset{\text{iff}}{\Longleftrightarrow}$ $t|a$ and $b^m \equiv 1 \bmod p$.

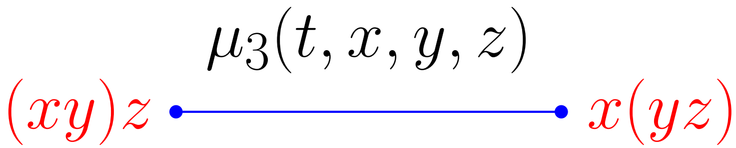
Definition. [Sugawara 1957], [Stasheff 1963]

A space X is an A_n-space $\overset{\text{def}}{\Longleftrightarrow}$

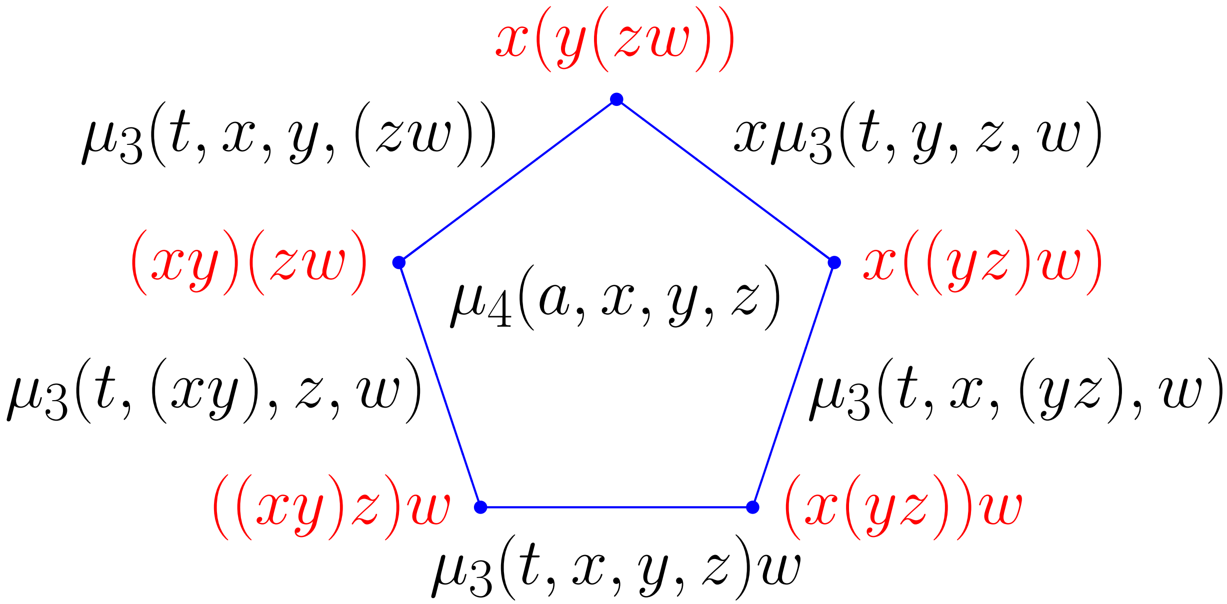
$$^{\exists}\{\mu_i \colon K_i \times X^i \to X\}_{1 \le i \le n}$$

with some relations, where $\{K_i\}_{i \ge 1}$ denote the associahedra constructed by [Stasheff 1963].

K_3



K_4



- X is an A_2-space $\stackrel{\text{iff}}{\Longleftrightarrow}$ X is an H-space
- X is an A_3-space $\stackrel{\text{iff}}{\Longleftrightarrow}$ X is a homotopy associative H-space
- X is an A_∞-space $\stackrel{\text{iff}}{\Longleftrightarrow}$ $X \simeq \Omega(BX)$ for some space BX by [Sugawara 1957] and [Stasheff 1963]

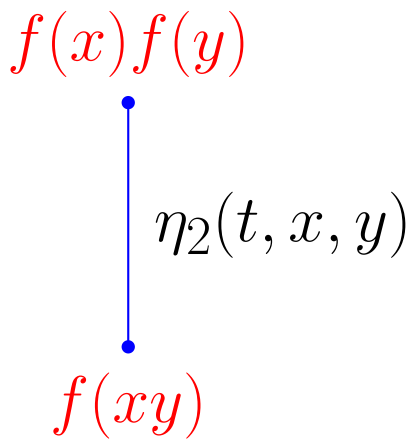
Definition. [Sugawara 1960], [Stasheff 1970], [Iwase-Mimura 1989]

Let X, Y be A_n-spaces. A map $f\colon X \to Y$ is an A_n-map $\overset{\text{def}}{\Longleftrightarrow}$

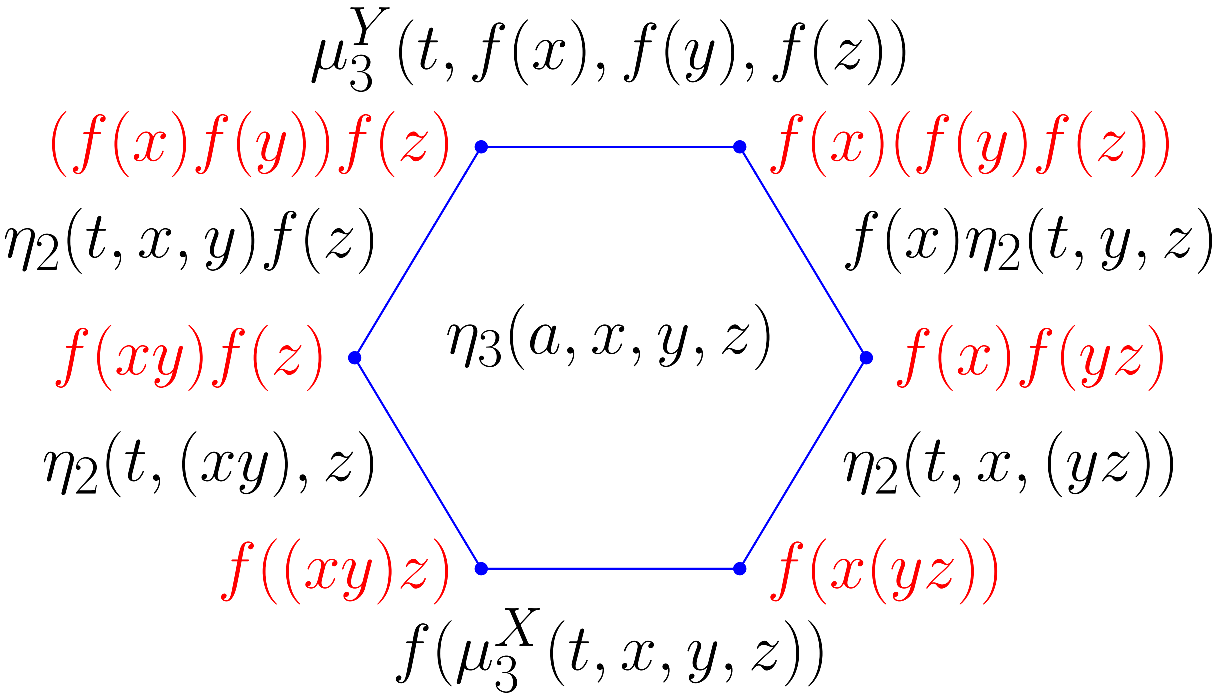
$$^{\exists}\{\eta_i\colon J_i \times X^i \to Y\}_{1\leq i\leq n}$$

with some relations, where $\{J_i\}_{i\geq 1}$ denote the multiplihedra constructed by [Iwase-Mimura 1989].

J_2



J_3



- $f\colon X \to Y$ is an A_2-map $\overset{\text{iff}}{\Longleftrightarrow}$ f is an H-map
- $f\colon X \to Y$ is an A_3-map $\overset{\text{iff}}{\Longleftrightarrow}$ f is an H-map preserving homotopy associativity homotopically

- $f\colon X \to Y$ is an A_∞-map $\overset{\text{iff}}{\Longleftrightarrow}$ $f \simeq \Omega(Bf)$ for some map $Bf\colon BX \to BY$ by [Sugawara 1960], [Stasheff 1970] and [Iwase-Mimura 1989]

In this talk, we study the condition for the power map on an A_n-space to be an A_n-map. The higher homotopy associativity of the power maps $\{\Phi_\lambda^X\}_{\lambda\in\mathbb{Z}}$ measures a lack of higher homotopy commutativity of (X,μ).

Theorem A.

Let p be an odd prime and $t \geq 1$ with $t|(p-1)$. Put $m = (p-1)/t$. The power maps $\{\Phi_\lambda^{W_t}\}_{\lambda \in \mathbb{Z}}$ on W_t satisfy the following:

(1) $\Phi_\lambda^{W_t}$ is an A_m-map for any $\lambda \in \mathbb{Z}$.

(2) $\Phi_\lambda^{W_t}$ is an A_{m+1}-map $\overset{\text{iff}}{\Longleftrightarrow} \lambda(\lambda^m - 1) \equiv 0 \bmod p$.

Remark 5.

- If $t = p - 1$, then Theorem A (2) is the same as Theorem 1.
- When $t = (p-1)/2$, Theorem A (2) is proved by [McGibbon 1982].
- When $\lambda \not\equiv 0 \bmod p$, $\Phi_\lambda^{W_t}$ is an A_{m+1}-map $\overset{\text{iff}}{\Longleftrightarrow}$ $\Phi_\lambda^{W_t}$ is a loop map by Theorem A (2) and Corollary 4.

Theorem B.

Let p, t and m be as in Theorem A. Assume that $\lambda \equiv 0 \bmod p$ and $2 \leq j \leq t$. The power maps $\{\Phi_\lambda^{W_t}\}_{\lambda \in \mathbb{Z}}$ on W_t satisfy the following:

(1) If $\Phi_\lambda^{W_t}$ is an $A_{(j-1)m+1}$-map, then it is also an A_{jm}-map.

(2) $\Phi_\lambda^{W_t}$ is an A_{jm+1}-map $\overset{\text{iff}}{\Longleftrightarrow}$ $\lambda \equiv 0 \bmod p^j$.

From Theorems A (2) and B (2) and Corollary 4, we have the following corollary:

Corollary 6.

Let p, t and m be as in Theorem A. The power map $\Phi_\lambda^{W_t}$ on W_t is an A_p-map $\overset{\text{iff}}{\Longleftrightarrow}$ $\lambda \equiv 0 \bmod p^t$ or $\lambda^m \equiv 1 \bmod p$.

Definition.

- A space X is $\mathbb{F}_p$-finite $\overset{\text{def}}{\Longleftrightarrow}$ $H^*(X;\mathbb{F}_p)$ is finite-dimensional as a vector space over $\mathbb{F}_p$.
- A space X is $\mathbb{F}_p$-acyclic $\overset{\text{def}}{\Longleftrightarrow}$ $\widetilde{H}^*(X;\mathbb{F}_p) = 0$.

Theorem C.

Let p be an odd prime. Assume that X is a simply connected $\mathbb{F}_p$-finite A_p-space and λ is a primitive $(p-1)$-st root of unity $\bmod p$. If the reduced power operations $\{\mathscr{P}^i\}_{i\geq 1}$ act trivially on the indecomposable module $QH^*(X;\mathbb{F}_p)$ and the power map Φ_λ^X on X is an A_n-map with $n > (p-1)/2$, then X is $\mathbb{F}_p$-acyclic.

Remark 7.

- The condition for λ cannot be removed. In fact:

  (1) If $\lambda \equiv 0 \bmod p$, then the power map $\Phi_\lambda^{W_2}$ on W_2 is an $A_{(p+1)/2}$-map by Theorem A (2)

  (2) Assume that $\lambda^k \equiv 1 \bmod p$ for some k with $1 \leq k < p-1$ and $k|(p-1)$. Put $t = (p-1)/k > 1$. Then the power map $\Phi_\lambda^{W_t}$ on W_t is a loop map by Corollary 4.

- Since the power maps $\{\Phi_\lambda^{W_2}\}_{\lambda \in \mathbb{Z}}$ on W_2 are $A_{(p-1)/2}$-maps by Theorem A (1), the assumption "$n > (p-1)/2$" cannot be relaxed in Theorem C.

Definition.

An H-space is p-regular $\overset{\text{def}}{\iff}$

$$X_{(p)} \simeq S^{2t_1-1}_{(p)} \times \cdots \times S^{2t_\ell-1}_{(p)} \quad (1 \leq t_1 \leq \cdots \leq t_\ell) \qquad \cdots (*)$$

Theorem. [Hubbuck-Mimura 1987], [Iwase 1989]

Let p be an odd prime. If X is a connected p-regular A_p-space with $(*)$, then $t_\ell \leq p$.

Theorem D.

Let p and λ be as in Theorem C. Assume that X is a simply connected p-regular A_p-space with $(*)$. If the power map Φ_λ^X on X is an A_n-map with $n > [p/t_\ell]$, then X is $\mathbb{F}_p$-acyclic.

Remark 8.

Since the power maps $\{\Phi_\lambda^{W_t}\}_{\lambda\in\mathbb{Z}}$ on W_t are A_m-maps by Theorem A (1) and $[p/t] = m$, the assumption "$n > [p/t_\ell]$" cannot be relaxed in Theorem D.

Proof of Theorem A (1).

By induction on i, we construct an A_m-form $\{\eta_i\}_{1\leq i\leq m}$ on $\Phi_\lambda^{W_t}$. Put $\eta_1 = \Phi_\lambda^{W_t}$. Assume inductively that $\{\eta_j\}_{1\leq j<i}$ is constructed for some $i \leq m$. Let $\Gamma_i(W_t) = \partial J_i \times (W_t)^i \cup J_i \times (W_t)^{[i]}$, where $X^{[i]}$ denotes the i-fold fat wedge of a space X defined as

$$X^{[i]} = \{(x_1, \ldots, x_i) \in X^i \mid x_j = * \text{ for some } j \text{ with } 1 \leq j \leq i\}.$$

Then $(J_i \times (W_t)^i)/\Gamma_i(W_t) \simeq S_{(p)}^{2ti-1}$.

We define $\widetilde{\eta}_i\colon \mathit{\Gamma}_i(W_t) \to W_t$ using $\{\eta_j\}_{1\le j<i}$. The obstructions to obtain $\eta_i\colon J_i \times (W_t)^i \to W_t$ with $\eta_i|_{\mathit{\Gamma}_i(W_t)} = \widetilde{\eta}_i$ appear in the cohomology groups

$$H^{k+1}(J_i \times (W_t)^i, \mathit{\Gamma}_i(W_t); \pi_k(W_t)) \cong \widetilde{H}^k(S^{2ti-2}_{(p)}; \pi_k(W_t)) \quad \text{for } k \ge 1.$$

The above is non-trivial only if k is an even integer with $k < 2p-2$ since $ti \le tm = p-1$. On the other hand, $\pi_k(W_t) = 0$ for any even integer k with $k < 2p-2$ by [Toda 1962]. Then we have a map η_i. This completes the induction, and we have an A_m-form $\{\eta_i\}_{1\le i\le m}$ on $\mathit{\Phi}^{W_t}_\lambda$.

Let X be an A_n-space. According to [Stasheff 1963], we have the projective spaces $\{P_i(X)\}_{0\le i\le n}$ with the following properties:

- There is a fibration

$$X \longrightarrow \Sigma^{i-1}X^{\wedge i} \xrightarrow{\gamma_{i-1}} P_{i-1}(X) \quad \text{for } 1 \le i \le n$$

- There is a long cofibration sequence:

$$\Sigma^{i-1}X^{\wedge i} \xrightarrow{\gamma_{i-1}} P_{i-1}(X) \xrightarrow{\iota_{i-1}} P_i(X) \xrightarrow{\rho_i} \Sigma^i X^{\wedge i} \xrightarrow{\Sigma\gamma_{i-1}} \cdots \quad \text{for } 1 \le i \le n,$$

  where $X^{\wedge i}$ denotes the i-fold smash product of X.

- $P_0(X) = \{*\}$ and $P_1(X) = \Sigma X$.
- When X is an A_∞-space, $P_\infty(X) = BX$.

Theorem. [Stasheff 1970], [Iwase-Mimura 1989], [Hemmi 2007]

Let X, Y be A_n-spaces.

(1) If $f\colon X \to Y$ is an A_n-map, then

$$^{\exists}\{P_i(f)\colon P_i(X) \to P_i(Y)\}_{1\le i\le n}$$

with $P_1(f) = \Sigma f$ and $P_i(f)\iota_{i-1} = P_{i-1}(f)\iota_{i-1}$ for $2 \le i \le n$.

(2) If Y is an A_{n+1}-space, then the converse of (1) also holds.

Put $\varepsilon_{i-1} = \iota_{i-1}\cdots\iota_1\colon \Sigma X = P_1(X) \to P_i(X)$ for $i \ge 2$.

Proof of the " only if " part of Theorem A (2).

It is known that

$$H^*(P_{m+1}(W_t);\mathbb{F}_p) \cong \mathbb{F}_p[\boldsymbol{u}]/(\boldsymbol{u}^{m+2}) \quad \text{with } \deg \boldsymbol{u} = 2t$$

and

$$\mathscr{P}^1(\boldsymbol{u}) = \xi \boldsymbol{u}^{m+1} \quad \text{with } \xi \not\equiv 0 \bmod p.$$

If $\Phi_\lambda^{W_t}$ is an A_{m+1}-map, then

$${}^\exists P_{m+1}(\Phi_\lambda^{W_t})\colon P_{m+1}(W_t) \to P_{m+1}(W_t)$$

with $P_{m+1}(\Phi_\lambda^{W_t})\varepsilon_m \simeq \varepsilon_m(\Sigma\Phi_\lambda^{W_t})$. This implies

$$P_{m+1}(\Phi_\lambda^{W_t})^*(\boldsymbol{u}) = \lambda \boldsymbol{u}.$$

Since

$$\mathscr{P}^1 P_{m+1}(\Phi_\lambda^{W_t})^*(\boldsymbol{u}) = \xi\lambda\boldsymbol{u}^{m+1}$$

and

$$P_{m+1}(\Phi_\lambda^{W_t})^*\mathscr{P}^1(\boldsymbol{u}) = \xi\lambda^{m+1}\boldsymbol{u}^{m+1},$$

we have $\lambda(\lambda^m - 1) \equiv 0 \bmod p$.

Proof of the " if " part of Theorem A (2).

According to [Toda 1962], we have

$$\pi_{2t+2(p-1)-2}(W_t) \cong \mathbb{Z}/p\{\alpha\}.$$

Let $C(\varphi)$ be the cofiber of $\varphi = \Sigma\alpha\colon S_{(p)}^{2t+2(p-1)-1} \to \Sigma W_t$. Then

$$H^*(C(\varphi);\mathbb{F}_p) = \mathbb{F}_p\{\boldsymbol{z},\boldsymbol{w}\} \quad \text{as an } \mathbb{F}_p\text{-algebra}$$
$$\text{with } \deg \boldsymbol{z} = 2t \text{ and } \deg \boldsymbol{w} = 2t + 2(p-1)$$

and
$$\mathscr{P}^1(\boldsymbol{z}) = \zeta \boldsymbol{w} \quad \text{with } \zeta \not\equiv 0 \bmod p.$$

Since $\varphi = \Sigma\alpha$ is a suspension map, we have a map $\Lambda\colon C(\varphi) \to C(\varphi)$ with the following commutative diagram:

$$\begin{array}{ccccc}
S^{2t+2(p-1)-1}_{(p)} & \xrightarrow{\varphi} & S^{2t}_{(p)} & \to & C(\varphi) \\
{\scriptstyle [\lambda]}\downarrow & & \downarrow{\scriptstyle [\lambda]} & & \downarrow{\scriptstyle \Lambda} \\
S^{2t+2(p-1)-1}_{(p)} & \xrightarrow{\varphi} & S^{2t}_{(p)} & \to & C(\varphi),
\end{array}$$

where $[\lambda]$ denote the self-maps of degree λ.

Since $\Phi_\lambda^{W_t}$ is an A_m-map,

$$^\exists P_m(\Phi_\lambda^{W_t})\colon P_m(W_t) \to P_m(W_t)$$

with $P_m(\Phi_\lambda^{W_t})\varepsilon_{m-1} \simeq \varepsilon_{m-1}(\Sigma\Phi_\lambda^{W_t})$.

Let $\widetilde{\varphi} = \varepsilon_{m-1}\varphi\colon S_{(p)}^{2t+2(p-1)-1} \to P_m(W_t)$. Since there is a fibration

$$W_t \longrightarrow S_{(p)}^{2t+2(p-1)-1} \xrightarrow{\gamma_m} P_m(W_t),$$

we have

$$\pi_{2t+2(p-1)-1}(P_m(W_t)) \cong \mathbb{Z}_{(p)}\{\gamma_m\} \oplus \mathbb{Z}/p\{\widetilde{\varphi}\}.$$

Put $X = C(\widehat{\varphi})$, where $\widehat{\varphi} = \iota_m\widetilde{\varphi} = \varepsilon_m\varphi\colon S_{(p)}^{2t+2(p-1)-1} \to P_{m+1}(W_t)$.
Then $C(\varphi) \subset X$ and $\pi_{2t+2(p-1)-1}(X) = 0$.

Since $P_{m+1}(W_t) = C(\gamma_m)$, we have a map $\widetilde{\Psi}\colon P_{m+1}(W_t) \to X$ with the following commutative diagram:

$$\begin{array}{ccccccc}
S^{2t}_{(p)} & = & \Sigma W_t & \xrightarrow{\varepsilon_{m-1}} & P_m(W_t) & \xrightarrow{\iota_m} & P_{m+1}(W_t) \\
{\scriptstyle [\lambda]}\downarrow & & {\scriptstyle \Sigma\Phi^{W_t}_\lambda}\downarrow & & \downarrow{\scriptstyle P_m(\Phi^{W_t}_\lambda)} & & \downarrow{\scriptstyle \widetilde{\Psi}} \\
S^{2t}_{(p)} & = & \Sigma W_t & \xrightarrow[\varepsilon_{m-1}]{} & P_m(W_t) & \xrightarrow[\widetilde{\iota}_m]{} & X,
\end{array}$$

where $\widetilde{\iota}_m$ denotes the composition of ι_m and the inclusion $P_{m+1}(W_t) \subset X$. Define a self-map $\Psi\colon X \to X$ by $\Psi|_{P_{m+1}(W_t)} = \widetilde{\Psi}$ and $\Psi|_{C(\varphi)} = \Lambda$.

$$
\begin{tikzcd}
S^{2t}_{(p)} \arrow[r, equal] \arrow[d, "{[\lambda]}"'] & \Sigma W_t \arrow[r, "\varepsilon_{m-1}"] \arrow[d, "\Sigma\Phi^{W_t}_\lambda"'] & P_m(W_t) \arrow[r, "\iota_m"] \arrow[d, "P_m(\Phi^{W_t}_\lambda)"] & P_{m+1}(W_t) \arrow[r, "\subset"] & X \arrow[d, "\Psi"] & C(\varphi) \arrow[l, "\supset"'] \arrow[d, "\Lambda"] \\
S^{2t}_{(p)} \arrow[r, equal] & \Sigma W_t \arrow[r, "\varepsilon_{m-1}"'] & P_m(W_t) \arrow[r, "\iota_m"'] & P_{m+1}(W_t) \arrow[r, "\subset"'] & X & C(\varphi), \arrow[l, "\supset"]
\end{tikzcd}
$$

From the definition,

$$H^*(X;\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[x]/(x^{m+2}) \oplus \mathbb{Z}_{(p)}\{y\} \quad \text{as a } \mathbb{Z}_{(p)}\text{-algebra}$$

with $\deg x = 2t$ and $\deg y = 2t + 2(p-1)$.

Since $\Psi|_{C(\varphi)} = \Lambda$, the induced homomorphism

$$\Psi^*\colon H^*(X;\mathbb{Z}_{(p)}) \to H^*(X;\mathbb{Z}_{(p)})$$

is given by $\Psi^*(x) = \lambda x$ and $\Psi^*(y) = \lambda y + \eta x^{m+1}$ for some $\eta \in \mathbb{Z}_{(p)}$.

Lemma.

If $\lambda(\lambda^m - 1) \equiv 0 \bmod p$, then $\eta \equiv 0 \bmod p$.

Proof.

$$H^*(P_{m+1}(W_t); \mathbb{F}_p) \longleftarrow H^*(X; \mathbb{F}_p) \longrightarrow H^*(C(\varphi); \mathbb{F}_p)$$

Write $\mathscr{P}^1(\boldsymbol{x}) = \xi \boldsymbol{x}^{m+1} + \zeta \boldsymbol{y}$ with $\xi, \zeta \not\equiv 0 \bmod p$. Since

$$\mathscr{P}^1 \Psi^*(x) = \lambda \xi \boldsymbol{x}^{m+1} + \lambda \zeta \boldsymbol{y}$$

and

$$\Psi^* \mathscr{P}^1(x) = \lambda^{m+1} \xi \boldsymbol{x}^{m+1} + \lambda \zeta \boldsymbol{y} + \eta \zeta \boldsymbol{x}^{m+1},$$

we have $\xi\lambda(\lambda^m - 1) + \eta\zeta \equiv 0 \bmod p$. Then $\eta \equiv 0 \bmod p$.

Let $\boldsymbol{a}, \boldsymbol{b} \in H_{2t+2(p-1)}(X; \mathbb{Z}_{(p)})$ denote the Kronecker duals of $x^{m+1}, y \in H^{2t+2(p-1)}(X; \mathbb{Z}_{(p)})$, respectively. Using the duality, we can show that

$$\Psi_*(\boldsymbol{a}) = \lambda^{m+1}\boldsymbol{a} + \eta\boldsymbol{b}$$

and

$$\Psi_*(\boldsymbol{b}) = \lambda\boldsymbol{b}.$$

Consider the homomorphism

$$\mathscr{E}\colon H_{2t+2(p-1)}(X;\mathbb{Z}_{(p)}) \to \pi_{2t+2(p-1)-1}(P_m(W_t))$$

defined by the following composition:

$$\begin{aligned} H_{2t+2(p-1)}(X;\mathbb{Z}_{(p)}) \;\longrightarrow\; & H_{2t+2(p-1)}(X, P_m(W_t);\mathbb{Z}_{(p)}) \\ & \xrightarrow[\cong]{\mathscr{H}^{-1}} \pi_{2t+2(p-1)}(X, P_m(W_t)) \xrightarrow{\partial} \pi_{2t+2(p-1)-1}(P_m(W_t)), \end{aligned}$$

where $\mathscr{H}$ denotes the Hurewicz isomorphism. Then $P_m(\Phi_\lambda^{W_t})_\#\mathscr{E} = \mathscr{E}\Psi_*$.

Since $\mathscr{E}(\boldsymbol{a}) = \gamma_m$ and $\mathscr{E}(\boldsymbol{b}) = \widetilde{\varphi}$, we have that

$$P_m(\Phi_\lambda^{W_t})_\#(\gamma_m) = \lambda^{m+1}\gamma_m + \eta\widetilde{\varphi} = \lambda^{m+1}\gamma_m \quad \text{by Lemma}.$$

This implies that $\iota_m P_m(\Phi_\lambda^{W_t})\gamma_m$ is null-homotopic, and so there is a self-map $\psi\colon P_{m+1}(W_t) \to P_{m+1}(W_t)$ with $\psi\iota_m \simeq \iota_m P_m(\Phi_\lambda^{W_t})$. Then $\Phi_\lambda^{W_t}$ is an A_{m+1}-map.

Remark.

Theorem B is proved in a similar way to the proof of Theorem A. In the proof, we use the Brown-Peterson cohomology instead of the $\bmod p$ cohomology.