Mapping spaces from projective spaces

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1. Introduction
   ▶ $A_n$-map
   ▶ Main theorem
**A$_n$-map**

$G, G', G''$: topological monoids

**A$_n$-map ($n = 1, 2, \ldots, \infty$)**

A family of maps \( \{f_i : [0, 1]^{i-1} \times G^i \to G'\}_{i=1}^n \) is called an A$_n$-form of $f$ if

1. $f_1 = f$,

2. $f_i(t_1, \ldots, t_{i-1}; g_1, \ldots, g_i) = \begin{cases} f_{i-1}(t_1, \ldots, \hat{t}_k, \ldots, t_{i-1}; g_1, \ldots, g_k g_{k+1}, \ldots, g_i) & \text{for } t_k = 0 \\ f_k(t_1, \ldots, t_{k-1}; g_1, \ldots, g_k) f_{i-k}(t_{k+1}, \ldots, t_{i-1}; g_{k+1}, \ldots, g_i) & \text{for } t_k = 1 \end{cases}$

3. $f_i(t_1, \ldots, t_{i-1}; g_1, \ldots, *, \ldots, g_i) = f_{i-1}(t_1, \ldots, \max\{t_{k-1}, t_k\}, \ldots, t_{i-1}; g_1, \ldots, \hat{g}_k, \ldots, g_i)$.

A triple $(f, \{f_i\}, \ell)$ is called an A$_n$-map. We denote the space of A$_n$-maps from $G$ to $G'$ by $A_n(G, G')$.

There is a composition of A$_n$-maps

$$A_n(G', G'') \times A_n(G, G') \to A_n(G, G'').$$
1. Introduction

2. $A_n$-maps

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4. Related topics
$G$: topological monoid, $G'$: grouplike topological monoid. Both of them are CW complex.

Main Theorem (recognition theorem for $A_n$-maps) (T)

The composition

$$
\mathcal{A}_n(G, G') \xrightarrow{B_n} \text{Map}_0(B_nG, B_nG') \xrightarrow{(i_n)^\#} \text{Map}_0(B_nG, BG')
$$

is a natural weak equivalence, where $i_n : B_nG' \to BG'$ is the natural inclusion.
2. Recognitions of $A_n$-map

- Projective spaces
- Recognitions of $A_n$-map
**Projective spaces**

$G$: topological monoid

The $n$-th projective space $B_nG$ is defined by

$$B_nG := \left( \bigsqcup_{0 \leq i \leq n} \Delta^i \times G^i \right) / \sim,$$

where $\sim$ is the usual simplicial identification.

$$* = B_0G \subset \Sigma G = B_1G \subset B_2G \subset \cdots \subset B_\infty G = BG.$$

An $A_n$-map induces a based map between the $n$-th projective spaces:

$$B_n: \mathcal{A}_n(G, G') \to \text{Map}_0(B_nG, B_nG').$$
Recognitions of $A_n$-map

$G$: topological monoid, $G'$: grouplike topological monoid.
Both of them are CW complex.

**Theorem (Stasheff, 1963)**

A based map $f: G \to G'$ admits an $A_n$-form if and only if the composite

$$\Sigma G \xrightarrow{\Sigma f} \Sigma G' \xrightarrow{i_1} BG'$$

extends to a map $B_nG \to BG'$.

**Theorem (Fuchs, 1965)**

There is a one-to-one correspondence between $\pi_0(\mathcal{A}_\infty(G, G'))$ and the homotopy set of based maps $[BG, BG']$.

**Theorem**

The model categories of simplicial groups and reduced simplicial sets are Quillen equivalent by the Kan’s loop group construction.
3. Main theorem

- Main theorem
- Adjointness of $B_n$ and $\Omega$
Main theorem

$G$: topological monoid, $G'$: grouplike topological monoid. Both of them are CW complex.

**Main Theorem (recognition theorem for $A_n$-maps) (T)**

The composition

$$A_n(G, G') \xrightarrow{B_n} \text{Map}_0(B_nG, B_nG') \xrightarrow{(i_n)^\#} \text{Map}_0(B_nG, BG')$$

is a natural weak equivalence, where $i_n : B_nG' \to BG'$ is the natural inclusion.
Proof of Theorem

When \( n = 1 \), this is the well-known adjunction of \( \Sigma \) and \( \Omega \). Suppose this is true for \( A_{n-1} \)-maps. Consider the following commutative diagram of homotopy fiber sequences:

\[
\begin{array}{ccc}
F & \longrightarrow & \mathcal{A}_n(G, G') \\
\downarrow & & \downarrow \\
F' & \longrightarrow & \text{Map}_0(B_nG, BG')
\end{array}
\]

\[
\begin{array}{ccc}
& & \\
& & \\
\longrightarrow & \longrightarrow & \longrightarrow \\
& & \\
\mathcal{A}_{n-1}(G, G') & \longrightarrow & \text{Map}_0(B_{n-1}G, BG')
\end{array}
\]

\( \cong \)

In fact, the map \( F \rightarrow F' \) coincides with the composite

\( F \cong \text{Map}_0(S^{n-1} \wedge G^n, G') \cong \text{Map}_0(S^n \wedge G^n, BG') \cong F' \). Then by the five lemma, we have the desired conclusion.
Adjointness of $B_n$ and $\Omega$

$G$: topological monoid which is a CW complex, $X$: a based space.

Corollary (adjointness of $B_n$ and $\Omega$) (T)

There is a natural weak equivalence

$$\mathcal{A}_n(G, \Omega X) \tilde{\rightarrow} \text{Map}_0(B_nG, X).$$

$G'$: grouplike topological monoid which is a CW complex.

Corollary

The following map is a weak equivalence.

$$\mathcal{A}_\infty(G, G') \tilde{\rightarrow} \text{Map}_0(BG, BG').$$
4. Related topics

- Evaluation fiber sequence
- Higher homotopy commutativity
- Gottlieb-type filtration on $\pi_*(X)$
1. Introduction

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### Evaluation fiber sequence

Let $X, Y$ be based CW complexes. The homotopy fiber sequence

$$
\cdots \to \Omega Y \to \text{Map}_0(X, Y) \to \text{Map}(X, Y) \to Y
$$

is called the **evaluation fiber sequence**. In general, this fiber sequence does not extend to the right.

If $Y = X$, there is a homotopy fiber sequence

$$
\cdots \to \Omega X \to \text{Map}_0(X, X)_{\text{id}} \to \text{Map}(X, X)_{\text{id}} \to X
$$

$$
\quad \to B \text{Map}_0(X, X)_{\text{id}} \to B \text{Map}(X, X)_{\text{id}}
$$

where the subspaces $\text{Map}_0(X, Y)_f \subset \text{Map}_0(X, Y)$ and $\text{Map}(X, Y)_f \subset \text{Map}(X, Y)$ consist of maps **freely** homotopic to a based map $f : X \to Y$. 

$G$: topological group which is a CW complex. The conjugation on $G$ defines an action on $BG$ and hence on $\text{Map}_0(X, BG)$. On the other hand, the conjugation defines a “homomorphism”

$$\alpha : G \to \mathcal{A}_n(G, G).$$

**Theorem (T)**

There is a homotopy fiber sequence

$$G \to \text{Map}_0(B_nG, BG)_{i_n} \to \text{Map}(B_nG, BG)_{i_n} \to BG \xrightarrow{B\alpha} B\mathcal{A}_n(G, G)_{\alpha}$$

where $\mathcal{A}_n(G, G)_{\alpha}$ is the union of path-components containing the image of $\alpha$.

This theorem follows from the fact that the weak equivalence

$$\mathcal{A}_n(G, G) \xrightarrow{\sim} \text{Map}_0(B_nG, BG)$$

is $G$-equivariant.
**Higher homotopy commutativity**

$G$: topological monoid, $N_{r,s}$: resultohedron \ ((r, s \geq 0)

**Definition (Kishimoto–Kono, 2010)**

If there is a family of maps $\{Q_{r,s}: N_{r,s} \times G^{r+s} \to G\}_{0 \leq r \leq k, 0 \leq s \leq \ell}$ satisfying appropriate compatibility, $G$ is said to be a $C(k, \ell)$-space.

$C(1, 1)$-space $\iff$ homotopy commutative
$G$: topological group which is a CW complex

**Theorem (Kishimoto–Kono, 2010)**

The following are equivalent:

1. $G$ is a $C(k, \ell)$-space,
2. $(i_k, i_\ell): B_k G \vee B_\ell G \to BG$ extends over $B_k G \times B_\ell G$,
3. $i_k^* \text{Map}(S^1, BG)$ is trivial as a fiberwise $A_\ell$-space.

**Theorem (T)**

$G$ is a $C(k, \ell)$-space if and only if the map $\alpha: G \to \mathcal{A}_\ell(G, G)$ is homotopic to the trivial map as an $A_k$-map.
Gottlieb-type filtration

X: based connected CW complex

Definition

Define a subgroup \( G_n^{(k)}(X) \subset \pi_n(X) \) by

\[
G_n^{(k)}(X) = \text{im}(ev_* : \pi_n(\text{Map}(B_k \Omega X, X)_{i_k}) \to \pi_n(X)).
\]

The group \( G_n(X) := G_n^{(\infty)}(X) \) is called the \( n \)-th Gottlieb group.

\[
G_n(X) = G_n^{(\infty)}(X) \subset \cdots \subset G_n^{(2)}(X) \subset G_n^{(1)}(X) \subset Z(\pi_n(X)) \subset \pi_n(X)
\]

For \( \alpha \in \pi_n(X) \), \( \alpha \in G_n^{(k)}(X) \) if and only if

\[
S^n \vee B_k \Omega X \xrightarrow{(\alpha, i_k)} X
\]

\[
S^n \times B_k \Omega X
\]
Example

If $X$ is an $H$-space, then $G_n^{(\infty)}(X) = \pi_n(X)$ for any $n \geq 1$. More generally, if $\Omega X$ is a $C(1, k)$-space, then $G_n^{(k)}(X) = \pi_n(X)$ for any $n \geq 1$.

Example (T)

For an odd prime $p$ and $\frac{r(p-1)}{2} \leq k < \frac{(r+1)(p-1)}{2}$, the subgroup

$$G_4^{(k)}(B SU(2))_{(p)} \subset \pi_4(B SU(2))_{(p)} \cong \mathbb{Z}_p$$

has index $p^r$ and $G_4^{(\infty)}(B SU(2)) = 0$.

Example (Kishimoto–T)

$G$: compact connected simple Lie group
Suppose that $H^*(BG; \mathbb{Q})$ is a polynomial algebra on the generators of degree $2n_1, \ldots, 2n_\ell$. If $p > 2n_\ell$, then the subgroup

$$0 \neq G_{2n_i}^{(n_\ell+p-1)}(BG)_{(p)} \subset \pi_{2n_i}(BG)_{(p)} \cong \mathbb{Z}_p$$

has index $\geq p$. 