Talk 2: A family of non-formal star products

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1. A family of star products

In this talk, the deformation parameter is taken as a number, not a formal parameter.

We give a brief review on a family of non-formal star products.

1. We introduce a family of star products on polynomials.
2. We give a bundle structure to this family of star products and obtain a gometric picure of Weyl algebra.
3. We introduce a topology and take a completion.
4. We consider a star exponential functions.

Based on the joint works with H. Omori, Y. Maeda, N. Miyazaki,

1.1. Background

Starting from the Weyl algebra, we naturally obtain a star product on polynomials.

Typical star products

We start with typical star products.

Let $\mathcal{P}(\mathbb{C}^{2m})$ the set of complex polynomails of variables

$$\boldsymbol{z} = (u_1, \ldots, u_m, v_1, \ldots, v_m)$$

We assume $\nu = i\hbar$ and $\hbar > 0$, positive number.

Moyal product

Moyal product

$$f *_o g = f \exp \frac{i\hbar}{2} \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) g$$
$$= fg + \frac{i\hbar}{2} f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) g + \frac{1}{2!} \left(\frac{i\hbar}{2}\right)^2 f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right)^2 g$$
$$+ \cdots + \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right)^k g + \cdots$$

Remark

The product has meanings, that is, it is convergent on polynomials.

It is an associative product.

We see

$$u_j *_o v_j = u_j v_j - \frac{i\hbar}{2}, \quad v_j *_o u_j = u_j v_j + \frac{i\hbar}{2},$$

Normal product

$$
\begin{aligned}
f *_N g &= f \exp i\hbar \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} \right) g \\
&= fg + i\hbar f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} \right) g + \frac{1}{2!} (i\hbar)^2 f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} \right)^2 g \\
&\qquad + \cdots + \frac{1}{k!} (i\hbar)^k f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} \right)^k g + \cdots
\end{aligned}
$$

$$
u_j *_N v_j = u_j v_j, \quad v_j *_N u_j = u_j v_j + i\hbar,
$$

Anti-normal product

$$f *_A g = f \exp i\hbar \left(-\overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) g$$

$$= fg + i\hbar f \left(-\overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) g + \frac{1}{2!} (i\hbar)^2 f \left(-\overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right)^2 g$$

$$+ \cdots + \frac{1}{k!} (i\hbar)^k f \left(-\overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right)^k g + \cdots$$

$$u_j *_A v_j = u_j v_j - i\hbar, \quad v_j *_A u_j = u_j v_j,$$

CCR

We have associative algebras

$$(\mathcal{P}(\mathbb{C}^{2m}), *_{_O}), \ (\mathcal{P}(\mathbb{C}^{2m}), *_{_N}), \ (\mathcal{P}(\mathbb{C}^{2m}), *_{_A}),$$

Proposition

These algebras satisfy the canonical commutation relations:

$$[u_j, u_k]_* = u_j *_{_\Lambda} u_k - u_k *_{_\Lambda} u_j = [v_j, v_k]_* = 0, \quad j, k = 1, \cdots, m,$$
$$[u_j, v_k]_* = -i\hbar\delta_{jk}, \quad * = *_{_O}, *_{_N}, *_{_A}$$

Mutually isomorphic

These algebras are generated by the elements satisfying the canonical commutation relations, then mahematically we can say

Proposition

These are mutally isomorphic, and isomorphic to the Weyl algebra.

Intertwiners

Acutally for these algebras we have Intertwiners.
For example,

Proposition

An intertwiner (algebra isomorphism) from $(\mathcal{P}(\mathbb{C}^{2m}), *_O)$ *to* $(\mathcal{P}(\mathbb{C}^{2m}), *_N)$*,*

$$I_O^N : (\mathcal{P}(\mathbb{C}^{2m}), *_O) \rightarrow (\mathcal{P}(\mathbb{C}^{2m}), *_N)$$

is given explicitly by

$$I_O^N(p) = \exp\left(\frac{i\hbar}{2}\partial_u \cdot \partial_v\right)(p) = p + \frac{i\hbar}{2}\partial_u \cdot \partial_v(p) + \frac{1}{2}\left(\frac{i\hbar}{2}\partial_u \cdot \partial_v\right)^2(p) + \cdots ,$$

This is well defined, namely convergent on polynomials.

Other intertwiners are given similarly as

$$I_O^A(p) = \exp\left(\frac{-i\hbar}{2}\partial_u \cdot \partial_v\right)(p)$$

$$I_N^A(p) = \exp\left(-i\hbar\partial_u \cdot \partial_v\right)(p)$$

It is easy to see

Proposition

$$I_{n_1}^{n_3} = I_{n_2}^{n_3} I_{n_1}^{n_2}, \quad n_1, n_2, n_3 = O, N, A$$

Ordering problem in Weyl algebra and star products

We can see these typical star products are naturally given from the Weyl algebra as follows:

1. Weyl algebra

The Weyl algebra W is an associative algebra over a complex numbers generated by elements

$$x_1, \cdots, x_m, y_1, \cdots, y_m \tag{1}$$

satisfying the canonical commutation relations with respect to the commuatator

$$[x_j, x_k]_* = [y_j, y_k]_* = 0, \quad j, k = 1, \cdots, m,$$
$$[x_j, y_k]_* = -i\hbar\delta_{jk},$$

2. Ordering

Since the algebra W is non-commutative, we need to fix the ordering when we express elements. we have standard ordering as follows:

For any element $a \in W$ we can write uniquely in the following three ways resepctively

- Weyl ordering $a = \sum a_{kl}\, x^k \circ y^l, \quad a_{kl} \in \mathbb{C}, \quad$ (finte sum)
- Normal ordering $a = \sum b_{kl}\, x^k y^l, \quad a_{kl} \in \mathbb{C}, \quad$ (finte sum)
- Anti-normal ordering $a = \sum c_{kl}\, y^k x^l, \quad a_{kl} \in \mathbb{C}, \quad$ (finte sum)

where $x^k \circ y^l$ means the complete symmetrization.

3. identification to polynomials

Using these expressions we have a linear isomorphism of $W \to \mathcal{P}(\mathbb{C}^{2m})$ respectively

- $\sigma_O(a) = \sum a_{kl}\, u^k v^l$ Weyl ordering $a = \sum a_{kl}\, x^k \circ y^l$,
- $\sigma_N(a) = \sum b_{kl}\, u^k v^l$ normal ordering $a = \sum b_{kl}\, x^k y^l$,
- $\sigma_A(a) = \sum c_{kl}\, u^l v^k$ anti-normal ordering $a = \sum c_{kl}\, y^k x^l$,

4. Weyl algebra induces typical star products

These identification induce a product on polynomials, e.g.
$f *_o g = \sigma_O((\sigma_O^{-1} f)((\sigma_O^{-1} g))$

Namely we have

Proposition

1.
   - *The Moyal product is induced by the Weyl ordering,*
   - *the normal product induced by normal ordering*
   - *the anti-normal product form anti-normal ordering, respectively.*
2. *The intertwiners are also induce by*
   - $I_O^A = \sigma_A \sigma_O^{-1}$
   - $I_O^N = \sigma_N \sigma_O^{-1}$
   - $I_N^A = \sigma_A \sigma_N^{-1}$

1.2. A family of star products

In this section,

- we introduce a family of star products parametrized by the space of all complex symmetric matrices. Mathematical Extension, Beyond Orderings
- Using the intertwiners and the family of star products, we give a geometric picture of the Weyl algebra.

Star products

Matrix

For simplicity, we consider star products of 2 variables $(u, v) = (u_1, u_2)$.
The general case for $(u_1, \cdots, u_m, u_1, \cdots, v_m)$ is similar.

In order to obtain the Weyl algebra, we fix the skew symmetric matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For an arbitrary **complex symmetric matrix** $K \in \mathcal{S}_C(2)$ we put

$$\Lambda = J + K = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$$

Bi-derivation

and consider a bi-derivation acting on complex polynoimals

$$p_1(u_1, u_2), p_2(u_1, u_2) \in \mathcal{P}(\boldsymbol{C}^2)$$

such that

$$p_1 \left(\overleftarrow{\partial} \Lambda \overrightarrow{\partial} \right) p_2 = p_1 \left(\sum_{k,l=1}^{2} \lambda_{kl} \overleftarrow{\partial}_{u_k} \overrightarrow{\partial}_{u_l} \right) p_2$$

$$= \sum_{k,l=1}^{2} \lambda_{kl} \partial_{u_k} p_1 \partial_{u_l} p_2$$

Star product

Now we define a product $*_K$ on the space of complex polynomials $p_1(u_1, u_2), p_2(u_1, u_2)$ by

$$p_1 *_K p_2 = p_1 \exp\left(\frac{i\hbar}{2}\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)p_2$$
$$= p_1 p_2 + \frac{i\hbar}{2}p_1\left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)p_2$$
$$+ \cdots + \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n p_1\left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)^n p_2 + \cdots$$

where $\Lambda = J + K$.

Proposition

For an arbitrary complex symmetric matrix $K \in \mathcal{S}_C(2)$ the product $*_K$ is associtaive on the space of all complex polynomials $\mathcal{P}(\mathbf{C}^2)$.

Standard star products

We remark here the definition of star products $*_K$ is an extension of star products given by standard ordering problems. For example, if we put

- $K = 0$, then the product becomes the Moyal product,
- $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we obtain the normal product
- $K = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ the anti-normal product.

A family of star product algebras

A family of star product algebras $\{(\mathcal{P}(\mathbb{C}^2), *_K)\}_{K\in\mathcal{S}_{\mathbf{C}}(2)}$ are all isomorphic to the Weyl algebra. In fact, we have

Proposition

For an artibrary $K \in \mathcal{S}_{\mathbf{C}}(2)$, the product $*_K$ satisfies the canonical commutation relations

$$[u_1, u_2]_{*_K} = u_1 *_K u_2 - u_2 *_K u_1 = -i\hbar$$

Intertwiners of star product algebras

It follows that all algebras $(\mathcal{P}(\boldsymbol{C}^2), *_K)$ are isomorphic to the Weyl algebra W_2 of two generators u_1, u_2. Actually, we have algebra isomorphis (intertwiners)

$$I_{K_1}^{K_2} : (\mathcal{P}(\boldsymbol{C}^2), *_{K_1}) \to (\mathcal{P}(\boldsymbol{C}^2), *_{K_2})$$

such that

$$I_{K_1}^{K_2}(p) = \exp\left(\frac{i\hbar}{4}(K_2 - K_1)\partial^2\right) p$$

where

$$(K_2 - K_1)\partial^2 = \sum_{kl=1}^{2} (K_2 - K_1)_{kl} \partial_{u_k} \partial_{u_l}$$

Chain rule

We have the relations

Proposition

(i) $I_{K_2}^{K_3} I_{K_1}^{K_2} = I_{K_1}^{K_3}$

(ii) $(I_{K_1}^{K_2})^{-1} = I_{K_2}^{K_1}$

Infinitesimal intertwiners

By differentiating the intertwiner with respect to K, we obtain an infinitesimal intertwiner at K

$$\nabla_\kappa(p) = \frac{d}{dt} I_K^{K+t\kappa}(p)_{|_{t=0}} = \frac{i\hbar}{4}\,\kappa\partial^2 p, \quad \kappa \in T_K(\mathcal{S}_{\mathbb{C}(2)}) = \mathcal{S}_{\mathbb{C}(2)}$$

where

$$\kappa\partial^2 p = \sum_{i,j}^{2} \kappa_{ij}\partial_i\partial_j p.$$

Then the infinitesimal intertwiner satisfies

$$\nabla_\kappa(p_1 *_K p_2) = \nabla_\kappa(p_1) *_K p_2 + p_1 *_K \nabla_\kappa(p_2)$$

for any $p_1(u_1, u_2), p_2(u_1, u_2) \in \mathcal{P}(\boldsymbol{C}^2)$.

2. Geometric expression of Weyl algebra

In the star product algebras $\left\{(\mathcal{P}(\boldsymbol{C}^2), *_K)\right\}_{K\in\mathcal{S}_C(2)}$, the algebras $(\mathcal{P}(\boldsymbol{C}^2), *_{K_1})$ and $(\mathcal{P}(\boldsymbol{C}^2), *_{K_2})$ are mutually isomorphic by the intertwiner $I_{K_1}^{K_2}$ and the elements $p_1 \in (\mathcal{P}(\boldsymbol{C}^2), *_{K_1})$ and $p_2 \in (\mathcal{P}(\boldsymbol{C}^2), *_{K_2})$ are identified when

$$p_2 = I_{K_1}^{K_2}(p_1) \tag{2}$$

It follows naturally a geometric picture to the family of star product algebras $\left\{(\mathcal{P}(\boldsymbol{C}^2), *_K)\right\}_{K\in\mathcal{S}_C(2)}$.

Algebra bundle

Product bundle and algebra structure

To describe this, we introduce an algebra bundle over $\mathcal{S}_{\mathbf{C}}(2)$ whose fibres consisit of the Weyl algebra in the following way.

We consider the the trivial bundle

$$\pi : \mathbb{P} = \mathcal{P}(\mathbf{C}^2) \times \mathcal{S}_{\mathbf{C}}(2) \to \mathcal{S}_{\mathbf{C}}(2) \tag{3}$$

We set the product $*_{K}$ in the fiber at $K \in \mathcal{S}_{\mathbf{C}}(2)$, that is

$$\pi^{-1}(K) = (\mathcal{P}(\mathbf{C}^2), *_{K}) \tag{4}$$

Isomorphisms between fibers

There is the identification map between fibers $\pi^{-1}(K_1) = (\mathcal{P}(\boldsymbol{C}^2), *_{K_1})$ and $\pi^{-1}(K_2) = (\mathcal{P}(\boldsymbol{C}^2), *_{K_2})$ such as

$$I_{K_1}^{K_2}; \pi^{-1}(K_1) \to \pi^{-1}(K_2)$$

satisfying

$$I_{K_2}^{K_3} I_{K_1}^{K_2} = I_{K_1}^{K_3}, \quad (I_{K_1}^{K_2})^{-1} = I_{K_2}^{K_1}$$

Each fiber is isomorphic to the Weyl algebra.

Flat connection and parallel translation

We set the infinitesimal intertwiner by

$$\nabla_\kappa(p) = \frac{d}{dt} I_K^{K+t\kappa}(p)_{|t=0} = \frac{i\hbar}{4}\,\kappa\partial^2 p$$

On this bundle, we regard the infinitesimal intertwiner ∇ as a flat connection and the intertwiner $I_{K_1}^{K_2}$ as its parallel translation.

We consider $\Gamma(\mathbb{P})$ the sections of this bundle.
By definition, a parallel section $p \in \Gamma(\mathbb{P})$ is given by

$$\nabla_\kappa p(K) = 0, \quad \forall \kappa, K \in \mathcal{S}_C(2)$$

It is easy to see

Lemma

A section $p \in \Gamma(\mathbb{P})$ is parallel if and only if

$$\tilde{p}(K_2) = I_{K_1}^{K_2}(\tilde{p}(K_1)), \quad \forall K_1, K_2 \in \mathcal{S}_C(2)$$

q-number polynomials

We denote by $\mathcal{P}(\mathbb{P})$ the space of all parallel sections, and call an element $p \in \mathcal{P}(\mathbb{P})$ a *q-number polynomial*.

Due to the identies $I_{K_2}^{K_3} I_{K_1}^{K_2} = I_{K_1}^{K_3}$ and $(I_{K_1}^{K_2})^{-1} = I_{K_2}^{K_1}$ we see,

$$\begin{aligned} I_{K_1}^{K_2}(p(K_1) *_{K_1} q(K_1)) &= (I_{K_1}^{K_2}(p(K_1)) *_{K_2} (I_{K_1}^{K_2}(q(K_1)) \\ &= p(K_2) *_{K_2} q(K_2), \quad \forall p, q \in \mathcal{P}(\mathbb{P}), \ \forall K_1, K_2 \in \mathcal{S}_C(2) \end{aligned}$$

We have

Proposition

The q-number polynomials $\mathcal{P}(\mathbb{P})$ naturally has an associative product $*$ by

$$p * q(K) = p(K) *_{K} q(K), \quad \forall K \in \mathcal{S}_{\mathbf{C}}(2)$$

With respect to this product, the algebra $(\mathcal{P}(\mathbb{P}), *)$ is isomorphic to the Weyl algebra.

Then the algebra $(\mathcal{P}(\mathbb{P}), *)$ is regarded as a geometric realization of the Weyl algebra.

3. Topology and completion

We consider to extend the star product to some space of functions.

We have two directions.

1. One is formal star product– star product on the space of all formal power series of $\hbar$ with coefficients in smooth functions
2. another is nonformal deformation.

3.1. Formal extension

We take a completion of the space of polynomials with uniform convergence on every compact set to obtain the space of smooth functions on $\mathbb{R}^{2m}$. And we extend the star product $*_\Lambda$ to the space of all formal power series with coefficients in smooth functions on $\mathbb{R}^{2m}$.

Let us consider the space of all formal power series

$$\mathcal{A}_\hbar = C^\infty(\mathbb{R}^{2m})[[\hbar]] \tag{5}$$

Then we have

Proposition

The star product $*_\Lambda$ is well-defined on $\mathcal{A}_\hbar$ such that

$$f *_\Lambda g = fg + \frac{i\hbar}{2}\{f,g\} + \cdots + \hbar^n C_n(f,g) + \cdots \tag{6}$$

where $\{f,g\}$ is the Poisson bracket and C_n is a bidifferential operator. And we have an associative algebra $(\mathcal{A}_\hbar, *_\Lambda)$.

This extension is very successful. Actually, we extend the notion of deformation quantization from on Euclidean space to general Poisson manifolds. Main reason is we can construct a Weyl diffeomorphism to patch the local star product algebras.

3.2. Non-formal extension

We are interested in this extension. But this seems too restrictive in some sense.

- Maybe we cannot glue the algebra of convergent star products.
- Maybe the function space is give by functions defined on the whole Euclidean space.

3.2.1. Frechet topology

We introduce a topology into $\mathcal{P}(\boldsymbol{C}^2)$ by a system of semi-norms in the following way.
Let ρ be a positive number. For every $s > 0$ we define a semi-norm for polynomials by

$$|p|_s = \sup_{u \in \boldsymbol{C}^2} |p(u_1, u_2)| \exp\left(-s|u|^{\rho}\right)$$

Then the system of semi-norms $\{|\cdot|_s\}_{s>0}$ defines a locally convex topology $\mathcal{T}_\rho$ on $\mathcal{P}(\boldsymbol{C}^2)$.

3.2.2. Fréchet space $\mathcal{E}_\rho(\mathbf{C}^2)$

Definition We take the completion of $\mathcal{P}(\mathbf{C}^2)$ with respect to the topology $\mathcal{T}_\rho$, we obtain a Fréchet space $\mathcal{E}_\rho(\mathbf{C}^2)$.

Proposition

For a positive number ρ, the Fréchet space $\mathcal{E}_\rho$ consists of entire functions on the complex plane $\mathbf{C}^2$ with finite semi-norm for every $s > 0$, namely,

$$\mathcal{E}_\rho(\mathbf{C}^2) = \left\{ f \in \mathcal{H}(\mathbf{C}^2) \mid |f|_s < +\infty, \ \forall s > 0 \right\} \tag{7}$$

Continuity for the case $0 < \rho \leq 2$

As to the continuitiy of star products and intertwiners, the space $\mathcal{E}_\rho(\boldsymbol{C}^2)$, $0 < \rho \leq 2$ is very good, namely, we have the following

Theorem

On $\mathcal{E}_\rho(\boldsymbol{C}^2)$, $0 < \rho \leq 2$ every product $*_K$ is continuous, and every intertwiner $I_{K_1}^{K_2} : (\mathcal{E}_\rho(\boldsymbol{C}^2), *_{K_1}) \to (\mathcal{E}_\rho(\boldsymbol{C}^2), *_{K_2})$ is continuous.

Continuity as a bimodule for the case $\rho > 2$

As to the spaces $\mathcal{E}_\rho(\boldsymbol{C}^2)$ for $\rho > 2$, the situation is no so good, but still we have the following.

Theorem

For $\rho > 2$, take $\rho' > 0$ such that

$$\frac{1}{\rho'} + \frac{1}{\rho} = 1$$

then every star product $*_K$ defines a continuous bilinear product

$$*_K : \mathcal{E}_\rho(\boldsymbol{C}^2) \times \mathcal{E}_{\rho'}(\boldsymbol{C}^2) \to \mathcal{E}_\rho(\boldsymbol{C}^2),\ \mathcal{E}_{\rho'}(\boldsymbol{C}^2) \times \mathcal{E}_\rho(\boldsymbol{C}^2) \to \mathcal{E}_\rho(\boldsymbol{C}^2)$$

This means that $(\mathcal{E}_\rho(\boldsymbol{C}^2), *_K)$ is a continuous $\mathcal{E}_{\rho'}(\boldsymbol{C}^2)$-bimodule.

3.2.3. q-number functions

The case $0 < \rho \leq 2$

Due to the previous theorem, we can introduce a similar concept as q-number polynomials into the Fréchet spaces.
Namely, the star product $*_K$ is well defined on $\mathcal{E}_\rho(\boldsymbol{C}^2)$ and then we consider the trivial bundle

$$\pi : \mathbb{E}_\rho = \mathcal{E}_\rho(\boldsymbol{C}^2) \times \mathcal{S}_C(2) \rightarrow \mathcal{S}_C(2) \tag{8}$$

with fibre over the point $K \in \mathcal{S}_C(2)$ consists of

$$\pi^{-1}(K) = (\mathcal{E}_\rho(\boldsymbol{C}^2), *_K) \tag{9}$$

The intertwiners $I_{K_1}^{K_2}$ are well defined for any $K_1, K_2 \in \mathcal{S}_{\boldsymbol{C}}(2)$ and then the bundle $\mathbb{E}_\rho$ has a flat connection ∇ and the parallel translation is the intertwiner.
The space of flat sections of the bundle denoted by $\mathcal{F}_\rho$ naturally has the product $*$ and can be regarded as a completion of the Weyl algebra W_2.

Remark to the case $\rho > 2$

For the case $\rho > 2$, at present it is not clear whether the intertwiners are well-defined and whether the product $*_K$ are well defined. However the flat connection ∇ is still well defined on $\pi : \mathbb{E}_\rho = \mathcal{E}_\rho(\boldsymbol{C}^2) \times \mathcal{S}_{\boldsymbol{C}}(2) \to \mathcal{S}_{\boldsymbol{C}}(2)$, so we can define a space $\mathcal{F}_\rho$ of all parallel sections of this bundle even for $\rho > 2$.

For $\rho > 2$, we are trying to extend the product $*_K$ and also the intertwiners $I_{K_1}^{K_2}$ by means of some regularizations, for example, Borel-Laplace transform, or finite part regularization. I hope to construct such a concept in near future.

4. Star exponentials

The space of q-number functions $\mathcal{F}_\rho$ is a complete topological algebra for $0 < \rho \leq 2$ (even $\rho > 2$ for future under some regularization). We can consider exponential element

$$\exp_* t\left(\frac{H}{i\hbar}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \underbrace{\frac{H}{i\hbar} * \cdots * \frac{H}{i\hbar}}_{n} \tag{10}$$

in this algebra.

For a q-number polynomial $H \in \mathcal{P}(\mathbb{P})$, we define the star exponenial $\exp_* t(H/i\hbar)$ by the differential equation

$$\frac{d}{dt}\exp_* t\left(\frac{H}{i\hbar}\right) = \frac{H}{i\hbar} * \exp_* t\left(\frac{H}{i\hbar}\right),\ \exp_* t\left(\frac{H}{i\hbar}\right)|_{t=0} = 1 \tag{11}$$

Remark

We set the Fréchet space

$$\mathcal{E}_{\rho+}(\boldsymbol{C}^2) = \cap_{\lambda>\rho}\mathcal{E}_{\lambda}(\boldsymbol{C}^2)$$

and we donote by $\mathfrak{E}_{\rho+}$ the correponding bundle and by $\mathcal{F}_{\rho+}$ the space of the flat sections of this bundle.
When $H \in \mathcal{P}(\mathbb{P})$ is a linear element, then $\exp_* t\left(\frac{H}{i\hbar}\right)$ belongs to the good space $\mathcal{F}_{1+}(\subset \mathcal{F}_2)$.
On the other hand, the most interesting case is given by quadratic form $H \in \mathcal{P}(\mathbb{P})$. In this case we can solve the differential equation explicitly, but the star exponential belongs to the space $\mathcal{F}_{2+}$, which is difficult to treat at present.
Although general theory related to the space $\mathcal{F}_{2+}$ is not yet established, we illustrate the concrete example of the star expoenential of the quadratic forms and its application.

4.1. Examples

We very the parameter $K \in \mathcal{S}_C(2)$ and at some K we can obtain interesting identities in the algebra of $*_K$ product.

Linear case

We consider a liner q-number polynomial. It is written in general form as

$$H = a_1 u + a_2 v = \langle \boldsymbol{a}, \boldsymbol{u} \rangle, \quad a_1, a_2 \in \mathbb{C}.$$

Star exponential of H belongs to the space of q-number function $\mathcal{F}_{1+}$. The star exponetial $\exp_* t\left(\frac{H}{i\hbar}\right)$ at K, which is denoted by $\exp_{*_K} t\left(\frac{H}{i\hbar}\right)$, is explicitly given as

$$\exp_{*_K} t\left(\frac{H}{i\hbar}\right) = \exp \frac{t^2}{4\hbar}\langle \boldsymbol{a}K, \boldsymbol{a} \rangle \exp \frac{t}{i\hbar}\langle \boldsymbol{a}, \boldsymbol{u} \rangle.$$

Hence if the real part satisfies an inequality such as

$$\Re \frac{1}{4\hbar}\langle \boldsymbol{a}K, \boldsymbol{a}\rangle < 0 \tag{A}$$

the term $\exp \frac{t^2}{4\hbar}\langle \boldsymbol{a}K, \boldsymbol{a}\rangle$ is rapidlly decreasing with respect to t and then we can consider an integral

$$\int_{-\infty}^{\infty} e^{-e^t} \exp_{*_K} t\left(z + \frac{H}{i\hbar}\right) dt$$

Then we define star gamma function by

$$\Gamma_*(z) = \int_{-\infty}^{\infty} e^{-e^t} \exp_* t\left(z + \frac{H}{i\hbar}\right) dt \tag{12}$$

This is evaluated at every K and the value $\Gamma_{*_K}(z)$ of the star gamma function at K is given by the integral, where K satisfies the condition (A). We have the identity for K satisfying (A)

$$\Gamma_{*_K}(z+1) = \left(z + \frac{H}{i\hbar}\right) *_K \Gamma_{*_K}(z).$$

Quadratic case

For a generic point in $\mathcal{S}_{C}(2)$

$$K = \begin{pmatrix} \tau' & \kappa \\ \kappa & \tau \end{pmatrix} \in \mathcal{S}_{C}(2)$$

In the star product $*_K$ algebra, we write the generator $u = u_1, v = u_2$ satisfying

$$[u, v]_{*_K} = -i\hbar$$

Then the star exponential of $H = 2u * v$ is explicitly given at a general point K as

$$\exp_{*_K} t\left(\frac{2u * v}{i\hbar}\right) = \frac{2e^{-t}}{\sqrt{D}} \exp\left[\frac{e^t - e^{-t}}{i\hbar D}\Big((e^t - e^{-t})\tau u^2 + 2\triangle uv + (e^t - e^{-t})\tau' v^2\Big)\right]$$

where

$$D = \triangle^2 - (e^t - e^{-t})\tau'\tau, \ \triangle = e^t + e^{-t} - \kappa(e^t - e^{-t}) \tag{13}$$