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E弦理論とNekrasov公式

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Seiberg-Witten解

(Seiberg-Witten '94)

4次元 $\mathcal{N}=2$ 超対称ゲージ理論の低エネルギー厳密解

低エネルギー有効理論： $\mathbf{U(1)}^n$ ゲージ理論

$$\mathcal{L}_{\text{eff}} = \frac{1}{8\pi}\mathrm{Im}\left[\int d^4\theta \frac{\partial F_0(A)}{\partial A^i}\bar{A}^i + \int d^2\theta \frac{1}{2}\frac{\partial^2 F_0(A)}{\partial A^i \partial A^j} W_\alpha^i W^{\alpha j}\right]$$

プレポテンシャル $F_0(a_1, \ldots, a_n)$ ：正則関数

例：pure SU(2)理論

$$F_0(a) = \underbrace{\boxed{\frac{1}{2}\tau_0 a^2}}_{\text{classical}} + \underbrace{\boxed{\frac{i}{\pi}a^2 \log\frac{a^2}{\Lambda^2}}}_{\text{1-loop}} + \underbrace{\boxed{\frac{1}{2\pi i}a^2\sum_{k=1}^{\infty} F_{0,k}\left(\frac{\Lambda}{a}\right)^{4k}}}_{\text{instanton part}}$$

Nekrasov partition function

(Nekrasov '02)

(for pure SU(N) theory; instanton part; $\epsilon_1 = -\epsilon_2 = \hbar$)

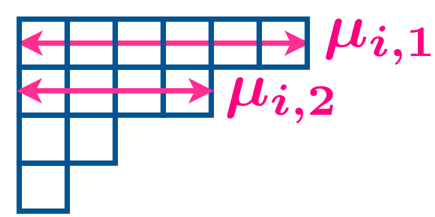
for 5D theory (in $\mathbb{R}^4 \times S^1$)

$$Z = \sum_{R} \Lambda^{|R|} \prod_{i,j=1}^{N} \prod_{k,l=1}^{\infty} \frac{\sinh \beta \left(a_{ij} + \hbar(\mu_{i,k} - \mu_{j,l} + l - k)\right)}{\sinh \beta \left(a_{ij} + \hbar(l - k)\right)}$$

$$R = (R_1, \ldots, R_N)$$

R_i : partition

$(a_{ij} := a_i - a_j)$



$$Z = \exp \sum_{g=0}^{\infty} F_g \hbar^{2g-2}$$

$\Rightarrow$ F_0 : prepotential

(Nekrasov-Okounkov '03)
(Nakajima-Yoshioka '03)

Seiberg-Witten解の一般化

- ゲージ群の一般化
- 物質場の一般化

  物質場なし（pure super Yang-Mills)

  基本表現（いくつか加えられる）

  随伴表現（一つまで）

  その他

- 高次元化 (5D = $\mathbb{R}^4 \times \boldsymbol{S}^1$, 6D = $\mathbb{R}^4 \times \boldsymbol{T}^2$)

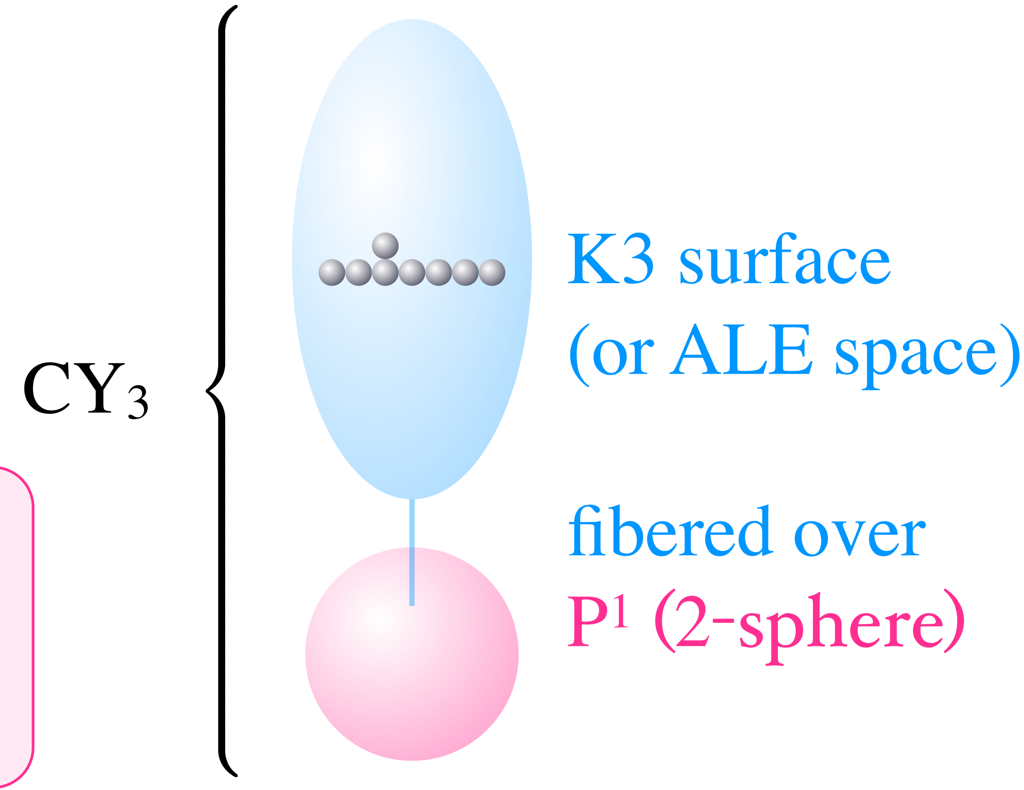
これら一般の場合：弦理論を用いた解析が非常に有効

弦理論による4次元 $\mathcal{N}=2$ ゲージ理論の実現

- Calabi-Yauコンパクト化

ADE型特異点

⇨ ADE型ゲージ場



CY$_3$ 内の正則2サイクルに巻き付いた2ブレーン

⇨ BPS状態

IIA string theory on CY$_3$ ⇨ 4次元ゲージ理論

M-theory on CY$_3$ ⇨ 5次元ゲージ理論

F-theory on CY$_3$ ⇨ 6次元場の理論

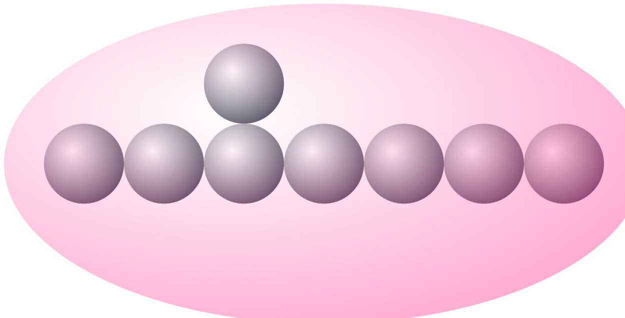
プレポテンシャル ＝ BPS分配関数 ＝ 位相的弦振幅 (g=0)

BPS states counting in M-theory on CY_3 (Gopakumar-Vafa '98)

5D $\mathcal{N}=1$ theory in $\mathbb{R}^4 \times S^1$

BPS states

= M2-branes wrapped over holomorphic 2-cycles in CY_3



Counting of BPS states = Counting of holomorphic 2-cycles

Seiberg-Witten prepotential of 5D $\mathcal{N}=1$ theory in $\mathbb{R}^4 \times S^1$

||

BPS partition function

||

(genus-zero) topological string amplitude of the CY_3

今日の話

ゲージ群の階数が1の場合（～SU(2)ゲージ理論）に限定し、

物質場の数を増やす拡張（４次元理論の範囲を超えて）

＝フレーバー対称性の拡張（ADE型）を考える

Nekrasov分配関数の知られている範囲を超えて
プレポテンシャル・分配関数を理解したい

- 弦理論からの興味：E弦理論

  6次元で最小の場の構成を持つ非自明超対称場の理論

- 有理楕円曲面、del Pezzo曲面の系列との綺麗な対応

  非トーリックCalabi-Yau多様体上の位相的弦振幅

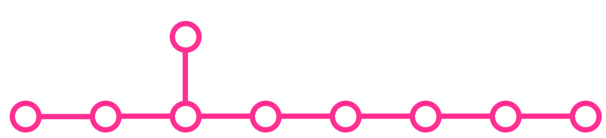
Flavor symmetries of possible effective U(1) theories

$$\begin{tikzcd}
\text{6D} & \hat{E}_8 \arrow[dd] & & & & & & & & \\
& & & & & & & & E_1 \arrow[r] & E_0 \\
\text{5D} & E_8 \arrow[r] \arrow[dd] & E_7 \arrow[r] \arrow[dd] & E_6 \arrow[r] \arrow[dd] & E_5 \arrow[r] \arrow[dd] & E_4 \arrow[r] \arrow[dd] & E_3 \arrow[r] \arrow[dd] & E_2 \arrow[ur] \arrow[dr] \arrow[dd] & & \\
& & & & & & & & \tilde{E}_1 \arrow[d] & \\
\text{4D} & E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[r] & D_4 \arrow[r] & D_3 \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d] & D_1 \arrow[r] \arrow[d] & D_0 & \\
& & & & & A_2 \arrow[r] & A_1 \arrow[r] & A_0 & &
\end{tikzcd}$$

(Minahan-Nemeschansky '96)

(Argyres-Plesser-Seiberg-Witten '95)

$$E_9 = \hat{E}_8$$



Flavor symmetries of possible effective U(1) theories

- Seiberg-Witten curve --- known for all theories

(Seiberg-Witten '94) (Minahan-Nemeschansky-Warner '97) (Eguchi-K.S. '02)

- BPS partition function (including higher genus parts):

6D The $\hat{E}_8$ theory needs further investigation.

$$
\begin{tikzcd}
\text{6D} & \hat{E}_8 \arrow[d] & & & & & & & & \\
\text{5D} & E_8 \arrow[r]\arrow[dd] & E_7 \arrow[r]\arrow[dd] & E_6 \arrow[r]\arrow[dd] & E_5 \arrow[r]\arrow[dd] & E_4 \arrow[r]\arrow[dd] & E_3 \arrow[r]\arrow[dd] & E_2 \arrow[r]\arrow[dr]\arrow[dd] & E_1 \arrow[r] & E_0 \\
 & & & & & & & & \tilde{E}_1 \arrow[d] & \\
\text{4D} & E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[r] & D_4 \arrow[r] & D_3 \arrow[r]\arrow[d] & D_2 \arrow[r]\arrow[d] & D_1 \arrow[r]\arrow[d] & D_0 & \\
 & & & & & A_2 \arrow[r] & A_1 \arrow[r] & A_0 & &
\end{tikzcd}
$$

Partition functions are constructed by "the ruled vertex formalism"

Nekrasov partition functions are known for these theories.

(Diaconescu-Florea '05) (Konishi-Minabe '06)

String Theory predicts the existence of nontrivial QFTs in 6D.

- The worldvolume theory of multiple M5 branes

  --- (2,0) SUSY (16 supercharges)

What about theories with (1,0) SUSY?

- Heterotic string theories on K3 (16 $\longrightarrow$ 8 supercharges)

What happens when instantons in K3 shrink to zero size?

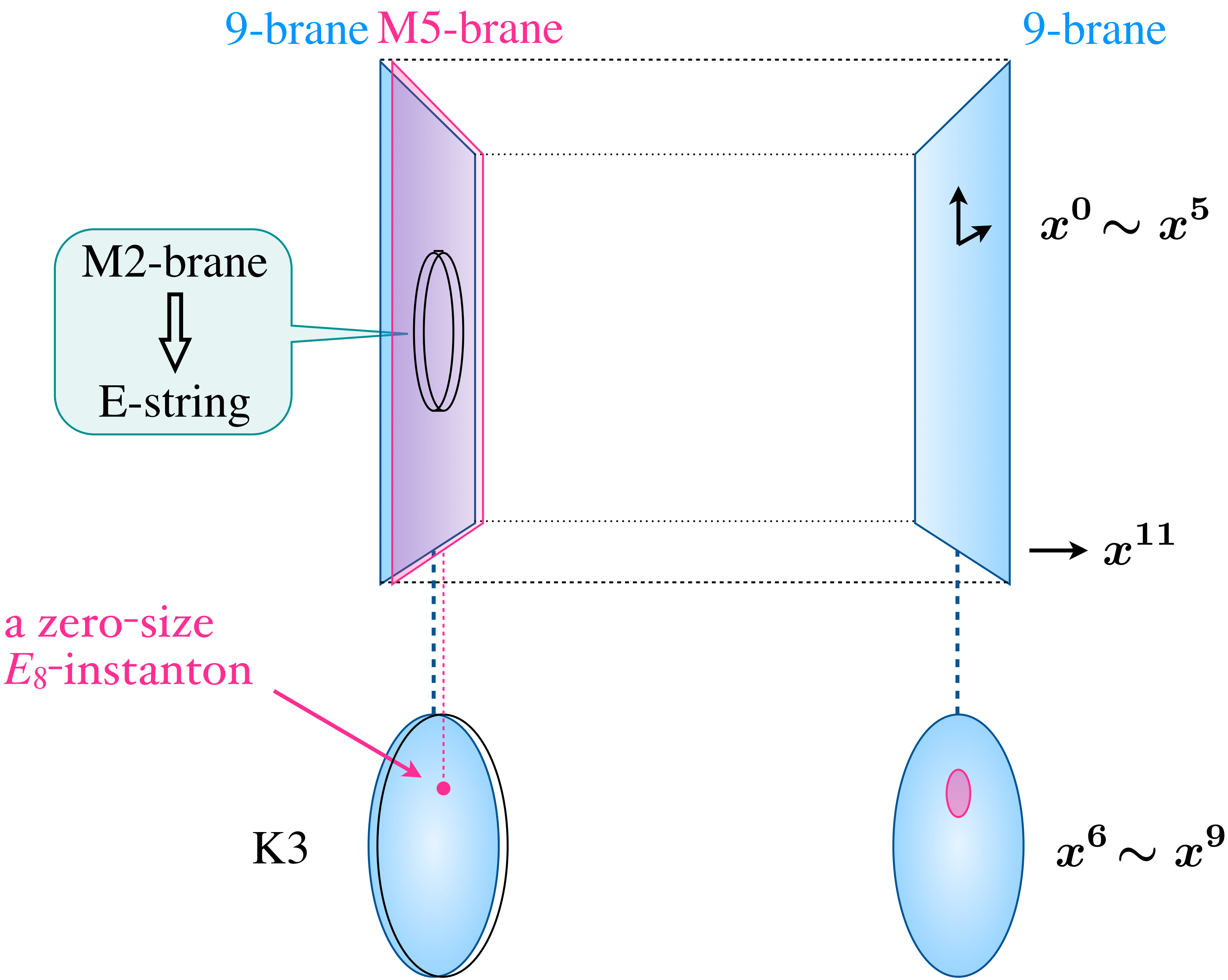
- Small SO(32) instantons

  $\Rightarrow$ extra Sp(n) gauge symmetry (Witten '95)

  $\Leftrightarrow$ worldvolume theory of n type I D5-branes

- What about small E_8 instantons?

M-theory description of the $E_8 \times E_8$ heterotic string theory on K3



E-string theory

(Ganor-Hanany '96) (Seiberg-Witten '96) (Klemm-Mayr-Vafa '96) (Ganor-Morrison-Seiberg '96) (Minahan-Nemeschansky-Vafa-Warner '98)

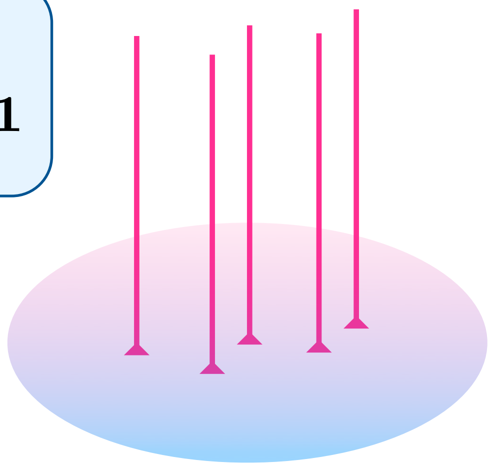
- 6D (1,0)-supersymmetric local QFT
- decoupled from gravity
- no vector multiplets
- Coulomb branch --- a tensor multiplet
- Higgs branch $\cong$ the moduli space of an E_8 instanton
- fundamental excitations --- strings
- global E_8 symmetry

The simplest interacting QFT with (1,0) SUSY in 6D!?

del Pezzo surfaces $\mathcal{B}_n (= S_{9-n}) \quad (n \leq 8)$

- complex surfaces with $c_1 > 0$

obtained by blowing up n points in $\mathbb{P}^2$
or (n-1) points in $\mathbb{P}^1 \times \mathbb{P}^1$

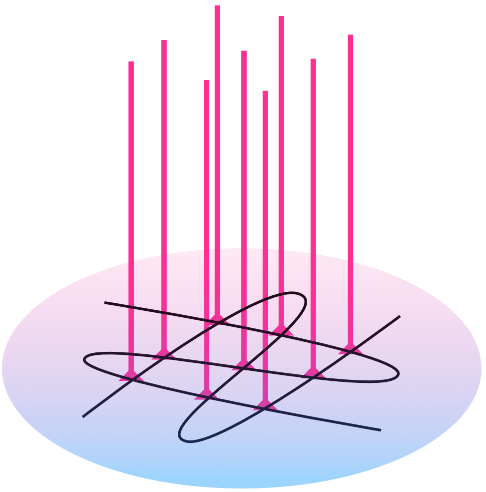


$\frac{1}{2}$K3 surfaces

= rational elliptic surfaces

= almost del Pezzo surfaces $\mathcal{B}_9$

obtained by blowing up 9 base points
of a pencil of cubic curves in $\mathbb{P}^2$



5D $\mathcal{N}=1$ SU(2) gauge theory with an E_n flavor symmetry

$$\boldsymbol{E_9} \to E_8 \to E_7 \to E_6 \to E_5 \to E_4 \to E_3 \to E_2 \begin{array}{l} \nearrow E_1 \to E_0 \\ \searrow \tilde{E}_1 \end{array}$$

$= \hat{E}_8$

del Pezzo surfaces

$= \mathbb{F}_1$ $\quad$ $= \mathbb{P}^2$

$$\boldsymbol{\mathcal{B}_9} \to \mathcal{B}_8 \to \mathcal{B}_7 \to \mathcal{B}_6 \to \mathcal{B}_5 \to \mathcal{B}_4 \to \mathcal{B}_3 \to \mathcal{B}_2 \begin{array}{l} \nearrow \mathcal{B}_1 \to \mathcal{B}_0 \\ \searrow \tilde{\mathcal{B}}_1 \end{array}$$

$= \frac{1}{2}\mathrm{K3}$

$= \mathbb{F}_0$

$= \mathbb{P}^1 \times \mathbb{P}^1$

M-theory on local $\mathcal{B}_n$

the total space of the canonical bundle of $\mathcal{B}_n$

$\Downarrow$ (low energy)

5D $\mathcal{N}=1$ SU(2) gauge theory with an E_n flavor symmetry

2nd homology of $\frac{1}{2}$K3 surfaces

	$\frac{1}{2}$K3	K3
$h^{0,0}$	1	1
$h^{1,0}$ $h^{0,1}$	0 0	0 0
$h^{2,0}$ $h^{1,1}$ $h^{0,2}$	0 10 0	1 20 1
$h^{2,1}$ $h^{1,2}$	0 0	0 0
$h^{2,2}$	1	1

e_0 : a line in $\mathbb{P}^2$ $\quad$ $e_j,\ j = 1, \ldots, 9$: exceptional curves

$$e_i \cdot e_j = \mathrm{diag}(+1, -1, \ldots, -1)$$

$$H_2(\tfrac{1}{2}\mathrm{K3}, \mathbb{Z}) = \Gamma^{9,1}$$

odd, self-dual lattice

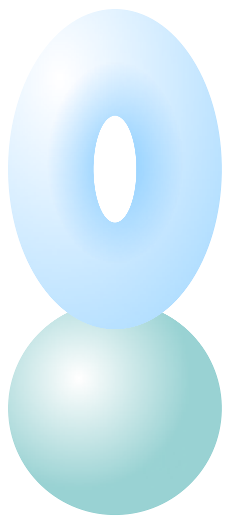
$$= \Gamma^{1,1} \oplus \Gamma_8$$

2nd homology of $\frac{1}{2}$K3 surfaces

$$\begin{aligned} H_2(\tfrac{1}{2}\mathbf{K3}, \mathbb{Z}) &= \Gamma^{9,1} \\ &= \Gamma^{1,1} \oplus \Gamma_8 \end{aligned}$$

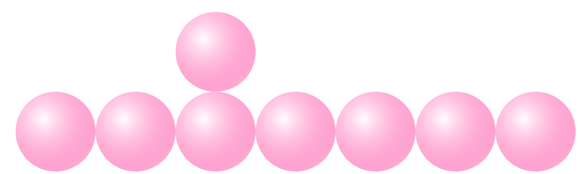
odd, self-dual lattice

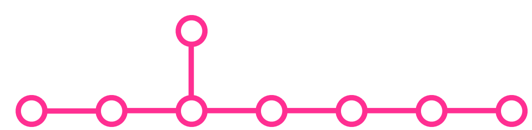
root lattice of E_8



E (fiber)

B (base)



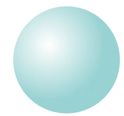
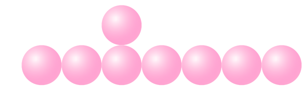


$$B \cdot B = -1$$
$$B \cdot E = 1$$
$$E \cdot E = 0$$

$$\alpha_i \cdot \alpha_j = -C_{ij}^{E_8}$$

Topological string amplitudes for the local $\frac{1}{2}$K3 surface

$$F(\varphi, \tau, \vec{\mu}; x) =$$

φ  τ $\vec{\mu}$ 

$$\sum_{r=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{\vec{\lambda} \in P_+} \sum_{\vec{w} \in \mathcal{O}_{\vec{\lambda}}} N^r_{n,k,\vec{\lambda}} \sum_{m=1}^{\infty} \frac{1}{m} \left(2 \sin \frac{mx}{2}\right)^{2r-2} e^{2\pi i m (n\varphi + k\tau + \vec{w}\cdot\vec{\mu})}$$

(Gopakumar-Vafa '98)

$N^r_{n,k,\vec{\lambda}}$: multiplicity of BPS states (Gopakumar-Vafa inv.)

n : winding number

k : momentum

$\vec{\lambda}$: weight of E_8

r : genus

$$F = \sum_{g=0}^{\infty} F_g x^{2g-2}$$

F_g : topological string amplitude at genus g

Prepotential = BPS partition function of wound E-strings

$$F_0(\varphi,\tau) = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} N_{n,k} \sum_{m=1}^{\infty} \frac{1}{m^3} e^{2\pi i m(n\varphi+k\tau)}$$

n : winding number $\qquad$ k : momentum

$N_{n,k}$: BPS multiplicity (instanton number)

(Klemm-Mayr-Vafa '96)

n \ k	0	1	2	3	4	5	$\cdots$
1	1	252	5130	54760	419895	2587788	
2	0	0	−9252	−673760	−20534040	−389320128	
3	0	0	0	848628	115243155	6499779552	
4	0	0	0	0	−114265008	−23064530112	
5	0	0	0	0	0	18958064400	
$\vdots$							$\ddots$

Winding number expansion of the prepotential

$$F_0(\varphi, \tau) = \sum_{n=1}^{\infty} Q^n Z_n(\tau) \qquad Q := e^{2\pi i \varphi + \pi i \tau}$$

φ : scalar vev (tension) $\quad$ τ : complex modulus of T^2

$$Z_1 = \frac{E_4}{\eta^{12}}, \qquad Z_2 = \frac{E_2 E_4^2 + 2E_4 E_6}{24\eta^{24}}, \qquad \cdots$$

$$Z_n = \frac{P_{6n-2}(E_2, E_4, E_6)}{\eta^{12n}}$$

$E_{2n}(\tau)$: Eisenstein series $\quad$ $\eta(\tau)$: Dedekind eta function

- Z_n at low orders can be determined by using the Seiberg-Witten curve or the modular anomaly equation

(Minahan-Nemeschansky-Warner '97)

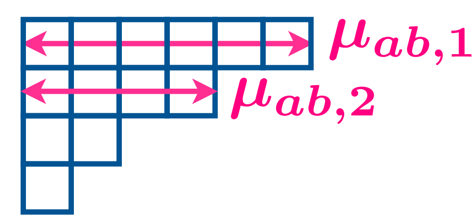
Nekrasov-type expression for the prepotential

(K.S. '12)

$$F_0 = \left(2\hbar^2 \ln \mathcal{Z}\right)\Big|_{\hbar=0}$$

$$\mathcal{Z} = \sum_{\vec{R}} Q^{|\vec{R}|} \prod_{a,b,c,d} \prod_{(i,j)\in R_{ab}} \frac{\vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar, \tau\right)^2}{\vartheta_{1-|a-c|,1-|b-d|}\left(\frac{1}{2\pi}h_{ab,cd}(i,j)\hbar, \tau\right)^2}$$

$\vec{R} = (R_{11}, R_{10}, R_{00}, R_{01})$ $\quad$ R_{ab} : partition

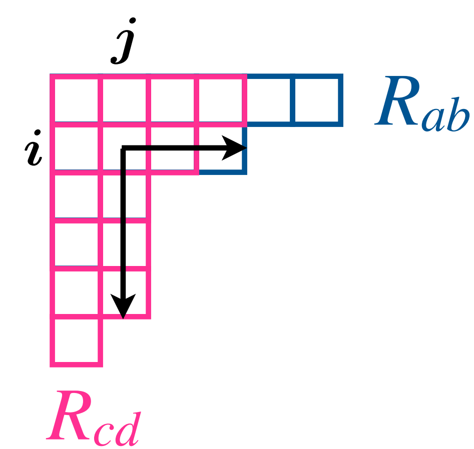


$a, b, c, d = 0, 1$

$\vartheta_{ab}(z, \tau)$: Jacobi theta functions

$h_{ab,cd}(i,j) := \mu_{ab,i} + \mu^{\vee}_{cd,j} - i - j + 1$

(relative hook-length)



The prepotential represents $g = 0$ topological string amplitude

$$F_0 = F_0^{\frac{1}{2}\mathrm{K3}}$$

However,

$$\mathcal{Z} \neq \mathcal{Z}^{\frac{1}{2}\mathrm{K3}}$$

$$:= \exp\left(\sum_{g=0}^{\infty} \hbar^{2g-2} F_g^{\frac{1}{2}\mathrm{K3}}\right)$$

Difference of modular anomalies

$$\partial_{E_2}\mathcal{Z} = \frac{1}{12}\hbar^2 \partial_\phi^2 \mathcal{Z}$$

$$\partial_{E_2}\mathcal{Z}^{\frac{1}{2}\mathrm{K3}} = \frac{1}{24}\hbar^2 \partial_\phi(\partial_\phi + 1)\mathcal{Z}^{\frac{1}{2}\mathrm{K3}}$$

(Hosono-Saito-Takahashi '99)

$\Downarrow$ $\Downarrow$

$$\partial_{E_2} F_0 = \frac{1}{24}\left(\partial_\phi F_0\right)^2$$

$$\partial_\phi = \frac{1}{2\pi i}\partial_\varphi = Q\partial_Q$$

Compactification with nontrivial Wilson line parameters

- One can introduce eight Wilson line parameters

$$\vec{\mu} = (m_1, \ldots, m_8) \in \mathbb{C}^8$$

and break the E_8 global symmetry

- Today we consider the cases with

$$\vec{\mu} = (m_1, m_2, m_3, m_4, m_1, m_2, m_3, m_4)$$

$\Longleftrightarrow$

$$\vec{\mu} = (0, 0, 0, 0, m_1 + m_2, m_1 - m_2, m_3 + m_4, m_3 - m_4)$$

- Notation

$$\vec{m} = (m_1, m_2, m_3, m_4)$$

Nekrasov-type expression

(K.S. '12)

$$\mathcal{Z} = \sum_{\vec{R}^{(4)}} Q^{|\vec{R}^{(4)}|} \prod_{a,b,c,d} \prod_{(i,j)\in R_{ab}} \frac{\vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar + m_{cd}, \tau\right) \vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar - m_{cd}, \tau\right)}{\vartheta_{1-|a-c|,1-|b-d|}\left(\frac{1}{2\pi} h_{ab,cd}(i,j)\hbar, \tau\right)^2}$$

$$\boxed{\vec{m} = (m_1, m_2, m_3, m_4) = (m_{11}, m_{10}, m_{00}, m_{01})}$$

$$F_0 = \left(2\hbar^2 \ln \mathcal{Z}\right)\Big|_{\hbar=0}$$

This is in perfect agreement with the prepotential computed from the Seiberg-Witten curve (checked up to order Q^{10}).

- Elliptic analogue of the Nekrasov partition function for the SU(N) gauge theory with $N_{\mathrm{f}} = 2N$ flavors

(Nekrasov '02) (Hollowood-Iqbal-Vafa '03)

$$\mathcal{Z}^{\mathrm{SU}(N)}_{N_{\mathrm{f}}=2N}(\hbar;\varphi,\tau;a_1,\ldots,a_N;m_1,\ldots,m_{2N})$$

$$:= \sum_{\vec{R}^{(N)}} \left(-e^{2\pi i\varphi}\right)^{|\vec{R}^{(N)}|} \prod_{k=1}^{N} \prod_{(i,j)\in R_k} \frac{\prod_{n=1}^{2N} \vartheta_1\left(a_k + m_n + \frac{1}{2\pi}(j-i)\hbar, \tau\right)}{\prod_{l=1}^{N} \vartheta_1\left(a_k - a_l + \frac{1}{2\pi}h_{kl}(i,j)\hbar, \tau\right)^2}$$

- Our formula can be expressed as a special case of this function

$$\mathcal{Z} = \mathcal{Z}^{\mathrm{SU}(4)}_{N_{\mathrm{f}}=8}\left(\hbar;\varphi,\tau;0,\frac{1}{2},-\frac{1+\tau}{2},\frac{\tau}{2};m_1,m_2,m_3,m_4,-m_1,-m_2,-m_3,-m_4\right)$$

Global symmetries and counting of holomorphic curves in CY_3

Wilson line parameters	unbroken global symmetry	local geometry
$\vec{m} = (0,0,0,0)$	E_8	E_8 del Pezzo
$\vec{m} = \left(0,0,0,\frac{1}{2}\right)$	$E_7 \oplus A_1$	E_7 del Pezzo
$\vec{m} = \left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$E_6 \oplus A_2$	E_6 del Pezzo
$\vec{m} = \left(0,\frac{1}{4},\frac{1}{4},\frac{1}{2}\right)$	$E_5 \oplus A_3$	E_5 del Pezzo
$\vec{m} = \left(0,0,\frac{1}{2},\frac{1}{2}\right)$	D_8	$\mathbb{P}^1 \times \mathbb{P}^1$

- Example: the $E_7 \oplus A_1$ case

$$\mathcal{Z} = \sum_{\vec{R}^{(2)}} Q^{|\vec{R}^{(2)}|} \prod_{k,l=1}^{2} \prod_{(i,j)\in R_k} \frac{\vartheta_{k+2}\left(\frac{1}{2\pi}(j-i)\hbar, \tau\right)^2}{\vartheta_{|k-l|+1}\left(\frac{1}{2\pi}h_{kl}(i,j)\hbar, \tau\right)^2}$$

$$\left(Q = e^{2\pi i \varphi + \pi i \tau}\right)$$

$$= \mathcal{Z}^{\mathrm{SU}(2)}_{N_f=4}\left(\hbar; \varphi, \tau; -\frac{1+\tau}{2}, \frac{\tau}{2}; 0,0,0,0\right)$$

$$F_0(\varphi,\tau) = \left(2\hbar^2 \ln \mathcal{Z}\right)\Big|_{\hbar=0}$$

$$= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} N_{n,k} \sum_{m=1}^{\infty} \frac{1}{m^3} e^{2\pi i m(n\varphi + k\tau)}$$

$2\times$ BPS multiplicities

$\Leftrightarrow$ Instanton numbers of local E_7 del Pezzo surface

Summary

- We have found a Nekrasov-type expression for the Seiberg-Witten prepotential for the E-string theory.

- We have generalized the expression to the cases with four Wilson line parameters.

- Our formulas give very simple, closed expressions for the genus-zero topological string amplitudes of some basic non-toric Calabi-Yau threefolds.