Integrable Systems on Grassmannians and Potential Functions

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Homological Mirror Symmetry for Fano varieties

Mirror of a Fano manifold

$$X \ : \ \text{Fano} \qquad \longleftrightarrow \qquad (X^\vee, W) \ : \ \text{Landau-Ginzburg model}$$

$$\begin{pmatrix} c_1(X) > 0 \\ \Leftrightarrow \text{Ric} > 0 \end{pmatrix} \qquad \begin{pmatrix} X^\vee\text{: non-cpt cpx mfd} \\ W : X^\vee \to \mathbb{C} \text{ holomorphic} \end{pmatrix}$$

Homological Mirror Symmetry

$$D^b\mathcal{F}(X) \cong D^d Sing(W) \cong H^0 MF(W),$$
$$D^b Coh(X) \cong D^b\mathcal{F}(W).$$

"Classical" Mirror Symmetry

$$QH(X;K) \cong \text{Jac}(W) = \frac{R[y_1, \dots, y_n]}{\left(y_i \frac{\partial W}{\partial y_i} \;\middle|\; i = 1, \dots, n\right)}$$

Strominger-Yau-Zaslow conjecture

"Conjecture" (T-duality). X and $X^\vee$ admit dual (special) Lagrangian torus fibration

$$\begin{tikzcd} X \arrow[dr, "\pi"'] & & X^\vee \arrow[dl, "\pi^\vee"] \\ & B & \end{tikzcd} \qquad (\pi^\vee)^{-1}(b) = (\pi^{-1}(b))^\vee \cong T^n.$$

Any Lagrangian fibration is (locally) given by a completely integrable system $(\varphi_1, \dots, \varphi_n) : X \to \mathbb{R}^n$:

$d\varphi_1, \dots, d\varphi_n$ linearly independent (on an open dense subset),

Poisson commutativity: $\{\varphi_i, \varphi_j\} = 0, \quad i, j = 1, \dots, n$

Remark (Auroux). In the Fano case, "special" Lagrangian fibration is defined on the complement of an anti-canonical divisor of X.

Toric case

Let X be a projective toric manifold:
$(\mathbb{C}^*)^n \subset X$ open dense, and $(\mathbb{C}^*)^n \curvearrowright (\mathbb{C}^*)^n$ extends to $(\mathbb{C}^*)^n \curvearrowright X$.

$(S^1)^n$ acts on X in Hamiltonian fashion, and the moment map

$$\Phi : X \longrightarrow \Delta \subset \mathbb{R}^n = \left(\mathrm{Lie}(S^1)^n\right)^*$$

gives a Lagrangian torus fibration on $\mathrm{Int}\,\Delta \subset \Delta$:

- Δ is a convex polytope (the moment polytope),
- Each fiber $L(u) = \Phi^{-1}(u)$ is a torus orbit,
- If u is in a relative interior of a k-dimensional face, then $L(u)$ is an isotropic submanifold of dim $= k$.
  In particular, $L(u)$ are Lagrangian tori for $u \in \mathrm{Int}\,\Delta$.

Lagrangian Intersection Floer theory for torus fibers (Cho-Oh, Fulaya-Oh-Ohta-Ono,...):

- Potential function (superpotential),
- Floer homology,
- $QH(X) \cong \mathrm{Jac}(W)$.

$\vdots$

This talk: A-model on the Grassmannians $\mathrm{Gr}(2, n) = \mathrm{Gr}(2, \mathbb{C}^n)$ of 2-planes in $\mathbb{C}^n$.

Plan

- Lagrangian Intersection Floer Theory
- Lagrangian Intersection Floer Theory for Toric Manifolds
- Integrable Systems on Grassmannians
- Potential Functions for Grassmannians

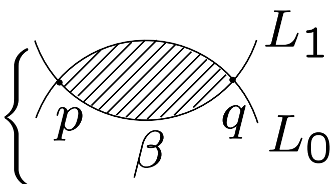
Lagrangian Intersection Floer Theory

Let (X, ω) be a compact symplectic manifold of $\dim = 2n$,
$L_0, L_1 \subset X$ Lagrangian submanifolds (i.e. $\dim L_i = n$ and $\omega|_{L_i} = 0$).

$$CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} Rp, \quad R = \mathbb{Z}, \mathbb{C}, \Lambda_0 = \text{Novikov ring}, \ldots$$

$\mathfrak{m}_1 = \partial : CF(L_0, L_1) \to CF(L_0, L_1)$ (and higher products $\mathfrak{m}_k$) is given by counting holomorphic disks with Lagrangian boundary condition:

$$\mathfrak{m}_1(p) = \sum_{q \in L_0 \cap L_1} \sum_{\beta \in \pi_2(X, L)} \# \Big\{$$



$$\Big\} \exp\Big(- \int_\beta \omega \Big) \cdot q$$

Under suitable assumptions, we have ${\mathfrak{m}_1}^2 = 0$, and Floer homology is defined by

$$HF(L_0, L_1) := H(CF(L_0, L_1), \mathfrak{m}_1).$$

Hamiltonian invariance

Theorem. *Under suitable conditions,*

$$HF(\varphi_0(L_0), \varphi_1(L_1)) \cong HF(L_0, L_1)$$

for Hamiltonian diffeomorphism φ_0, φ_1.

Define

$$HF(L, L) := HF(L, \varphi(L))$$

by a Hamiltonian diffeomorphism φ on X such that L and $\varphi(L)$ are transverse.

Corollary. *If* L *is displaceable, i.e.* $L \cap \varphi(L) = \emptyset$ *for some Hamiltonian diffeomorphism,*

$$HF(L, L) \neq 0.$$

Deformation of A_∞-structure $\{\mathfrak{m}_k\}_k$

For $b \in H^1(L; R/2\pi\sqrt{-1}\mathbb{Z})$, one can define "twisted" products $\{\mathfrak{m}_k^b\}_k$:

$$\mathfrak{m}_1^b(\bullet) = \sum_k \mathfrak{m}_k(b, \ldots, b, \bullet, b, \ldots, b)$$

Remark.

$H^1(L; \sqrt{-1}\mathbb{R}/2\pi\sqrt{-1}\mathbb{Z}) \cong \{$flat $U(1)$-connections over $L\}/$gauge.

If $(\mathfrak{m}_1^b)^2 = 0$, define $HF((L,b),(L,b)) := H(CF(L,L), \mathfrak{m}_1^b)$.

Definition. (L,b) is *balanced* if $HF((L,b),(L,b)) \neq 0$.

Remark. displaceable $\Rightarrow$ not balanced.

Potential Functions

$(\mathfrak{m}_1^b)^2 \neq 0$ in general (because of disk bubbles):

$$(\mathfrak{m}_1^{b_0,b_1})^2 = \mathfrak{m}_2^{b_0,b_1}(\mathfrak{m}_0^{b_0}(L_0), \cdot) - \mathfrak{m}_2^{b_0,b_1}(\cdot, \mathfrak{m}_0^{b_1}(L_1)).$$

$\mathfrak{m}_0^b(L)$ is "defined" by counting holo. disks $u : (D^2, \partial D^2) \to (X, L)$:

$$\mathfrak{m}_0^b(L) = \sum_{\beta \in \pi_2(X,L)} \#\{u; [u] = \beta\} \exp\Big(-\int_\beta \omega\Big) \operatorname{hol}_b(\partial\beta),$$

where $\operatorname{hol}_b(\partial\beta)$ is the holonomy of b along $\partial\beta$.

$HF((L_0, b_0), (L_1, b_1))$ is defined if $\mathfrak{m}_0^{b_0}(L_0) = \mathfrak{m}_0^{b_1}(L_1)$.

Definition (Potential function).

$$\mathfrak{PO} : \bigcup_L H^1(L; R/2\pi\sqrt{-1}\mathbb{Z}) \longrightarrow R, \quad (L, b) \longmapsto \mathfrak{m}_0^b(L).$$

Lagrangian Floer Theory for Toric Fano Manifolds

Theorem (Cho-Oh, Fukaya-Oh-Ohta-Ono). *Let X be a toric Fano manifold with moment polytope*

$$\Delta = \{u \in \mathbb{R}^n \mid \ell_i(u) = \langle v_i, u\rangle - \tau_i \geq 0, i = 1, \ldots, m\}.$$

Then the potential function for $L(u) = \Phi^{-1}(u)$, $u \in \operatorname{Int}\Delta$ is

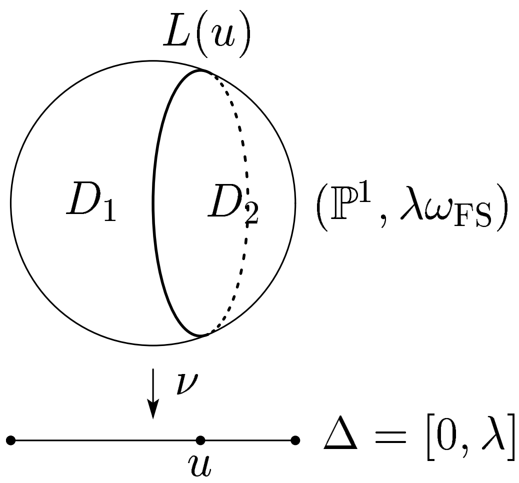
$$\mathfrak{PO}(u,b) = \mathfrak{PO}(L(u),b) = \sum_{i=1}^{m} e^{\langle v_i, b\rangle} e^{-\ell_i(u)},$$

and $\mathfrak{PO}$ gives the superpotential of the LG mirror of X.

Example. $X = \mathbb{P}^1$ with $b = 0$,

$$\mathfrak{PO}(L(u)) = e^{-\mathrm{Area}(D_1)} + e^{-\mathrm{Area}(D_2)}$$
$$= e^{-u} + e^{-(\lambda - u)} = y + \frac{Q}{y},$$

where $y = u^{-u}$, $Q = e^{-\lambda}$.



Lagrangian Floer Theory for Toric Fano Manifolds

Theorem (Fukaya-Oh-Ohta-Ono).

$$QH(X) \cong \mathrm{Jac}(\mathfrak{PO}) = \frac{R[y_1, \ldots, y_n]}{\left(y_i \frac{\partial \mathfrak{PO}}{\partial y_i} \;\middle|\; i = 1, \ldots, n \right)}$$

In particular, $\#\,\mathrm{Crit}(\mathfrak{PO}) = \mathrm{rank}\, H^*(X)$.

Theorem (Fukaya-Oh-Ohta-Ono). *For* $L(u) = \Phi^{-1}(u)$ *(*$u \in \mathrm{Int}\,\Delta$*) and* $b \in H^1(L(u))$*, the followings are equivalent:*

- $(u, b) \in \mathrm{Crit}(\mathfrak{PO})$,
- $(L(u), b)$ *is balanced, i.e.* $HF((L(u), b), (L(u), b)) \neq 0$,
- $HF((L(u), b), (L(u), b)) \cong H^*(L(u))$.

Remark.

$$\#\,\mathrm{Crit}(\mathfrak{PO}) = \#\text{ of balanced Lagrangian fibers.}$$

Grassmannian $\mathrm{Gr}(2,n) = \mathrm{Gr}(2,\mathbb{C}^n)$

$\mathrm{Gr}(2,n)$ **as a symplectic reduction**

$$\mathrm{Gr}(2,n) = \mathbb{C}^{n\times 2}/\!\!/U(2) = \mu_{U(2)}^{-1}(\sqrt{-1}I)/U(2),$$

where $\mu_{U(2)} : \mathbb{C}^{n\times 2} \to \mathfrak{u}(2)$ is the moment map of the $U(2)$-action:

$$(z,w) = \begin{pmatrix} z_1 & w_1 \\ \vdots & \vdots \\ z_n & w_n \end{pmatrix} \longmapsto \sqrt{-1}\sum_{i=1}^{n} \begin{pmatrix} |z_i|^2 & \overline{z_i}w_i \\ z_i\overline{w_i} & |w_i|^2 \end{pmatrix}.$$

Note: $(z,w) \in \mu_{U(2)}^{-1}(\sqrt{-1}I) \iff \|z\| = \|w\| = 1$ and $\langle z,w\rangle = 0$.

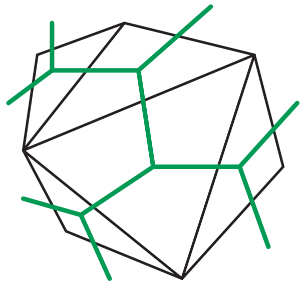
Plücker embedding: $\mathrm{Gr}(2,n) \longrightarrow \mathbb{P}(\Lambda^2\mathbb{C}^n)$,

Example. $\mathrm{Gr}(2,4) \subset \mathbb{P}(\Lambda^2\mathbb{C}^4) = \mathbb{P}^5$ is given by

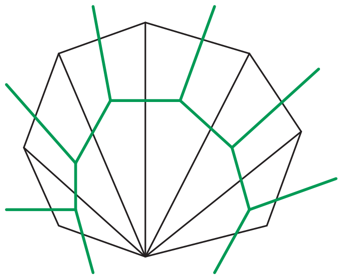
$$\mathrm{Gr}(2,5) = \{[Z_{ij}]_{1\le i<j\le 4} \in \mathbb{P}^5 \mid Z_{12}Z_{34} - Z_{13}Z_{24} + Z_{14}Z_{23} = 0\}.$$

Completely integrable systems on $\mathrm{Gr}(2, n)$

Theorem. *For each triangulation* Γ *of a convex* n*-gon, one can define a completely integrable system* $\Phi_\Gamma : \mathrm{Gr}(2, n) \to \mathbb{R}^{2(n-2)}$.



Remark (Hausmann-Knutson)**.** In the case where the triangulation is the “caterpillar”, Φ_Γ is the Gelfand-Cetlin system (Guillemin-Sternberg).



Construction of Φ_Γ

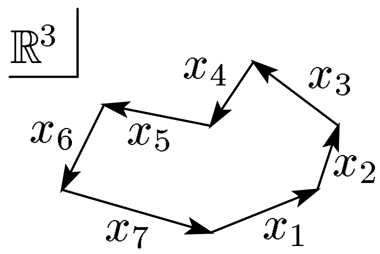
Consider the map $\nu : \mathbb{C}^{n\times 2} \to \mathfrak{su}(2)^n \cong (\mathbb{R}^3)^n$ given by

$$(z,w) = \begin{pmatrix} z_1 & w_1 \\ \vdots & \vdots \\ z_n & w_n \end{pmatrix} \longmapsto \left(x_i = \frac{\sqrt{-1}}{2} \begin{pmatrix} |z_i|^2 - |w_i|^2 & 2\overline{z_i} w_i \\ 2 z_i \overline{w_i} & |w_i|^2 - |z_i|^2 \end{pmatrix} \right)_{i=1,\dots,n}.$$

Then

$$(z,w) \in \mu_{U(2)}^{-1}(\sqrt{-1}I) \Longrightarrow \sum_{i=1}^{n} x_i = 0.$$

i.e. $(x_1, \dots, x_n)$ defines a spatial n-gon in $\mathfrak{su}(2) \cong \mathbb{R}^3$.

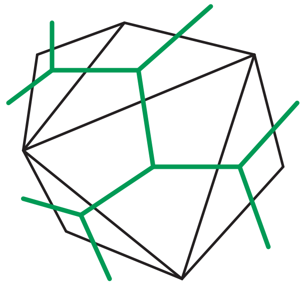


Construction of Φ_Γ

ν induces

$$\nu : \mathrm{Gr}(2,n) \longrightarrow \text{moduli of } n\text{-gons with fixed perimeter 2.}$$

A triangulation Γ of n-gon gives a set of $n-3$ diagonals.



Then $\Phi_\Gamma : \mathrm{Gr}(2,n) \to \mathbb{R}^{2(n-2)} = \mathbb{R}^{n-1} \times \mathbb{R}^{n-3}$ is given by

$$\Phi_\Gamma([z,w]) = \text{lengths of } n-1 \text{ edges and } n-3 \text{ diagonals of } \nu([z,w])$$

The image $\Delta_\Gamma := \Phi_\Gamma(\mathrm{Gr}(2,n))$ is a convex polytope defined by triangle inequalities.

$\mathrm{Gr}(2, n)$ and the polygon space

Remark. ν induces

$$\nu : \mathrm{Gr}(2, n)/T \cong \text{moduli of } n\text{-gons with fixed perimeter 2},$$

where $T \subset U(n)$ is a maximal torus. The symplectic reduction

$$\mathrm{Gr}(2, n)//T \cong \text{moduli of } n\text{-gons in } \mathbb{R}^3 \text{ with fixed side lengths}$$

is called the polygon space.

$$\begin{aligned} \mathrm{Gr}(2, n)//T &\cong \text{moduli space of weighted } n\text{-points in } \mathbb{P}^1 \\ &\cong \text{moduli of parabolic } SU(2)\text{-bundles on } \mathbb{P}^1 \\ &\qquad \text{(if parabolic weights are "small")} \end{aligned}$$

Φ_Γ is a lift of bending Hamiltonians on the polygon space (Kapovich-Millson, Klyachko).

Toric Degeneration of $\mathrm{Gr}(2, n)$

A toric degeneration of $\mathrm{Gr}(2, n)$ is given by deforming the Plücker relations into binomials.

Example. Toric degeneration of $\mathrm{Gr}(2, 4)$ is given by

$$tZ_{12}Z_{34} - Z_{13}Z_{24} + Z_{14}Z_{23} = 0.$$

Theorem (Speyer-Sturmfels).

$$\{\text{toric degenerations of } \mathrm{Gr}(2, n)\ \} \overset{\text{1-1}}{\longleftrightarrow} \{\text{triangulations of } n\text{-gon}\}.$$

Let $f : \mathfrak{X}^{\Gamma} \to S (= \mathbb{C} \text{ or } \mathbb{C}^{n-3})$ denote the toric degeneration corresponding to Γ, and $X_0^{\Gamma} = f^{-1}(0)$ its central fiber (toric variety).

Toric Degeneration of Integrable Systems

Proposition. $\Delta_\Gamma = \Phi_\Gamma(\mathrm{Gr}(2,n))$ *is the moment polytope of* X_0^Γ.

Theorem. $\Phi_\Gamma : \mathrm{Gr}(2,n) \to \Delta_\Gamma$ *can be deformed into a toric moment map* $X_0^\Gamma \to \Delta_\Gamma$.

Theorem. X_0^Γ *is a (singular) toric Fano variety and admits a small resolution* $\pi : \widetilde{X_0^\Gamma} \to X_0^\Gamma$, *i.e.*

$$\mathrm{codim}_{\mathbb{C}}\, \pi^{-1}(\textit{singular locus}) \geq 2$$

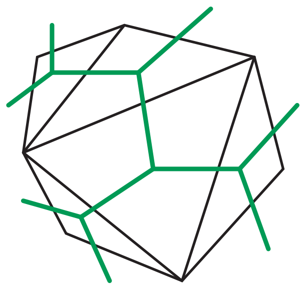
This enables us to compare holomorphic disks in $\mathrm{Gr}(2,n)$ and X_0^Γ.

Potential Function

Theorem. *Fix a triangulation* Γ *of the reference polygon. Then the potential function of the Lagrangian torus fibers* $L(u)$ *is given by*

$$\mathfrak{PO}_\Gamma = \sum_{\text{triangles}} \left(\frac{y(b)y(c)}{y(a)} + \frac{y(a)y(c)}{y(b)} + \frac{y(a)y(b)}{y(c)} \right),$$

where $y(a)$ *is a Laurent monomial in* $Q = e^{-\omega}$ *and* $y_i = e^{b_i}$ *(*b_i *are basis of* $H^1(L(u))$*) associated to an edge or a diagonal* a*, and the sum is taken over all triangles in the triangulation* Γ*.*



Theorem. *For any pair* (Γ, Γ') *of triangulations of the reference polygon, there is a piecewise-linear automorphism*

$$T_{\Gamma,\Gamma'} : \mathbb{R}^{2n-4} \to \mathbb{R}^{2n-4}$$

of the affine space such that $T_{\Gamma,\Gamma'}(\Delta_{\Gamma'}) = \Delta_{\Gamma}$*. The map* $T_{\Gamma,\Gamma'}$ *is defined over* $\mathbb{Z}$ *if* Δ_Γ *is an integral polytope.*

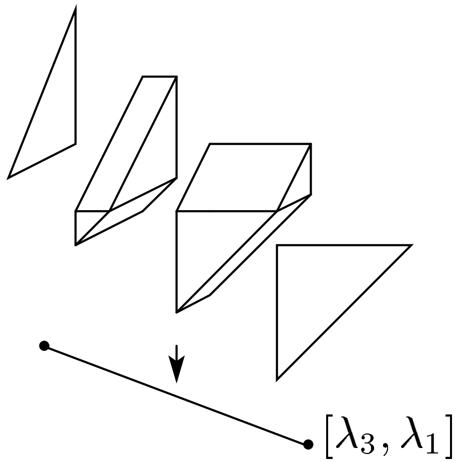
Theorem. *For any pair* (Γ, Γ') *of triangulations of the reference polygon, the potential functions* $\mathfrak{PO}_\Gamma$ *and* $\mathfrak{PO}_{\Gamma'}$ *are related by a subtraction-free rational change of variables whose "tropicalization" is the piecewise-linear transformation* $T_{\Gamma,\Gamma'}$*.*

$$y' = y \cdot \frac{y_1 y_4 + y_2 y_3}{y_1 y_2 + y_3 y_4}.$$

Example: the case of Gr(2, 4)

(After a linear coordinate change), Δ_Γ is given by

$$u_1 \le u_2 \le \lambda_1,$$
$$\lambda_3 \le u_3 \le u_1,$$
$$u_3 \le u_4 \le u_2.$$



The above figure is the projection $\Delta_\lambda \to [\lambda_3, \lambda_1]$, $(u_i)_i \mapsto u_1$.

Δ_Γ has an edge on which

- Δ_Γ is not Delzant,
- X_0^Γ is singular, and
- Fibers of Φ_Γ : Gr(2, 4) $\to \mathbb{R}^4$ are Lagrangian $U(2) \cong S^3 \times S^1$.

Potential function for Gr(2, 4)

The potential function is given by

$$\begin{aligned}\mathfrak{PO} &= e^{-x_2}T^{-u_2+\lambda_1} + e^{-x_1+x_2}T^{-u_1+u_2} + e^{x_1-x_3}T^{u_1-u_3} \\ &\quad + e^{x_3}T^{u_3-\lambda_3} + e^{x_2-x_4}T^{u_2-u_4} + e^{-x_3+x_4}T^{-u_3+u_4} \\ &= \frac{Q_1}{y_2} + \frac{y_2}{y_1} + \frac{y_1}{y_3} + \frac{y_3}{Q_3} + \frac{y_2}{y_4} + \frac{y_4}{y_3}.\end{aligned}$$

This coincides with the superpotential $W : (\mathbb{C}^*)^4 \to \mathbb{C}$ obtained by Eguchi-Hori-Xiong.

$\mathfrak{PO}$ has 4 critical points

$$y_1 = y_4 = \pm\sqrt{Q_1Q_3},\ y_3 = \pm\sqrt{2Q_3y_1},\ y_2 = Q_1Q_3/y_3.$$

Hence there exist 4 balanced $(L(u), b)$.
Note: $4 < \dim H^*(\mathrm{Gr}(2,4)) = 6$.

Eguchi-Hori-Xiong constructed a partial compactification of $(\mathbb{C}^*)^4$.

Floer homologies for Lagrangian Fibers on $\partial\Delta_\Gamma$

Identify the edge with $[-1,1]$ in Δ_Γ.

$$L_t = \Phi_\Gamma^{-1}(t) \cong U(2)$$

is a Lagrangian fiber for $t \in (-1,1)$.

Proposition. *There exists $g \in U(4)$ such that $g(L_t) = L_{-t}$. In particular, if $t \neq 0$, then L_t is displaceable from itself by a Hamiltonian diffeomorphism, and hence $HF(L_t, L_t) = 0$.*

Proposition. *L_0 is a fixed point set of an anti-holomorphic (and anti-symplectic) involution on* $\mathrm{Gr}(2,4)$.

Corollary (Iriyeh-Sakai-Tasaki)**.**

$$HF(L_0, L_0; \mathbb{Z}/2\mathbb{Z}) \cong H^*(L_0; \mathbb{Z}/2\mathbb{Z})(\cong H^*(U(2), \mathbb{Z}/2\mathbb{Z})).$$

Conjectures/Questions.

- There exists two $b \in H^1(L_0;\Lambda)$ such that $HF((L_0,b),(L_0,b);\Lambda) \neq 0$.
- $HF((L_t,b),(L_t,b);\Lambda) = 0$ for $t \neq 0$.
- $\mathfrak{PO}(L_t,b) \stackrel{?}{=} 0$ for any t.
  Relation to the partial compactification by Eguchi-Hori-Xiong?

- Higher dimensional case?