自由群の自己同型群の Andreadakis-Johnson filtrationについて

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Automorphism groups of free groups

- $F_n := \langle x_1, \ldots, x_n \rangle$: Free group of rank $n \geq 2$
- $H := F_n/[F_n, F_n] \cong \mathbb{Z}^{\oplus n}$: Abelianization of F_n

$$\rho : \mathrm{Aut}\, F_n \xrightarrow{\text{surj.}} \mathrm{Aut}(H) = \mathrm{GL}(n, \mathbb{Z})$$

Nielsen, 1924

- $\mathrm{IA}_n := \mathrm{Ker}(\mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbb{Z}))$

Free group analogue of the Torelli group

IA-automorphism groups

$$\mathrm{IA}_n := \mathrm{Ker}(\mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbb{Z}))$$

- (Magnus, 1935) IA_n is finitely generated by

$$K_{ij} : x_i \mapsto {x_j}^{-1} x_i x_j,$$

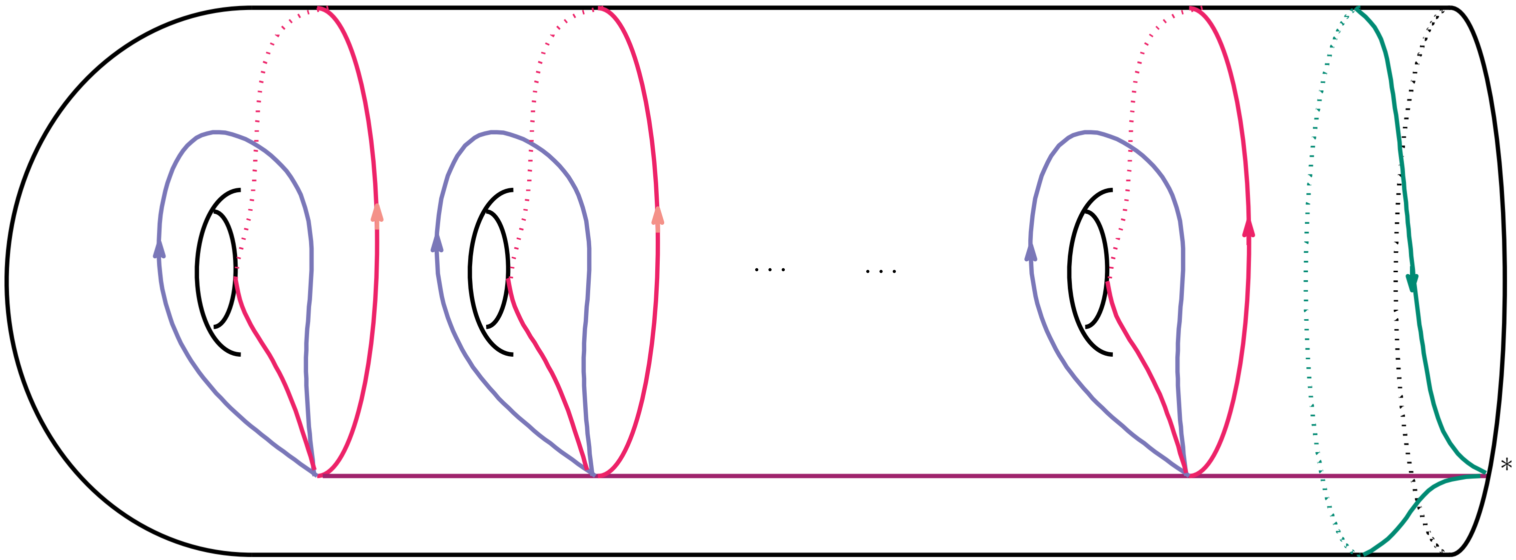
$$K_{ijk} : x_i \mapsto x_i\,[x_j, x_k], \quad j < k$$

Problem For $n \geq 3$,

Find a presentation for IA_n.

Mapping class groups of surfaces

- $\Sigma_{g,1} :=$



$$\pi_1(\Sigma_{g,1}, *) \cong F_{2g}$$

- $\mathcal{M}_{g,1} := \mathrm{Diff}^+(\Sigma_{g,1}, \partial)/\mathrm{isotopy}$

Theorem (Dehn, Nielsen) $g \geq 1$

$\exists \iota : \mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}\, F_{2g}$ s.t.,

$$\mathrm{Im}(\iota) = \{\sigma \in \mathrm{Aut}\, F_{2g} \mid \zeta^{\sigma} = \zeta\}$$

- **Torelli group**

$$H_1(\Sigma_{g,1}, \mathbb{Z}) \curvearrowleft \mathcal{I}_{g,1} := \mathrm{IA}_{2g} \cap \mathcal{M}_{g,1}$$

trivially

$$\begin{array}{ccccccccc}
1 & \to & \mathrm{IA}_{2g} & \to & \mathrm{Aut}\, F_{2g} & \xrightarrow{\rho} & \mathrm{GL}(2g, \mathbb{Z}) & \to & 1 \\
 & & \uparrow & & \iota \uparrow & & \uparrow & & \\
1 & \to & \mathcal{I}_{g,1} & \to & \mathcal{M}_{g,1} & \to & \mathrm{Sp}(2g, \mathbb{Z}) & \to & 1
\end{array}$$

Andreadakis-Johnson filtration of Aut F_n

- $\Gamma_n(k)$: lower central series of F_n

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n]$$

- $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$

Facts. (Magnus, Witt, Hall)

$\mathcal{L}_n := \bigoplus_{k\geq 1} \mathcal{L}_n(k)$ is the free Lie algebra generated by H.

- **Andreadakis-Johnson filtration** $\quad k \geq 1$

$$\mathcal{A}_n(k) := \mathrm{Ker}(\mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)))$$

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

Theorem (Andreadakis, 1965)

(1) $[\mathcal{A}_n(k), \mathcal{A}_n(l)] \subset \mathcal{A}_n(k+l)$

(2) $\displaystyle\bigcap_{k\geq 1} \mathcal{A}_n(k) = 1$

- $\mathrm{gr}^k(\mathcal{A}_n)$ **: sequence of approximations of** IA_n**.**

Johnson homomorphisms

- $H^* := \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$
- The k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to \mathrm{Hom}_{\mathbb{Z}}(H, \mathcal{L}_n(k+1)) = H^* \otimes \mathcal{L}_n(k+1)$$

$$\sigma \mod(\mathcal{A}_n(k+1)) \mapsto (x \mapsto x^{-1}x^{\sigma})$$

Fact.

Each of τ_k is injective and $\mathrm{GL}(n, \mathbb{Z})$-equivariant.

S. Morita, R. Hain, F. Cohen, B. Farb, ...

Theorem. (Cohen-Pakianathan, Farb, Kawazumi)

$$\tau_1 : \mathrm{gr}^1(\mathcal{A}_n) \to H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(2)$$

is surjective, and the abelianization of IA_n.

Theorem. (S., 2004)

There exists an exact sequence

$$0 \to \mathrm{gr}^2(\mathcal{A}_n) \xrightarrow{\tau_2} H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(3) \to S^2 H \to 0$$

of $\mathrm{GL}(n, \mathbb{Z})$-modules.

Problem 2. $k \geq 3$

Determine $\mathrm{Im}(\tau_k)$ and $\mathrm{Coker}(\tau_k)$.

- It is too difficult to obtain a generating set of

$$\mathcal{A}_n(k) \text{ and } \mathrm{gr}^k(\mathcal{A}_n).$$

Problem 3. $k \geq 2$.

Determine whether $\mathcal{A}_n(k)$ is finitely generated or not.

Andreadakis conjecture

- $\mathcal{A}'_n(k)$: Lower central series of IA_n

$$\begin{array}{ccccccc} \mathcal{A}_n(1) & \supset & \mathcal{A}_n(2) & \supset & \mathcal{A}_n(3) & \supset & \cdots \\ \| & & \|\ \text{Bachmuth} & & \cup\ \text{Finite index Pettet} & & \\ \mathcal{A}'_n(1) & \supset & \mathcal{A}'_n(2) & \supset & \mathcal{A}'_n(3) & \supset & \cdots \end{array}$$

Conjecture (Andreadakis) For any $n, k \geq 3$,

$$\mathcal{A}_n(k) = \mathcal{A}'_n(k)$$

Mapping class group case

- **Johnson filtration**

$$\mathcal{M}_{g,1}(k) := \mathcal{A}_{2g}(k) \cap \mathcal{M}_{g,1}$$

- **Lower central series of $\mathcal{I}_{g,1}$**

$$\mathcal{I}_{g,1} = \mathcal{M}_{g,1}(1) \supset \mathcal{M}_{g,1}(2) \supset \cdots$$

Facts $\mathcal{M}_{g,1}(k) \neq \mathcal{M}'_{g,1}(k)$

- $k = 2$: **Johnson**
- $k = 3$: **Morita**

"Upper-triangular" automorphism groups

- **Nielsen automorphisms**

$$E_{ij} : x_i \mapsto x_i x_j,$$

$$E_{i-1j} : x_i \mapsto x_j^{-1} x_i$$

$$\mathbf{A}_n^+ := \langle E_{ij},\, E_{i-1j} \,|\, 1 \leq j < i \leq n \rangle$$

$$\begin{array}{ccc} \rho : \mathrm{Aut}\, F_n & \xrightarrow{\text{surj.}} & \mathrm{Aut}(H) \\ \cup & & \| \\ \mathbf{A}_n^+ & & \Lambda_n \end{array}$$

Heisenberg group

$$\Lambda_n = \left\{ \begin{pmatrix} 1 & O & & O \\ a_{21} & 1 & & \ddots & \\ a_{31} & a_{32} & \ddots & & O \\ \vdots & & \ddots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & 1 \end{pmatrix} \middle| \, a_{ij} \in \mathbb{Z} \right\}$$

Theorem (Magnus) Λ_n is generated by F_{ij} for $(j < i)$ subject to relations:

- $[F_{ij}, F_{jk}] = F_{ik}$ for $i > j > k$,
- $[F_{ij}, F_{kl}] = 1$ for $j \neq k$ and $i \neq l$.

Theorem (Satoh, 2013) A_n^+ is generated by $E_{i\pm1\,j}$ for $(j < i)$ subject to relations:

- $[E_{ij}, E_{jk}] = E_{ik}$ for $i > j > k$,
- $[E_{i-1\,j}, E_{jk}] = E_{i-1\,k}$ for $i > j > k$,
- $[E_{ij}^{-1}, E_{j-1\,k}] = E_{ik}$ for $i > j > k$,
- $[E_{i-1\,j}^{-1}, E_{j-1\,k}] = E_{i-1\,k}$ for $i > j > k$,
- $[E_{i\pm1\,j}, E_{kl}] = 1$ for $i > k$, $i \neq l$ and $j \neq k$,
- $[E_{i\pm1\,j}, E_{k-1\,l}] = 1$ for $i > k$, $i \neq l$ and $j \neq k$,
- $[E_{ij}, E_{i-1\,k}] = 1$ for $i > j, k$.

“Upper-triangular” IA-automorphism groups

$$\mathrm{IA}_n^+ := \mathrm{Ker}(\mathrm{A}_n^+ \xrightarrow{\rho} \Lambda_n)$$

Theorem (Satoh, 2013) IA_n^+ is generated by

$$K_{ij} : x_i \mapsto x_j{}^{-1} x_i x_j, \quad j < i$$

$$K_{ijk} : x_i \mapsto x_i\,[x_j, x_k], \quad k < j < i$$

Moereover, we gave an infinite presentation for IA_n^+.

Andreadakis-Johnson filtration for IA_n^+

- $\mathcal{A}_n(k)^+ := \mathrm{IA}_n^+ \cap \mathcal{A}_n(k)$

$$\mathrm{IA}_n^+ = \mathcal{A}_n(1)^+ \supset \mathcal{A}_n(2)^+ \supset \cdots$$

(1) $[\mathcal{A}_n(k)^+, \mathcal{A}_n(l)^+] \subset \mathcal{A}_n(k+l)^+$

(2) $\displaystyle\bigcap_{k \geq 1} \mathcal{A}_n(k)^+ = 1$

The main theorem

- **Lower central series of** IA_n^+

$$\mathrm{IA}_n^+ = \mathcal{A}'_n(1)^+ \supset \mathcal{A}'_n(2)^+ \supset \cdots$$

- $\mathcal{A}'_n(k)^+ \subset \mathcal{A}_n(k)^+$

Theorem (Satoh, 2013) For any $n \geq 2$ and $k \geq 1$,

$$\mathcal{A}_n(k)^+ = \mathcal{A}'_n(k)^+.$$

Outline of the proof

- $\mathrm{gr}^k(\mathcal{A}_n^+) := \mathcal{A}_n(k)^+/\mathcal{A}_n(k+1)^+$.
- $\mathrm{gr}^k({\mathcal{A}'_n}^+) := \mathcal{A}'_n(k)^+/\mathcal{A}'_n(k+1)^+$

$$
\begin{array}{ccc}
\mathrm{gr}^k(\mathcal{A}_n^+) & \xrightarrow{\tau_k^+} & H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1) \\
\uparrow & & \| \\
\mathrm{gr}^k({\mathcal{A}'_n}^+) & \xrightarrow{{\tau'_k}^+} & H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1)
\end{array}
$$

- $V_n(k)$: the $\mathbb{Z}$-submodule of $H^* \otimes \mathcal{L}_n(k+1)$ generated by

$$x_i^* \otimes [x_{j_1}, x_{j_2}, \ldots, x_{j_k}, x_i], \quad j_1, \ldots, j_k < i$$

$$x_i^* \otimes [x_{j_1}, x_{j_2}, \ldots, x_{j_{k+1}}], \quad j_1, \ldots, j_{k+1} < i$$

where

$$[y_1, y_2, \ldots, y_k] = [[\cdots[y_1, y_2], y_3], \cdots], y_k].$$

Proposition For any $k \geq 1$,

$$\mathrm{Im}({\tau'_k}^+) = V_n(k).$$

Proposition For any $k \geq 1$,

$\mathrm{gr}^k({\mathcal{A}'_n}^+)$ **is generated by** $\mathrm{rank}_{\mathbb{Z}}(\mathrm{Im}({\tau'_k}^+))$ **elements.**

Hence,

$${\tau'_k}^+ : \mathrm{gr}^k({\mathcal{A}'_n}^+) \to V_n(k)$$

is an isomorphism.

- $\mathcal{A}'_n(1)^+ = \mathcal{A}_n(1)^+$

$$\begin{array}{ccc} \mathrm{gr}^1(\mathcal{A}_n^+) & \xrightarrow[\text{inj.}]{\tau_1^+} & V_n(1) \\ \uparrow & & \| \\ \mathrm{gr}^1(\mathcal{A}'^{\,+}_n) & \xrightarrow[\text{inj.}]{\tau'^{+}_1} & V_n(1) \end{array}$$

$\Longrightarrow$

$$\mathcal{A}'_n(2)^+ = \mathcal{A}_n(2)^+$$

- By using the induction on $k \geq 1$, we obtain the main theorem.

Some remarks

Theorem (Satoh, 2013)

- $H_1(\mathrm{IA}_n^+, \mathbb{Z}) \cong V_n(1)$, induced from

$$\tau_1^+ : \mathrm{gr}^1(\mathcal{A}_n^+) \to V_n(1).$$

- $\cup : \Lambda^2 H^1(\mathrm{IA}_n^+, \mathbb{Z}) \to H^2(\mathrm{IA}_n^+, \mathbb{Z})$

$$\mathrm{Im}(\cup) \cong \mathbb{Z}^{\oplus \frac{1}{72} n(n^2-1)(n-2)^2(n^2+5n+9)}$$

- $H^2(\mathrm{IA}_n^+, \mathbb{Z}) \neq \mathrm{Im}(\cup)$