Talk 1: Formal deformation quantizaton

Akira Yoshioka [2]

7 February 2013 Shinshu

[2]Dept. Math. Tokyo University of Science

In this talk, we give a brief review on formal deformation quantization.

1. We explain definition of formal deformation quantization and its basic facts.
2. We discuss existence and classification problems of formal deformation quantization on manifolds.
3. We explain Weyl manifolds, which give the existence of formal deformation quantization on symplectic manifolds.

Based on the joint works with H. Omori, Y. Maeda,

1.1. Background

Wigner introduced a phase space distribution, related to Weyl's quantization procedure.

[Wg]: E. P. Wigner, *On the quantum correction for thermodynamics equilibriun*, Phys. Rev. , **40** (1932) 749–759.

Moyal introduced the "sine-Poisson" bracket, now called the Moyal bracket, for functions on phase space.

[Moyal]: J. E. Moyal, *Quantum Mechanics as s statistical theory*, Proc. Cambridge Phil. Soc., **45** (1949) 99–124.

Deformation quntization is proposed by Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer to express the trasfer from classical mechanics to quantum mechanics, and also the opposite direction. They analyse a deformation quantization for fomal power series.

[BFFLS1] F Bayen, M Flato, C Fronsdal, A Lichnerowicz, D Sternheimer, *Deformation theory and quantization. I. Deformations of symplectic structures*, Annals of Physics, **111**(1978) 61–110.

1.2. Example: Moyal product

The Moyal product is a well-known example of star product.

$$
\begin{aligned}
f *_O g &= f \exp \tfrac{\nu}{2} \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) g = f \sum_{k=0}^{\infty} \frac{1}{k!} \left(\tfrac{\nu}{2}\right)^k \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right)^k g \\
&= fg + \tfrac{\nu}{2} f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) g + \tfrac{1}{2!} \left(\tfrac{\nu}{2}\right)^2 f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right)^2 g \\
&\qquad + \cdots + \tfrac{1}{k!} \left(\tfrac{\nu}{2}\right)^k f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right)^k g + \cdots
\end{aligned}
$$

Formal power series.

We assume ν is a formal parameter and we consider the space of all formal power series $\mathcal{A}_\nu$ of ν with coefficints in the space of all smooth functions on $\mathbb{R}^{2m}$:

$$\mathcal{A}_\nu = \left\{ f = \sum_{k=0}^{\infty} f_k(\boldsymbol{w})\, \nu^k \mid f_k(\boldsymbol{w}) \in C^\infty(\mathbb{R}^{2m}) \right\} \tag{1}$$

where $\boldsymbol{w}$ denotes the variables $\boldsymbol{w} = (u_1, \ldots, u_m, v_1, \ldots, v_m)$.

Bidervation

We consider a biderivation

$$\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} = \sum_{j=1}^{m} \left(\overleftarrow{\partial_{v_j}} \cdot \overrightarrow{\partial_{u_j}} - \overleftarrow{\partial_{u_j}} \cdot \overrightarrow{\partial_{v_j}} \right) \tag{2}$$

where left arrow means the differential $\overleftarrow{\partial}_{u_j}$ acts on the function on the left hand side and the right arrow on the right, nemely

$$\begin{aligned} f\left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) g &= f \sum_{j=1}^{n} \left(\overleftarrow{\partial_{v_j}} \cdot \overrightarrow{\partial_{u_j}} - \overleftarrow{\partial_{u_j}} \cdot \overrightarrow{\partial_{v_j}} \right) g \\ &= \sum_{j=1}^{n} \left(\partial_{v_j} f \partial_{u_j} g - \partial_{u_j} f \partial_{v_j} g \right) \end{aligned}$$

Moyal product

The Moyal product $f *_o g$ is given by the power series of the bidifferential operators $\left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)$ such that

$$
\begin{aligned}
f *_o g = f \exp \tfrac{\nu}{2} \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) g &= f \sum_{k=0}^{\infty} \frac{1}{k!} \left(\tfrac{\nu}{2}\right)^k \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)^k g \\
= fg + \tfrac{\nu}{2} f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) g &+ \tfrac{1}{2!} \left(\tfrac{\nu}{2}\right)^2 f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)^2 g \\
&+ \cdots + \tfrac{1}{k!} \left(\tfrac{\nu}{2}\right)^k f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)^k g + \cdots \qquad (3)
\end{aligned}
$$

Theorem

The Moyal product is a well-defined $\mathbb{C}[[\nu]]$-bilinear product on the space of all formal power series $\mathcal{A}_\nu$, and associative.

General product.

Biderivation.

We can consider a slight extension of the Moyal product.
For an arbitrary complex $2m \times 2m$ matrix $\Lambda = (\Lambda^{jk})$ we consider a biderivation such that

$$\overleftarrow{\partial} \Lambda \overrightarrow{\partial} = \sum_{jk} \Lambda^{jk} \overleftarrow{\partial}_{w_j} \overrightarrow{\partial}_{w_k} \tag{4}$$

where we put $\boldsymbol{w} = (w_1, \cdots, w_{2m}) = (u_1, \cdots, u_m, v_1, \cdots, v_m)$.

Product

Instead of the biderivation $\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}$ in the definition of the Moyal product, using the biderivation $\overleftarrow{\partial} \Lambda \overrightarrow{\partial}$ we define a bilinear product $f *_{\Lambda} g$ by

$$\begin{aligned} f *_{\Lambda} g &= f \exp \tfrac{\nu}{2} \left(\overleftarrow{\partial} \Lambda \overrightarrow{\partial} \right) g \\ &= fg + \tfrac{\nu}{2} f \left(\overleftarrow{\partial} \Lambda \overrightarrow{\partial} \right) g + \tfrac{1}{2!} \left(\tfrac{\nu}{2} \right)^2 f \left(\overleftarrow{\partial} \Lambda \overrightarrow{\partial} \right)^2 g + \cdots \\ &= fg + \nu C_1(f,g) + \nu^2 C_2(f,g) + \cdots + \nu^k C_k(f,g) + \cdots \end{aligned} \tag{5}$$

where $C_k(f,g)$ is a bidifferential operator given by

$$C_k(f,g) = f \left(\frac{1}{k!} \frac{1}{2^k} \left(\overleftarrow{\partial} \Lambda \overrightarrow{\partial} \right)^k \right) g$$

For any Λ, the product $*_{\Lambda}$ is associative. If Λ is symmetric, the product is commutative.

Poisson structure

Moreover, the skewsymmetric part of $C_1(f,g)$ gives a Poisson bracket

$$C_1^-(f,g) = C_1(f,g) - C_1(g,f)$$

Actually, if we write the skewsymmetric part of the matrix Λ as J then it is easy to see

$$C_1^-(f,g) = f\left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)g = \sum_{jk} J^{jk}\overleftarrow{\partial}_{w_j} f\overrightarrow{\partial}_{w_k} g$$

For a matrix $\Lambda = J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the star product $*_\Lambda$ is equal to the Moyal product.

Canonical commutation relations.

We see directly

Theorem

For the Moyal product or the star product $*_\Lambda$ *with* $J = J_0$, *the generators* $(u_1, \cdots, u_m, v_1, \cdots, v_m)$ *satisfy the canonical commutation relations.*

$$[u_j, u_k]_* = u_j *_\Lambda u_k - u_k *_\Lambda u_j = [v_j, v_k]_* = 0, \quad j, k = 1, \cdots, m,$$
$$[u_j, v_k]_* = -i\hbar$$

Idea of deformation quantizaion

Remark

We remark here that

- *we can construct a quantum mechanical system not by using operators but by using ordinary functions and deformed product.*
- *Moreover, if the parameter tends to 0, we see the product becomes an ordinary commutative multiplication of functions.*

This is the basic idea of deformation quantization.

1.3. Deformation quantization

The the product $*_\Lambda$ in the previous subsection:

$$\begin{aligned} f *_\Lambda g &= f \exp \tfrac{\nu}{2} \left(\overleftarrow{\partial} \Lambda \overrightarrow{\partial} \right) g \\ &= fg + \tfrac{\nu}{2} f \left(\overleftarrow{\partial} \Lambda \overrightarrow{\partial} \right) g + \tfrac{1}{2!} \left(\tfrac{\nu}{2} \right)^2 f \left(\overleftarrow{\partial} \Lambda \overrightarrow{\partial} \right)^2 g + \cdots \\ &= fg + \nu C_1(f,g) + \nu^2 C_2(f,g) + \cdots + \nu^k C_k(f,g) + \cdots \end{aligned}$$

where $C_k(f,g)$ is a bidifferential operator given by

$$C_k(f,g) = f \left(\frac{1}{k!} \frac{1}{2^k} \left(\overleftarrow{\partial} \Lambda \overrightarrow{\partial} \right)^k \right) g$$

and the skewsymmetric part of $C_1^-(f,g)$ gives a Poisson bracket.

Generalizing this product, we give a concept of deformation quantization on a general manifold.

Formal power seires.

Let M be an arbitrary manifold. We set the space of all formal power series $\mathcal{A}_\nu(M)$ of ν with coefficients in the smooth functions $C^\infty(M)$:

$$\mathcal{A}_\nu(M) = \left\{ f = \sum_{k=0}^{\infty} f_k \, \nu^k \mid f_k \in C^\infty(M) \right\} \tag{6}$$

Star product.

Assume we have a $\mathbb{C}[[\nu]]$-bilinear product $*_\nu$ on $\mathcal{A}_\nu(M)$ given in the form

$$f *_\nu g = fg + \nu C_1(f,g) + \nu^2 C_2(f,g) + \cdots + \nu^k C_k(f,g) + \cdots \tag{7}$$

where $C_k(f,g)$ is a bidifferential operator on M.

Definition (Star product)

A product $*_\nu$ is called a star product if it is associative and the constant function 1 is the unit element.

Poisson structure.

For any star product $*_\nu$, the associativity gives

$$[f, g *_\nu h] = [f, g] *_\nu h + g *_\nu [f, h]$$

and

$$\sum_{\text{cyclic}} [f, [g, h]] = 0$$

Here we write the skewsymmetric part of C_1 as $\{f, g\} = C_1^-(f, g)$. Substituting the expansion of the product (7) we see the lowest order terms give

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

and

$$\sum_{\text{cyclic}} \{f, \{g, h\}\} = 0$$

respectively.

Then we have

Proposition

The skewsymmtric part of the fist order term C_1 of a star product becomes a Poisson bracket of the manifold M.

Deformation quantizaton.

We are in a position to define a deformation quantization of a Poisson manifold.

Definition

For a Poisson manifold $(M, \{\ ,\ \})$*, star product* $*_\nu$ *is called a deformation quantization of* $(M, \{\ ,\ \})$ *when the skewsymmetric part* C_1^- *is equal to the Poisson bracket* $\{\ ,\ \}$*. And* $*_\nu$ *is called a star product of* $(M, \{\ ,\ \})$*.*

Remark

Remark that $*_\nu$ *is localized to an arbtrary domain* $U \subset M$*, that is, we have a star product* $f * g$ *for any* $f, g \in \mathcal{A}_\hbar(U)$*.*

1.4. Equivalence of formal star products

Assume we have star products $*_\nu$, $*'_\nu$ on M. Let us consider a $\mathbb{C}[[\nu]]$-linear map $T_\nu : \mathcal{A}_\nu(M) \to \mathcal{A}_\nu(M)$ of the form

$$T_\nu = T_0 + \nu T_1 + \cdots + \nu^k T_k + \cdots ,$$

where $T_k \ (k = 0, 1, \cdots)$ is a differential operator on M.

Definition

A linear isomorphism T_ν is called a an isomorphism of if

$$T_\nu(f *_\nu g) = T_\nu(f) *_\nu T_\nu(g)$$

2. Existence and Classification

There were existence problem and classification problem of a deformation quantization for Poisson manifolds.

2.1. Symplectic case

The problem established independetly:

- Dewilde-Lecomte: Step-wise and Cohomology

  [DL]: M. De Wilde, P. B. A. Lecomte *Existence of star products and of formal deformations of the Poisson Lie algebra of arbitrary sympletic manifolds*, Lett, Math, Phys. **7** (1983), 487–496
- Omori-Maeda-Yoshioka: Weyl algebra bundle and quantized contact structure

  [OMY1]: H. Omori, Y. Maeda, A. Yoshioka, *Weyl manifolds and deformation quantization*, Adv. in Math. **85** (1991) 224–255.
- Fedosov: Fedosov connection

  [F]: B. Fedosov, *A simple geometrical constructon of deformation quantization*, J. Differential Goem. **40** (1994), 213–1994

Localization and Darboux chart

When $(M, \{\ ,\ \})$ is symplectic, the deformation quantization $*$ has a nice property.

On a Darboux chart $(U, (u_1, \cdots, u_n, v_1, \cdots, v_n))$, the Poisson bracket is expressed in the form

$$\{f, g\} = \sum_i \frac{\partial f}{\partial_{v_i}} \frac{\partial g}{\partial_{u_i}} - \frac{\partial f}{\partial_{u_i}} \frac{\partial g}{\partial_{v_i}} = f \overleftarrow{\partial} \Lambda \overrightarrow{\partial} g, \tag{8}$$

where $\Lambda = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. Then we have

Proposition (Quantized Darboux theorem)

For any deformation quantization $*_\nu$ on a symplectic manifold $(M, \{\ ,\ \})$, locally the product $*_\nu$ is isomorphic to the Moyal product on $C^\infty(U)[[\hbar]]$.

Existence

On the other hand, by gluing local Moyal algebra we obtain a deformation quantizaion.

Theorem (DeWilde-Lecomte, OMY, Fedosov)

For any symplectic manifold $(M, \{\ ,\ \})$, there exists a deformation quantization.

Classification

Let M be a symplectic manifold with symplectic form ω. We consider a formal power series of closed two forms

$$\Omega_\nu = \omega + \nu\,\Omega_1 + \cdots + \nu^k \Omega_k + \cdots$$

on a symplectic manifold M where ω_0 is the symplectic form.

Theorem (Classification theorem)

For a symplectic manifold (M, ω), isomorphic classe of deformation quantizations have one-one correspondence with the cohomology classes $\{[\Omega_\nu]\}$.

See for example, [F], [OMY2]:

[OMY2], H. Omori, Y. Maeda, A. Yoshioka, *Poincare-Cartan class and deformation quantization of Kaehler manifolds*, Commun. Math. Phys. **194** (1998) 207–230.

Poisson case

- Kontsevich [K] shows the existence of a formal star product on any poisson manifold.

3. Weyl manifolds

Weyl manifold W_M is an algebra bundle over a symplectic manifold M whose fibre is a Weyl algebra W.

- Weyl algebra bundle
- Weyl diffeomorphisms, patching
- Deformation quantization

Trivial bundle

For a symplectic manifold M, we have an altas consisting of Darboux charts $\{(U, x^1, \cdots, x^m, y_1, \cdots, y_m)\}$.

We sometimes write as $(z^1, \cdots, z^{2m}) = (x^1, \cdots, x^m, y_1, \cdots, y_m)$.

We consider a trivial bundle for every Darboux chart

$$W_U = U \times W$$

Weyl continuation

We embedd a set $C^\infty(U)$ of all smooth functions on U into W_U as sections by

$$f(z) \mapsto f^\#(z) = \sum_k \frac{1}{k!} f^{(k)}(z) Z^k \in \Gamma(W_U)$$

where Z^k is a complete symmetrization of the product in W. The section $f^\#$ is called a *Weyl continuation* of f.

We denote by $\mathcal{F}(W_U)$ the space of all Weyl continuations on U.
We have

Proposition

$$f^\# * g^\# = (f *_\nu g)^\#$$

where $*$ is the multiplication in W and $*_\nu$ is the Moyal product with respect to the Darboux chart $(z^1, \cdots, z^{2m})$.

3.2. Weyl diffeomorphisms, patching

Weyl diffeomorphisms

For bundles W_U, $W_{U'}$, we consider a bundle isomorphism

$$\Phi : W_U \to W_{U'}$$

Definition

An isomorphism Φ is called a Weyl diffeomorphism if

1. $\Phi(\nu) = \nu$
2. $\Phi^* \mathcal{F}(W_{U'}) = \mathcal{F}(W_U)$
3. $\overline{\Phi^*(f^\#)} = \Phi^*(\overline{f^\#})$

Patching

We can patch $\{W_U\}$ by Weyl diffeomorphisms and we get the Weyl manifold.

Theorem

For a symplectic manifold M, there exist Weyl manifolds W_M.

Contact algebra

For patching, we need to deal with the center of the Weyl algebra. In order to do this, we introduce a quantized contact Lie algebra.
We consider an element τ such that

$$[\tau, \nu] = 2\nu, \ [\tau, Z^k] = \nu Z^k, \ k = 1, 2, \cdots, 2m.$$

We set a Lie algebra (called a *quantized contact algebra*)

$$\mathfrak{G} = \mathbb{R}\tau \oplus W$$

3.3. Deformation quantization

Global Weyl functions

Definition

A section $\tilde{f} \in \Gamma(W_M)$ *is called a global Weyl function of* W_M *if it is a Weyl function when restricted to a local trivialization. We denote by* $\mathcal{F}(W_M)$ *the algebra of all Weyl functions of* W_M.

Proposition

There exist a $\mathbb{C}[[\nu]]$*-linear isomorphism* σ *from* $C^\infty(M)[[\nu]]$ *to* $\mathcal{F}_M$.

Existence of Deformation quantizaion

Using the linear isomorphism $\sigma : C^\infty(M)[[\nu]] \to \mathcal{F}_M$ we can introduce an associative product on $C^\infty(M)[[\nu]]$ by

$$f *_\nu g = \sigma^{-1}(\sigma(f) * \sigma(g))$$

We have

Theorem

The product $f *_\nu g$ is a deformation quantization of a symplectic manifokd (M, ω), namely we have

$$f *_\nu g = fg + \nu\{f, g\} + \cdots + \nu^k C_k(f, g) + \cdots$$

Remark

We can also patch tivial algebra bundles with fiber $\mathfrak{G} = \mathbb{R}\tau \oplus W$ and we obtain a bundle $\mathfrak{G}_M$.
From this bundle, we can obtain a Fedosov connection by means of the contact structure τ.
*[Y]: A. Yoshioka, Contact Weyl manifolds over a symplectic manifold, Advnced Studies in Pure Mathematics, **37** (2002) 459–493.*

4. Rerences

[BFFLS1] F Bayen, M Flato, C Fronsdal, A Lichnerowicz, D Sternheimer, *Deformation theory and quantization. I. Deformations of symplectic structures*, Annals of Physics, **111**(1978) 61–110.

[BFFLS2] F Bayen, M Flato, C Fronsdal, A Lichnerowicz, D Sternheimer, *Deformation theory and quantization. II. Physical Applications*, Annals of Physics, **111**(1978) 111–151.

[DL] M. De Wilde, P. B. A. Lecomte, *Existence of star products and of formal deformations of the Poisson Lie algebra of arbitrary sympletic manifolds*, Lett, Math, Phys. **7** (1983), 487–496

[F] B. Fedosov, *A simple geometrical constructon of deformation quantization*, J. Differential Goem. **40** (1994), 213–1994

[K] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett, Math, Phys. **66** (2003), 157–216.

[Moyal] J. E. Moyal, *Quantum Mechanics as s statistical theory*, Proc. Cambridge Phil. Soc., **45** (1949) 99–124.

[OM] H. Omori, Y. Maeda, *Quantum Calculus (in Japanese)*, Springer, 2004

[OMY1] H. Omori, Y. Maeda, A. Yoshioka, *Weyl manifolds and deformation quantization*, Adv. in Math. **85** (1991) 224–255.

[OMY2] H. Omori, Y. Maeda, A. Yoshioka, *Poincare-Cartan class and deformation quantization of Kaehler manifolds*, Commun. Math. Phys. **194** (1998) 207–230.

[OMY3] H. Omori, Y. Maeda, A. Yoshioka, *Deformation quantizatons of Poisson algebras*, Contemp. Math., **179** (1994) 213–240.

[OMMY1] Omori H, Maeda Y., Miyazaki N. and Yoshioka A., *Orderings and Non-formal Deformation quantization*, Lett. Math. Phys. **82** (2007) 153–175. e

[OMMY2] Omori H, Maeda Y., Miyazaki N. and Yoshioka A., *Deformation of Expression of Algebras*, (preprint)

[Td] Toda M., *Introduction to Elliptic functions*, Nihon Hyoronshya, Tokyo, 2001. (in Japanese)

[TY] Tomihisa T. and Yoshioka A., *Star Products and Star Exponentials*, J. Geometry and Symmetry in Physics, **19** (2010) 99–111.

[Wg] E. P. Wigner, *On the quantum correctino for thermodynamics equilibriun*, Phys. Rev. , **40** (1932) 749–759.

[Y] A. Yoshioka, *Contact Weyl manifolds over a symplectic manifold*, Advnced Studies in Pure Mathematics, **37** (2002) 459–493.

[Y1] Yoshioka A., *A family of star products and its application*, In: XXVI workshop on geometrical methods in physics, Kielonowski P., Odzijewicz A. Schlichenmaier M. and Voronov T. (Eds), AIP. Conference Proceedings **956**, Melville, New York 2007, pp 37–42,

[Y2] Yoshioka A., *Examples of Star Exponentials*, In: XXVI workshop on geometrical methods in physics. Kielonowski P., S. T. Ali, Odzijewicz A. Schlichenmaier M. and Voronov T. (Eds), AIP. Conference Proceedings **1191**, Melville, New York 2009, pp 188–193,