AN OPERADIC MODEL FOR A MAPPING SPACE AND ITS ASSOCIATED SPECTRAL SEQUENCE

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1. Overview

Let X and Y be spaces (or simplicial sets) and $\mathcal{F}(X,Y)$ denote the mapping space. In [13], Haefliger has given a rational model for a mapping space $\mathcal{F}(X,Y)$ for which Y is a nilpotent space. Subsequently, Bousfield, Peterson and Smith [5] have constructed another rational model for a mapping space with a functorial way, more precisely, their model is obtained by using a division functor in the category of commutative differential $\mathbb{Z}$ graded algebras over the rational field. In the same paper, we are also aware of an interesting spectral sequence (henceforth BPS spectral sequence) converging to $H^*(\mathcal{F}(X,Y);\mathbb{Q})$, which is constructed with the algebraic model. Brown and Szczarba [6] have derived an accessible rational model for $\mathcal{F}(X,Y)$ by computing the division functor explicitly. The construction renders the model more computable. For more recent progress of the material, we refer the reader to [7], [8], [14], [18] and [19].

As for a p-adic model for a space, Mandell [20] has proved that the homotopy category of nilpotent, p-complete spaces of finite p-type is equivalent to a full subcategory of the homotopy category of algebras over an $\overline{\mathbb{F}}_p$-operad $\mathcal{E}$. Here $\overline{\mathbb{F}}_p$ denotes the closure of the finite field $\mathbb{F}_p$. This motivates us to construct an $\mathcal{E}$-algebra model for a mapping space $\mathcal{F}(X,Y)$. Recently, Fresse [12] has given such a model by means of a division functor in the category of algebras over an $\mathbb{F}_p$-operad under some finiteness condition on the homotopy group of X. One of the purposes of the paper [11] is to improve Fresse's model for a mapping space. Another one is to construct a spectral sequence converging to $H^*(\mathcal{F}(X,Y);\overline{\mathbb{F}}_p)$, which is regarded as a p-adic version of the BPS spectral sequence.

We recall briefly the algebraic model for a mapping space over an operad due to Fresse. Let $\mathcal{E}$ denote the Barratt-Eccles operad over a field $\mathbb{K}$, which is an E_∞-operad. Then we can regard the normalized cochain functor $C^*(-;\mathbb{K})$ as a functor from the category of simplicial sets to $\mathcal{E}$-Alg the category of $\mathcal{E}$-algebras ([2, 1.5],[21]). Let A be an $\mathcal{E}$-algebra and K an $\mathcal{E}$-coalgebra. The diagonal map on $\mathcal{E}$

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makes the dg-module $\mathrm{Hom}_{\mathbb{K}}(K,-)$ of homogeneous morphisms into an $\mathcal{E}$-algebra (see [12, 1.5] for details). We denote by $\mathcal{E}(\ ,\ )$ the hom set in $\mathcal{E}$-Alg.

Proposition 1.1. [12, 1.6.Proposition] *Let K be an $\mathcal{E}$-coalgebra. Then the functor $\mathrm{Hom}_{\mathbb{K}}(K,-)$ has a left adjoint. More explicitly, for A an $\mathcal{E}$-algebra, there is an $\mathcal{E}$-algebra $A \oslash K$ such that $\mathcal{E}(A \oslash K, -) \cong \mathcal{E}(A, \mathrm{Hom}_{\mathbb{K}}(K,-))$.*

Let K^* be an $\mathcal{E}$-algebra of finite type and K_* the $\mathcal{E}$-coalgebra which is the dual to K^*. Then, by definition, $A \oslash K_*$ is regarded as Lannes' functor $(A : K^*)_{\mathcal{E}\text{-Alg}}$ in the category of $\mathcal{E}$-algebras (see [22, 3.2 and 3.8] for the existence of the division functor, such as Lannes' T-functor). Moreover, if A is an almost free algebra $\mathcal{E}(V)$, then $A \oslash K$ is also an almost free algebra of the form $\mathcal{E}(V \otimes K)$. Since $\mathrm{Hom}_{\mathbb{K}}(K,-)$ preserves fibrations and acyclic fibrations, the total left derived functor $- \oslash^L K$ of $- \oslash K$ can be defined; that is, we have a natural bijection $\bar{h}\mathcal{E}(A \oslash^L K, -) \cong \bar{h}\mathcal{E}(A, \mathrm{Hom}_{\mathbb{K}}(K,-))$ for any $\mathcal{E}$-algebra A. Here $\bar{h}\mathcal{E}(\ ,\ )$ denotes the hom set in the homotopy category of $\mathcal{E}$-algebras. The functor $-\oslash^L K$ provides an $\mathcal{E}$-algebra model for a mapping space.

Theorem 1.2. [12, 1.10.Theorem] *Let X and Y be simplicial sets. We assume that X is finite and that $\pi_n(Y)$ is a finite p-group for $n \geq 0$. We have a quasi-isomorphism between $C^*(\mathcal{F}(X,Y);\mathbb{K})$ and $C^*(Y;\mathbb{K}) \oslash^L C_*(X;\mathbb{K})$, which is functorial with respect to X and Y.*

Henceforth, we work in the category of algebras over the Barratt-Eccles operad $\mathcal{E}$ defined in the field $\overline{\mathbb{F}}_p$. The chain and cochain complexes $C_*(X;\overline{\mathbb{F}}_p)$ and $C^*(X;\overline{\mathbb{F}}_p)$ are written as $C_*(X)$ and $C^*(X)$, respectively. In [11], we first show that $C^*(\mathcal{F}(X,Y))$ can be connected with $C^*(Y) \oslash^L C_*(X)$ by quasi-isomorphisms without assuming that $\pi_n(Y)$ is a finite p-group, subject to the connectedness of the mapping space $\mathcal{F}(X,Y)$. More precisely, the following theorem is established.

Theorem 1.3. [11, Theorem 1.3] *Let X be a finite simplicial set and Y a connected nilpotent simplicial set of finite type. Assume that the connectivity of Y is greater than or equal to the dimension of X. Then there exists an isomorphism between $C^*(\mathcal{F}(X,Y))$ and $C^*(Y) \oslash^L C_*(X)$, which is functorial with respect to X and Y, in the homotopy category of $\mathcal{E}$-algebras.*

As mentioned above, the functor $-\oslash K_*$ is regarded as Lannes' division functor $(- : K^*)_{\mathcal{E}\text{-Alg}}$. This fact enables us to construct a spectral sequence converging to the cohomology $H^*(\mathcal{F}(X,Y))$. In order to describe the spectral sequence more precisely, we recall that the generalized Steenrod algebra $\mathcal{B}$ is the free associative $\mathbb{F}_p$-algebra generated by the P^s and (if $p > 2$) the βB^s for $s \in \mathbb{Z}$ over the two sided ideal generated by the Adem relations (see [20, Section 11]). The result [20, Theorem 1.4] states that the quotient algebra $\mathcal{B}/(Id - P^0)$ is the usual Steenrod algebra $\mathcal{A}$. Let $\mathcal{K}$-$\overline{\mathbb{F}}_p$ be the category of unstable $\overline{\mathbb{F}}_p$-algebras over the generalized Steenrod algebra $\mathcal{B}$. We have a spectral sequence.

Theorem 1.4. [11, Theorem 1.4] (Compare with [5, Corollary 3.5]) *Let X be a finite simplicial set and Y a connected nilpotent simplicial set of finite type. Assume that the connectivity of Y is greater than or equal to the dimension of X. Then there exists a left-half plane spectral sequence $\{E_r, d_r\}$ with*

$$E_2^{s,*} \cong L_s(H^*(Y) : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$$

converging strongly to $H^*(\mathcal{F}(X,Y))$. Here $L_s(- : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ denotes the s^{th} left derived functor of the division functor $(- : H^*(X))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ in the category $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$. Moreover the spectral sequence is natural with respect to X and Y.

In what follows, we shall refer to the spectral sequence in Theorem 1.4 as the mod p BPS spectral sequence. For a $\mathcal{B}$-algebra B and a $\mathcal{B}$-algebra A of finite type, one can define the derived functor $L_s(B : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ using a simplicial resolution of B in the category $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$. Since the resolution is a complex in the category of unstable $\mathcal{B}$-modules, the functor $L_s(B : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ for any s inherits the $\mathcal{B}$-module structure from that of the complex. The same derived functor can be defined in the category $\mathcal{S}\text{-}\overline{\mathbb{F}}_p$ of unstable $\overline{\mathbb{F}}_p$-algebras over the usual Steenrod algebra $\mathcal{A}$. Observe that an object in $\mathcal{S}\text{-}\overline{\mathbb{F}}_p$ is regarded as one in $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$ with the natural projection $\mathcal{B} \to \mathcal{B}/(Id - P^0) = \mathcal{A}$. The following theorem allows us to work in the more familiar category $\mathcal{S}\text{-}\overline{\mathbb{F}}_p$ than $\mathcal{K}\text{-}\overline{\mathbb{F}}_p$ when computing the mod p BPS spectral sequence.

Theorem 1.5. [11, Theorem 1.5] *Let A and B be $\mathcal{A}$-algebras of finite type. Then $L_s(B : A)_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p}$ is isomorphic to $L_s(B : A)_{\mathcal{S}\text{-}\overline{\mathbb{F}}_p}$ as a $\mathcal{B}$-module for any s.*

This theorem implies that the mod p BPS spectral sequence is reducible in the second quadrant. Moreover we have

Assertion 1.6. *The mod p BPS spectral sequence possesses an unstable module structure on $\mathcal{B}$ and hence on $\mathcal{A}$.*

For the more precise statement concerning the Steenrod operations on the spectral sequence, see [11, Theorem 7.7].

Unfortunately, we are less successful in computing the BPS spectral sequence. However we firmly believe the spectral sequence to be of use in the study of mapping spaces. As for the edge homomorphism, we have an interesting example (Theorem 1.7 below). Let Y be a simply connected space whose cohomology is a polynomial algebra. Then the results [15, Remarks 3.4, 3.5] and [17, Theorem 1.6] enable one to determined explicitly the mod p cohomology of the free loop space $LY = \mathcal{F}(S^1, Y)$. To be exact, if $H^*(Y;\mathbb{F}_p) = \mathbb{F}_p[y_1, .., y_l]$, as an $H^*(Y;\mathbb{F}_p)$-algebra, then $H^*(LY;\mathbb{F}_p) \cong \mathbb{F}_p[y_1, .., y_l] \otimes \Lambda(\bar{y}_1, \bar{y}_2, .., \bar{y}_l)$ if $p \neq 2$, where $\deg \bar{y}_i = \deg y_i - 1$. In the case $p = 2$, we see that

$$H^*(LY;\mathbb{F}_2) \cong \mathbb{F}_2[y_1, .., y_l] \otimes \mathbb{F}_2[\bar{y}_1, \bar{y}_2, .., \bar{y}_l] \,/\, (\bar{y}_i^2 + \mathfrak{D}Sq^{\deg y_i - 1} y_i; i = 1, 2, .., l)$$

as an $H^*(Y;\mathbb{F}_2)$-algebra, for which $\mathfrak{D}$ is the derivation defined by $\mathfrak{D}(y_i) = \bar{y}_i$

Since the derivation $\mathfrak{D}$ is compatible with the Steenrod operations ([15, Remark 3.5]), we can determine explicitly the $\mathcal{A}$-algebra structure of $H^*(LY;\mathbb{F}_p)$ from that of the polynomial algebra $H^*(Y;\mathbb{F}_p)$ (see, for example, [15, Example 3.6]). The following theorem asserts that the $\mathcal{A}$-algebra structure of $H^*(LY;\overline{\mathbb{F}}_p)$ is also expressed via the Lannes' division functor.

Theorem 1.7. [11, Theorem 6.4] *Let Y be a simply-connected space whose mod p cohomology is a polynomial algebra. Then the edge homomorphism*

$$\mathrm{edge}_{(Y:S^1)} : (H^*(Y) : H^*(S^1))_{\mathcal{K}\text{-}\overline{\mathbb{F}}_p} \to H^*(LY)$$

is an isomorphism.

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