Talk 3: Examples: Star exponential

Akira Yoshioka [2]

7 February 2013 Shinshu

[2] Dept. Math. Tokyo University of Science

1. Star exponential

1. We recall the definition of star exponentials.
2. We show some construction of star exponentials.

Based on the joint works with H. Omori, Y. Maeda, N. Miyazaki,

1.1. Definition

The space of q-number functions $\mathcal{F}_\rho$ is a complete topological algebra for $0 < \rho \leq 2$ (even $\rho > 2$ for future under some regularization). We can consider exponential element

$$\exp_* t\left(\frac{H}{i\hbar}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \underbrace{\frac{H}{i\hbar} * \cdots * \frac{H}{i\hbar}}_{n}$$

in this algebra.
For a q-number polynomial $H \in \mathcal{P}(\mathbb{P})$, we define the star exponenial $\exp_* t(H/i\hbar)$ by the differential equation

$$\frac{d}{dt} \exp_* t\left(\frac{H}{i\hbar}\right) = \frac{H}{i\hbar} * \exp_* t\left(\frac{H}{i\hbar}\right), \ \exp_* t\left(\frac{H}{i\hbar}\right)|_{t=0} = 1$$

Remark

We set the Fréchet space

$$\mathcal{E}_{\rho+}(\boldsymbol{C}^2) = \cap_{\lambda>\rho}\mathcal{E}_{\lambda}(\boldsymbol{C}^2) \tag{1}$$

and we donote by $\mathfrak{E}_{\rho+}$ the correponding bundle and by $\mathcal{F}_{\rho+}$ the space of the flat sections of this bundle.
When $H \in \mathcal{P}(\mathbb{P})$ is a linear element, then $\exp_* t\left(\frac{H}{i\hbar}\right)$ belongs to the good space $\mathcal{F}_{1+}(\subset \mathcal{F}_2)$.
On the other hand, the most interesting case is given by quadratic form $H \in \mathcal{P}(\mathbb{P})$. In this case we can solve the differential equation explicitly, but the star exponential belongs to the space $\mathcal{F}_{2+}$, which is difficult to treat at present.
Although general theory related to the space $\mathcal{F}_{2+}$ is not yet established, we illustrate the concrete example of the star expoenential of the quadratic forms and its application.

1.2. Examples

We very the parameter $K \in \mathcal{S}_C(2)$ and at some K we can obtain interesting identities in the algebra of $*_K$ product.

Linear case

We consider a liner q-number polynomial. It is written in general form as

$$H = a_1 u + a_2 v = \langle \boldsymbol{a}, \boldsymbol{u} \rangle, \quad a_1, a_2 \in \mathbb{C}.$$

Star exponential of H belongs to the space of q-number function $\mathcal{F}_{1+}$. The star exponetial $\exp_* t\left(\frac{H}{i\hbar}\right)$ at K, which is denoted by $\exp_{*_K} t\left(\frac{H}{i\hbar}\right)$, is explicitly given as

$$\exp_{*_K} t\left(\frac{H}{i\hbar}\right) = \exp \frac{t^2}{4\hbar} \langle \boldsymbol{a}K, \boldsymbol{a} \rangle \exp \frac{t}{i\hbar} \langle \boldsymbol{a}, \boldsymbol{u} \rangle.$$

Hence if the real part satisfies an inequality such as

$$\Re \frac{1}{4\hbar}\langle \boldsymbol{a}K, \boldsymbol{a}\rangle < 0 \tag{A}$$

the term $\exp \frac{t^2}{4\hbar}\langle \boldsymbol{a}K, \boldsymbol{a}\rangle$ is rapidlly decreasing with respect to t and then we can consider an integral

$$\int_{-\infty}^{\infty} e^{-e^t} \exp_{*_K} t\left(z + \frac{H}{i\hbar}\right) dt$$

Then we define star gamma function by

$$\Gamma_*(z) = \int_{-\infty}^{\infty} e^{-e^t} \exp_* t\left(z + \frac{H}{i\hbar}\right) dt \tag{2}$$

This is evaluated at every K and the value $\Gamma_{*_K}(z)$ of the star gamma function at K is given by the integral, where K satisfies the condition (A). We have the identity for K satisfying (A)

$$\Gamma_{*_K}(z+1) = \left(z + \frac{H}{i\hbar}\right) *_K \Gamma_{*_K}(z).$$

2. Examples

In this section, we show some examples of star exponetials.

2.1. Path integral (iterated integral) construction of quadratic case

[YM]: Yoshioka A., T. Matsumoto, *Path integral for Star Exponential functons of quadratic forms*, In: Geometry, Integrability and Quantization. I. M. Mladenov, G. L. Naber (Eds), **4**, Coral Press Scientific Publishing, Sofia 2003, pp 330–340,

Notation

We first give the Moyal product on $\mathbb{C}^2$. Let x, y be coordinate functions of $\mathbb{C}^2$ and let $\hbar$ be a positive real parameter. The canoncial Poisson bracket is given by

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = f\left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) g \tag{3}$$

Here we use a notation such as

$$\{f, g\} = f\left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) g = f\left(\overleftarrow{\partial}_x \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \overrightarrow{\partial}_x\right) g \tag{4}$$

Using the binomial theorem formally, we set bidifferential operators

$$\left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right)^n = \sum_{l+m=n} \frac{n!}{l!m!}(-1)^m \left(\overleftarrow{\partial}_x \overrightarrow{\partial}_y\right)^l \left(\overleftarrow{\partial}_y \overrightarrow{\partial}_x\right)^m$$

and

$$\exp\left(\frac{i\hbar}{2}\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n \left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right)^n$$

The Moyal product is then given by

Definition

$$f *_0 g = f \exp\left(\frac{i\hbar}{2}\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) g$$

We remark here that the product $f *_0 g$ is not necessarily convergent for arbitrary smooth functions, however it is defined when at least one of f, g is a polynomial function.

With the Moyal product, we can define the $*$-exponential function of a quadratic form in the following way. Let us consider a quadratic form on $\mathbb{C}^2$ given by

$$H = ax^2 + 2bxy + cy^2, \quad a, b, c \in \mathbb{C}. \tag{5}$$

The $*$-exponential function is formally given by

$$e_*^{t\frac{H}{i\hbar}} * f = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\frac{H}{i\hbar}\right)_*^n * f \tag{6}$$

for every polynomial function $f(x, y)$ where $\left(\frac{H}{i\hbar}\right)_*^n = \underbrace{\frac{H}{i\hbar} * \cdots * \frac{H}{i\hbar}}_{n}$. In this note, we study the explicit form of $e_*^{t\frac{H}{i\hbar}}$ by means of the path-integral method.

Normal forms and the invariance of $*_0$

Given a quadratic function $H = ax^2 + 2bxy + cy^2$ on $\mathbb{C}^2$, we consider a descriminant

$$D = b^2 - ac. \tag{7}$$

We will show that H with nonvanishing descriminant is transformed into $x^2 - y^2$ via linear transformations by $SL(2, \mathbb{C})$.

First we consider the case where H has the descriminat $D = \frac{1}{4}$. We will prove

Proposition

There exists $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL(2, \mathbb{C})$ *such that*

$$H = -\tfrac{1}{2}w^2 + \tfrac{1}{2}z^2, \tag{8}$$

where

$$\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{9}$$

Such matrices $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ *are not unique, parametrized by* $\mathbb{C}$.

First we consider the case where H has the descriminat $D = \frac{1}{4}$. We will prove

Proposition

There exists $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL(2, \mathbb{C})$ *such that*

$$H = -\frac{1}{2}w^2 + \frac{1}{2}z^2, \tag{10}$$

where

$$\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{11}$$

Such matrices $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ *are not unique, parametrized by* $\mathbb{C}$.

Furthermore, by a direct calculation we have the following invariance property of the Moyal product:

Proposition

The Moyal product is expressed in terms of the coordinate functions w, z *such as*

$$f *_0 g = f \exp\left(\frac{i\hbar}{2} \overleftarrow{\partial}_w \wedge \overrightarrow{\partial}_z\right) g$$

Path Integral representation

In what follows, we give a path-integral representation of the $*$-exponential function of quadratic forms.

1. Fist we consider $H = -\frac{1}{2}x^2 + \frac{1}{2}y^2$.
2. Then we obtain the $*$-exponential functions for general H with $D \neq 0$.

$*$-exponential of $-\frac{1}{2}x^2 + \frac{1}{2}y^2$

First, we give the $*$-exponential function of $-\frac{1}{2}x^2 + \frac{1}{2}y^2$.

The basic tool is the following Mehler's formula:

Lemma

Let $H_n(t)$ be an Hermite polynomial of degree n. Then it holds

$$e^{-x^2-y^2}\sum_{n=0}^{\infty}\frac{z^n}{2^n n!}H_n(x)H_n(y) = \frac{1}{\sqrt{1-z^2}}\exp\left(-\frac{1}{1-z^2}(x^2+y^2-2zxy)\right)$$

For the first step of path integral, we consider the product $\exp\left(t\frac{H}{i\hbar}\right) *_0 \exp\left(s\frac{H}{i\hbar}\right)$ for $H = (-x^2 + y^2)/2$.

Mehler's formula gives the formula:

Proposition

$$\exp\left(t\,\frac{H}{i\hbar}\right) *_o \exp\left(s\,\frac{H}{i\hbar}\right) = \frac{1}{1+ts/4}\exp\left(\frac{t+s}{1+ts/4}\,\frac{H}{i\hbar}\right)$$

Direct calculation of star Moyal product gives

$$\exp\left(t\frac{H}{i\hbar}\right) * \exp\left(s\frac{H}{i\hbar}\right)$$

$$= \exp\left(t\frac{H}{i\hbar}\right)\sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{i\hbar}{2}\right)^{l}\sqrt{\frac{t}{2i\hbar}}^{\,l}H_l\left(\sqrt{\frac{t}{2i\hbar}}x\right)\sqrt{\frac{is}{2\hbar}}^{\,l}H_l\left(\sqrt{\frac{is}{2\hbar}}y\right)$$

$$\times\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{i\hbar}{2}\right)^{k}\sqrt{\frac{it}{2\hbar}}^{\,k}H_k\left(\sqrt{\frac{it}{2\hbar}}y\right)\sqrt{\frac{s}{2i\hbar}}^{\,k}H_k\left(\sqrt{\frac{s}{2i\hbar}}x\right)\exp\left(s\frac{H}{i\hbar}\right)$$

and then it holds

$$\exp\left(t\frac{H}{i\hbar}\right) * \exp\left(s\frac{H}{i\hbar}\right)$$

$$= \exp\left(\frac{t+s}{i\hbar}H\right)\sum_{l=0}^{\infty}\frac{1}{2^l l!}\left(\frac{i\sqrt{ts}}{2}\right)^{l}H_l\left(\sqrt{\frac{t}{2i\hbar}}x\right)H_l\left(\sqrt{\frac{is}{2\hbar}}y\right) \qquad (12)$$

$$\times\sum_{k=0}^{\infty}\frac{1}{2^k k!}\left(\frac{-i\sqrt{ts}}{2}\right)^{k}H_k\left(\sqrt{\frac{it}{2\hbar}}y\right)H_k\left(\sqrt{\frac{s}{2i\hbar}}x\right)$$

For the next step, we consider the iterated product of exponential functions $e^{t_1\tilde{H}} * \cdots * e^{t_n\tilde{H}}$, where we put $\tilde{H} = \frac{H}{i\hbar} = (-x^2 + y^2)/2i\hbar$.
In what follows, we will show the formula:

Proposition

$$e^{t_1\tilde{H}} * \cdots * e^{t_n\tilde{H}} = \frac{1}{c_n(t)} \exp\left(\frac{s_n(t)}{c_n(t)}\tilde{H}\right) \qquad \text{(a)}$$

where

$$c_n(t) = 1 + \sum_{k;2\le 2k\le n} \sum_{1\le i_1<i_2<\cdots<i_{2k}\le n} (t_{i_1}/2)\cdots(t_{i_{2k}}/2) \qquad \text{(b)}$$

and

$$s_n(t) = 2 \sum_{k;1\le 2k+1\le n} \sum_{1\le i_1<i_2<\cdots<i_{2k+1}\le n} (t_{i_1}/2)\cdots(t_{i_{2k+1}}/2) \qquad \text{(c)}$$

For $t > 0$, we devide the interval $[0, t]$ into N equal segments for every positive integer N. We put $\triangle t = t/N$. Then we have

$$c_N(\triangle t) = \sum_{k;0\leq 2k\leq N} a_k(t/2)^{2k}, \quad s_N(\triangle t) = 2\sum_{k;1\leq 2k+1\leq N} b_k(t/2)^{2k+1},$$

where the coefficients are given by

$$a_k = \frac{1}{(2k)!}\cdot\left(1-\tfrac{1}{N}\right)\cdot\left(1-\tfrac{2}{N}\right)\cdots\left(1-\tfrac{2k-1}{N}\right)$$

and

$$b_k = \frac{1}{(2k+1)!}\cdot\left(1-\tfrac{1}{N}\right)\cdot\left(1-\tfrac{2}{N}\right)\cdots\left(1-\tfrac{2k}{N}\right)$$

It is easy to see the coefficients of $c_N(\triangle t)$ and $s_N(\triangle t)$ converges as $N \to \infty$ and the limits are

$$\lim_{N\to\infty} c_N(\triangle t) = \cosh \tfrac{t}{2}, \quad \lim_{N\to\infty} s_N(\triangle t) = 2\sinh \tfrac{t}{2}$$

Thus, we have

Theorem

The N-itereted product converges

$$\lim_{N\to\infty} e^{(t/N)\tilde{H}} * \cdots * e^{(t/N)\tilde{H}} = \frac{1}{\cosh \frac{t}{2}} e^{2\tilde{H} \tanh \frac{t}{2}}$$

where $\tilde{H} = H/(i\hbar)$ *and* $H = -\frac{x^2}{2} + \frac{y^2}{2}$.

$*$-exponential for general case

We consider the $*$-exponential function of $H = ax^2 + 2bxy + cy^2$, $a, b, c \in \mathbb{C}$ with $D = b^2 - ac \neq 0$.

For this case we can show $H = 2\sqrt{D}(-\frac{1}{2}w^2 + \frac{1}{2}z^2)$, where $w = px + qy$, $z = rx + sy$ and $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL(2, \mathbb{C})$.
Notice the commutator of w, z satisfies $[w, z] = i\hbar$. Then the trasformation fomula yields

Theorem

The $*$-exponential function of H is

$$e_*^{t\frac{H}{i\hbar}} = \frac{1}{\cosh\sqrt{D}t}\exp\left(\frac{H}{\sqrt{D}}\tanh\sqrt{D}t\right)$$

2.2. Star Hermite polynomials

We recall the following.

For any 2×2 complex matrix $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \in M_2(\mathbb{C})$, we have a biderivation on polynomials

$$p_1 \overleftarrow{\partial} \Lambda \overrightarrow{\partial} p_2 = p_1 \left(\sum_{\alpha\beta} \lambda_{\alpha\beta} \overleftarrow{\partial_\alpha} \overrightarrow{\partial_\beta} \right) p_2 = \sum_{\alpha\beta} \lambda_{\alpha\beta} \partial_\alpha p_1 \, \partial_\beta p_2, \quad p_1, p_2 \in \mathcal{P}(\mathbb{C}^2).$$

We have

Proposition

For any $\Lambda \in M_2(\mathbb{C})$, the product $*_\Lambda$ is well-defined and associative on $\mathcal{P}(\mathbb{C}^2)$.

In this section, we consider the star product for the simple case where

$$\Lambda = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$$

Then we see easily that the star product is commutative and explicitly given by $p_1 *_\Lambda p_2 = p_1 \exp\left(\frac{i\hbar\rho}{2}\overleftarrow{\partial_{u_1}}\overrightarrow{\partial_{u_1}}\right) p_2$. This means that the algebra is essentially reduced to space of functions of one varible u_1. Thus, we consider functions $f(w), g(w)$ of one variable $w \in \mathbb{C}$ and we consider a commutative star product $*_\tau$ with complex parameter τ such that

$$f(w) *_\tau g(w) = f(w) e^{\frac{\tau}{2}\overleftarrow{\partial}_w \overrightarrow{\partial}_w} g(w)$$

A direct calculation gives

$$\exp_{*_\tau} itw = \exp(itw - (\tau/4)t^2)$$

We see the generating function of Hermite polynomilas gives

$$e^{\sqrt{2}tw-\frac{1}{2}t^2} = \sum_{n=0}^{\infty} H_n(w)\frac{t^n}{n!}$$

The left hand side is a star exponential function at $\tau = -1$, $\exp_{*_{-1}}(\sqrt{2}tw) = e^{\sqrt{2}tw-\frac{1}{2}t^2}$. Thus the expansion

$$\exp_{*_{-1}}(\sqrt{2}tw) = \sum_{n=0}^{\infty} \frac{1}{n!}(\sqrt{2}tw)^n_{*_{-1}}$$

yields

$$(\sqrt{2}w)^n_{*_{-1}} = H_n(w).$$

We define a star Hermite function by

$$\exp_{*_\tau}(\sqrt{2}tw) = \sum_{n=0}^{\infty} H_n(w,\tau)\frac{t^n}{n!}$$

The identity $\frac{d}{dt}\exp_{*_\tau}\sqrt{2}tw = \sqrt{2}w *_\tau \exp_{*_\tau}\sqrt{2}tw$ yields

$$\frac{\tau}{\sqrt{2}}H_n'(w,\tau) + \sqrt{2}w\,H_n(w,\tau) = H_{n+1}(w,\tau).$$

The exponetial law gives

$$\sum_{k+l=n}\frac{n!}{k!l!}H_k(w,\tau) *_\tau H_l(w,\tau) = H_n(w,\tau)$$

2.3. Star theta functions

For $\Re\tau > 0$ star exponential

$$\exp_{*_\tau} niw = \exp(niw - (\tau/4)n^2)$$

is rapidly decreasing with respect to n. Then we can consider summations for τ satisfying $\Re\tau > 0$

$$\sum_{n=-\infty}^{\infty} \exp_{*_\tau} 2niw$$

$$= \sum_{n=-\infty}^{\infty} \exp\left(2niw - \tau n^2\right) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niw}, \quad (q = e^{-\tau}).$$

This is Jacobi's theta function $\theta_3(w, \tau)$.

Then we set star theta functions as

$$\theta_{1*_\tau}(w) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n \exp_{*_\tau}(2n+1)iw,$$

$$\theta_{2*_\tau}(w) = \sum_{n=-\infty}^{\infty}\exp_{*_\tau}(2n+1)iw,$$

$$\theta_{3*_\tau}(w) = \sum_{n=-\infty}^{\infty}\exp_{*_\tau} 2niw,$$

$$\theta_{4*_\tau}(w) = \sum_{n=-\infty}^{\infty}(-1)^n \exp_{*_\tau} 2niw$$

We see

$$\exp_{*_\tau} 2iw *_\tau \theta_{k*_\tau}(w) = \theta_{k*_\tau}(w), \quad (k = 2, 3)$$
$$\exp_{*_\tau} 2iw *_\tau \theta_{k*_\tau}(w) = -\theta_{k*_\tau}(w), \quad (k = 1, 4)$$

Then using $\exp_{*_\tau} 2iw = e^{-\tau}e^{2iw}$ and the product formula directly we have the quasi-periodicity

$$e^{2iw-\tau}\theta_{k*_\tau}(w + i\tau) = \theta_{k*_\tau}(w), \quad (k = 2, 3)$$
$$e^{2iw-\tau}\theta_{k*_\tau}(w + i\tau) = -\theta_{k*_\tau}(w), \quad (k = 1, 4)$$

Star delta function

For τ with $\Re\tau > 0$, we consider

$$\delta_{*_\tau}(w-a) = \int_{-\infty}^{\infty} e_{*_\tau}^{it(w-a)} dt, \quad \forall a \in \mathbb{C}$$

We see easily for any star polynomial

$$p_*(w) = \sum_k a_k w_*^k, \quad a_k \in \mathbb{C}$$

it holds

$$p_*(w) *_\tau \delta_{*_\tau}(w-a) = p(a)$$

Using star delta functions we have

$$\theta_{1*_\tau} = \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\delta_{*_\tau}(w + \tfrac{\pi}{2} + n\pi)$$

$$\theta_{2*_\tau} = \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\delta_{*_\tau}(w + n\pi)$$

$$\theta_{3*_\tau} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta_{*_\tau}(w + n\pi)$$

$$\theta_{4*_\tau} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta_{*_\tau}(w + \tfrac{\pi}{2} + n\pi)$$

By a direct caluculation we have

$$\delta_{*_\tau}(w-a) = \frac{2\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\tfrac{1}{\tau}(w-a)^2\right)$$

We see $\theta_{3*_\tau}(w) = \theta_3(w,\tau)$ Then we have

$$\theta_3(w,\tau) = \sum_n \delta_{*_\tau}(w+n\pi)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\tau}} \sum_n \exp\left(-\tfrac{1}{\tau}(w+n\pi)^2\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\tau}} e^{-(1/\tau)w^2} \theta_3(2\pi w/i\tau, \pi^2/\tau)$$

2.3. Clifford algebra

In this subsection, we construct a Clifford algebra by means of the star exponential $\exp_* t(\frac{2u*v}{i\hbar})$ for certain K. In what follows, we decsribe a rough sketch of the construction.

We recall the star exponential for quadratic case.

Quadratic case

First we consider a generic point in $\mathcal{S}_C(2)$

$$K = \begin{pmatrix} \tau' & \kappa \\ \kappa & \tau \end{pmatrix} \in \mathcal{S}_C(2)$$

In the star product $*_K$ algebra, we write the generator $u = u_1, v = u_2$ satisfying

$$[u, v]_{*_K} = -i\hbar$$

Then the star exponential of $H = 2u * v$ is explicitly given at a general point K as

$$\exp_{*_K} t\left(\frac{2u * v}{i\hbar}\right) = \frac{2e^{-t}}{\sqrt{D}} \exp\left[\frac{e^t - e^{-t}}{i\hbar D}\Big((e^t - e^{-t})\tau u^2 + 2\triangle uv + (e^t - e^{-t})\tau' v^2\Big)\right]$$

where

$$D = \triangle^2 - (e^t - e^{-t})\tau'\tau, \ \triangle = e^t + e^{-t} - \kappa(e^t - e^{-t})$$

In the sequel, **we assum** $\tau' = 0$, that is, we take a point

$$K = \begin{pmatrix} 0 & \kappa \\ \kappa & \tau \end{pmatrix}$$

Then $\sqrt{D} = \triangle$.

We have a limit

$$\lim_{t\to-\infty} \varpi_{00} = \exp_{*_K} t\left(\frac{2u*v}{i\hbar}\right) = \frac{2}{1+\kappa}\exp\left(-\frac{1}{i\hbar(1+\kappa)}(2uv - \frac{\tau}{1+\kappa}u^2)\right) \qquad (13)$$

which we call a **vacuum**.
In fact, we have

Lemma

i) $\varpi_{00} *_K \varpi_{00} = \varpi_{00}$
ii) $v *_K \varpi_{00} = \varpi_{00} *_K u = 0.$

Putting $t = \pi i$, we have the identity

$$\exp_{*_K} \pi i \left(\frac{2u * v}{i\hbar} \right) = 1 \tag{14}$$

Using

$$v *_K (u *_K v) = (v *_K u) *_K v = (u *_K v + i\hbar) *_K u$$

we see that the star exponential satisfies

$$\begin{aligned} v *_K \exp_{*_K} t\left(\frac{2u * v}{i\hbar}\right) &= \exp_{*_K} t\left(\frac{2v * u}{i\hbar}\right) *_K v \\ &= \exp_{*_K} t\left(\frac{2u * v + 2i\hbar}{i\hbar}\right) *_K v \\ &= e^{2t} \exp_{*_K} t\left(\frac{2u * v}{i\hbar}\right) *_K v \end{aligned}$$

Then the integral $\frac{1}{2}\int_{-\infty}^{0}\exp_{*_K} t(\frac{2v*u}{i\hbar})dt$ converges and then we define

$$\frac{1}{2}\int_{-\infty}^{0}\exp_{*_K} t\left(\frac{2v*u}{i\hbar}\right)dt = (v *_K u)_+^{-1} \tag{15}$$

and

$$\mathring{v} = u *_K (v *_K u)_+^{-1}. \tag{16}$$

Then we have

Lemma

The element $\mathring{v}$ is the right inverse of v satisfying

$$v *_K \mathring{v} = 1, \quad \mathring{v} *_K v = 1 - \varpi_{00}$$

Now we fix an integer l. By putting

$$t = t_l = \frac{\pi i}{2^l}$$

we obtain 2^l roots of the unity

$$\Omega_l = \exp_{*_K} \frac{\pi i}{2^l}\left(\frac{2u*v}{i\hbar}\right), \ \varpi_l = \exp 2\left(\frac{\pi i}{2^l}\right) \tag{17}$$

such that

$$\Omega_{l *_K}^{2^l} = \underbrace{\Omega_l *_K \cdots *_K \Omega_l}_{2^l} = 1, \ \varpi_l^{2^l} = 1$$

Then we have

Lemma

These satisfy

$$\Omega_{l *_K}^k *_K u_{*_K}^m *_K \varpi_{00} *_K v_{*_K}^m = \varpi_l^{km} u_{*_K}^m *_K \varpi_{00} *_K v_{*_K}^m$$

Now we take appropriate complex numbers $a_0, a_1, \cdots, a_{2^l-1}$ so that an element

$$E = \sum_{k=0}^{2^l-1} a_k \Omega^k_{l*_K}$$

satisfies the identies

$$E *_K u^m_{*_K} *_K \varpi_{00} *_K v^m_{*_K} = \begin{cases} *_K u^m_{*_K} *_K \varpi_{00} *_K v^m_{*_K} & \cdots 0 \le m \le 2^{l-1} - 1 \\ 0 & \cdots 2^{l-1} \le m \le 2^l - 1 \end{cases}$$

We see this is equivalent to

$$\sum_{k=0}^{2^l-1} a_k \varpi_l^{km} = \begin{cases} 1 \cdots 0 \leq m \leq 2^{l-1} - 1 \\ 0 \cdots 2^{l-1} \leq m \leq 2^l - 1 \end{cases}$$

The complex numbers $a_0, a_1, \cdots, a_{2^l-1}$ are uniquely determined by these equations. Then we have

Lemma

The element E satisfies

$$E *_K E = 1$$

and the element $F = 1 - E$ satisfies

$$F *_K F = 1, \ E *_K F = F *_K E = 0$$

Further we have

Lemma

$$E *_K (v)^{2^{l-1}}_{*_K} = (v)^{2^{l-1}}_{*_K} *_K F, \quad (\overset{\circ}{v})^{2^{l-1}}_{*_K} *_K F = E * (\overset{\circ}{v})^{2^{l-1}}_{*_K}$$

where $(v)^{2^{l-1}}_{*_K} = \underbrace{v *_K \cdots *_K v}_{2^{l-1}}$ *and* $(\overset{\circ}{v})^{2^{l-1}}_{*_K} = \underbrace{\overset{\circ}{v} *_K \cdots *_K \overset{\circ}{v}}_{2^{l-1}}$

Now we set

$$\xi = E *_K (v)^{2^{l-1}}_{*_K}, \ \eta = (\mathring{v})^{2^{l-1}}_{*_K} *_K F$$

Then we have

Theorem

The elements ξ and η of the $*_K$ product algebra satisfies the identities

$$\xi *_K \xi = \eta *_K \eta = 0$$
$$\xi *_K \eta + \xi *_K \eta = 1$$

[OM] H. Omori, Y. Maeda, *Quantum Calculus (in Japanese)*, Springer, 2004

[TY] Tomihisa T. and Yoshioka A., *Star Products and Star Exponentials*, J. Geometry and Symmetry in Physics, **19** (2010) 99–111.

[YM] Yoshioka A., T. Matsumoto, *Path integral for Star Exponential functons of quadratic forms*, In: Geometry, Integrability and Quantization. I. M. Mladenov, G. L. Naber (Eds), **4**, Coral Press Scientific Publishing, Sofia 2003, pp 330–340,