Toward Riemannian diffeology

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KSS '25: Toward Riemannian diffeology, `arXiv:2505.04170`

12–16 May 2025, 9^{th} EACAT
Hebei Normal University

§1. Diffeology and a diffeological space

Definition 1.1 (Souriau '80)

For a set X, a set $\mathcal{D}$ of functions $U \to X$ for each open set U in $\mathbb{R}^n$ and for each $n \in \mathbb{N}$ is a *diffeology* of X if the following three conditions hold:

(i) (Covering) Every constant map $U \to X$ for all open set $U \subset \mathbb{R}^n$ is in $\mathcal{D}$.

(ii) (Compatibility) If $U \to X$ is in $\mathcal{D}$, then for any smooth map $V \to U$ from an open set $V \subset \mathbb{R}^m$, the composite $V \to U \to X$ is also in $\mathcal{D}$.

(iii) (Locality) If $U = \cup_i U_i$ is an open cover and $U \to X$ is a map such that each restriction $U_i \to X$ is in $\mathcal{D}$, then the map $U \to X$ is in $\mathcal{D}$.

An element of a diffeology $\mathcal{D}$ is called a *plot* of X. A *diffeological space* $(X, \mathcal{D})$ consists of a set X and a diffeology $\mathcal{D}$ of X.

Definition 1.2

Let $(X, \mathcal{D}^X)$ and $(Y, \mathcal{D}^Y)$ be diffeological spaces. A map $X \to Y$ is *smooth* if for any plot $p \in \mathcal{D}^X$, the composite $f \circ p$ is in $\mathcal{D}^Y$.

FACT : The category of diffeological spaces **Diff** is complete, cocomplete and cartesian closed.

Examples

- Let M be a manifold. The underlying set M and the *standard diffeology* $\mathcal{D}^M_{\text{std}}$ give rise to a diffeological space $m(M) := (M, \mathcal{D}^M_{\text{std}})$, where $\mathcal{D}^M_{\text{std}}$ is defined to be the set of all smooth maps $U \to M$ from domains to M in the usual sense.
- For $Y \in \textbf{Top}$, the diffeological space $C(Y)$ consists of the underlying set Y and the diffeology $\mathcal{D}^Y := \{\text{continuous maps } P\colon U_P \to Y\}$.
- For $(X, \mathcal{D}^X) \in \textbf{Diff}$, the topological space $D(X)$ consists of the underlying set X and the topology $\mathcal{O}$ defined by

$$\mathcal{O} := \{O \subset X \mid P^{-1}(O) \text{ is an open set of } U_P \text{ for each } P \in \mathcal{D}^X\}$$

which is called the *D-topology* of X.

Overview of Diffeology (What can we do with diffeology?)

- The de Rham theory for diffeology (P. Iglesias-Zemmour '13, K. '20)
- Sheaf theory (Nezhad–Ahmadi '19, Krepski–Watts–Wolbert '24)
- (Abstract) Homotopy theory (Christensen–Wu '14, Shimakawa–Yoshida–Haraguchi '18, Kihara '19 '23,)
- Rational homotopy theory for non-simply connected diffeological spaces (with local systems in the sense of Gómez-Tato–Halperin–Tanré) (K. '24)

There exists a commutative diagram of functors

$$\begin{tikzcd}
 & & \mathbf{Sets}^{\Delta^{op}} \arrow[d, shift left, "| \ |_D"] \arrow[dr, shift left, "| \ |"] & \\
\mathbf{Mfd} \arrow[r, "j", "\text{fully faithful}"'] \arrow[rr, bend left, "m\text{:fully faithful}"] \arrow[rrr, bend right, "\text{forgetful}"'] & \mathbf{DifferentialSp}(\text{Sikorski '71}) \arrow[r, "k"] & \mathbf{Diff} \arrow[u, shift left, "S^D"] \arrow[u, phantom, "\vdash"] \arrow[r, shift left, "D"] \arrow[r, phantom, "\perp"] & \mathbf{Top}. \arrow[l, shift left, "C"] \arrow[ul, shift left, "S"] \arrow[ul, phantom, "\vdash"]
\end{tikzcd}$$

Riemannian notions?

§2. A framework of Riemannian diffeology

By using the functor $\widehat{T}_2 \colon \mathbf{Euc} \to \mathbf{Diff}$ defined by

$$\widehat{T}_2(U) := U \times \mathbb{R}^{\dim U} \times \mathbb{R}^{\dim U},$$

we have the functor $T_2 := \mathbb{L}\mathcal{Y}\widehat{T}_2 \colon \mathbf{Diff} \to \mathbf{Diff}$ via the left Kan extension along the Yoneda functor $\mathcal{Y}$. Observe that as a diffeological space,

$$T_2(X) \cong \operatorname*{colim}_{P \in \mathcal{D}} (U_P \times \mathbb{R}^{\dim U_P} \times \mathbb{R}^{\dim U_P}).$$

$$\begin{tikzcd}
U_P \times \mathbb{R}^{\dim U_P} \times \mathbb{R}^{\dim U_P} \arrow[d, "\pi_{\widehat{T}_2(U_P)}"'] \arrow[dr, "g(P)\ :\ \text{the composite}"] & \\
T_2(X) \arrow[r, "g"'] & \mathbb{R}.
\end{tikzcd} \tag{1}$$

Definition 2.1

A map $g \colon T_2(X) \to \mathbb{R}$ is a *weak Riemannian metric* on X if the composite $g \circ \pi_{\widehat{T}_2(U_P)}$ is a symmetric and positive covariant 2-tensor on U_P for each plot P of X, where $\pi_{\widehat{T}_2(U_P)}$ denotes the canonical map $\widehat{T}_2(U_P) \to T_2(X)$.

Remark 2.2

If M is a manifold, then $T_2(M) \cong T(M) \times_M T(M)$ as a diffeological space.

A weak Riemannian metric $g\colon T_2(X) \to \mathbb{R}$ on a diffeological space X gives a pseudodistance d of X by applying the usual procedure; that is, the pseudodistance $d\colon X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is defined by

$$d(x,y) = \inf_{\gamma \in \mathrm{Path}(X;x,y)} \ell(\gamma), \quad \text{where} \quad \ell(\gamma) = \int_0^1 (g(\gamma)_t(1,1))^{\frac{1}{2}} dt \qquad (2)$$

and $d(x,y) = \infty$ if there is no smooth path connecting x and y. Here $\mathrm{Path}(X;x,y)$ denotes the set of smooth paths γ with $\gamma(0) = x$ and $\gamma(1) = y$.

Theorem 2.3 (KSS '25)

Let $d\colon X \times X \to \mathbb{R}_{\geq 0}$ be the pseudodistance on a diffeological space X defined by a weak Riemannian metric $g\colon T_2(X) \to \mathbb{R}$. Then the D-topology of X is finer than the topology $\mathcal{O}_d$ defined by d; that is,

$$\text{the D-topology of } X \supset \mathcal{O}_d.$$

In particular, the function $d : D(X) \times D(X) \to \mathbb{R}$ is continuous.

The definiteness of a weak Riemannian metric

Let $(X, \mathcal{D})$ be a diffeological space. A subset $\mathcal{G} \subset \mathcal{D}$ is a *generating family* of $\mathcal{D}$ if each plot $P \in \mathcal{D}$ locally factors through an element $Q \in \mathcal{G}$; that is,

$P|_W = Q \circ h$ for some smooth $h : W \to \mathrm{dom}(Q)$, an open subset $W \subset \mathrm{dom}(P)$.

$\mathcal{G}_{\text{atlas}} := \{i_\lambda \circ \varphi_\lambda^{-1} : \varphi_\lambda(V_\lambda) \to M \mid i_\lambda \text{ is the inclusion}, \lambda \in \Lambda\}$ with the atlas $\{(V_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ and the set of immersions $\mathcal{G}_{\text{imm}} := \{\text{immersions } f : U \looparrowright M\}$ are generating families of the standard diffeology $\mathcal{D}^M_{\text{std}}$.

Definition 2.4

A weak Riemannian metric $g : T_2(X) \to \mathbb{R}$ is *definite* if there exists a generating family $\mathcal{G}$ of the diffeology $\mathcal{D}$ of X such that the symmetric positive covariant 2-tensor $g(P) := g \circ \pi_{\widehat{T}_2(U_P)}$ is definite in the usual sense for every $P \in \mathcal{G}$.

$$\begin{tikzcd} U_P \times \mathbb{R}^{\dim U_P} \times \mathbb{R}^{\dim U_P} \arrow[d, "\pi_{\widehat{T}_2(U_P)}"'] \arrow[dr, "g(P)"] & \\ T_2(X) \arrow[r, "g"'] & \mathbb{R}. \end{tikzcd} \tag{3}$$

A Riemannian metric on a diffeological adjunction space

We consider the diffeological adjunction space $X \coprod_A Y$ obtained by two *inductions* i and j into weak Riemannian diffeological spaces

$$(X, g_X) \xleftarrow{i} A \xrightarrow{j} (Y, g_Y). \tag{4}$$

Here, $i : A \to X$ is induction $\overset{\text{def}}{\Longleftrightarrow}$ i is an inc. and $P \in \mathcal{D}_A$ iff $i \circ P \in \mathcal{D}_X$.

Theorem 2.5 (KSS '25)

Suppose that the diagram $\begin{array}{ccc} T_2(A) & \xrightarrow{j_*} & T_2(Y) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle g_Y} \\ T_2(X) & \xrightarrow[g_X]{} & \mathbb{R} \end{array}$ *is commutative. Then, the map* $g\colon T_2(X \coprod_A Y) \to \mathbb{R}$ *defined by*

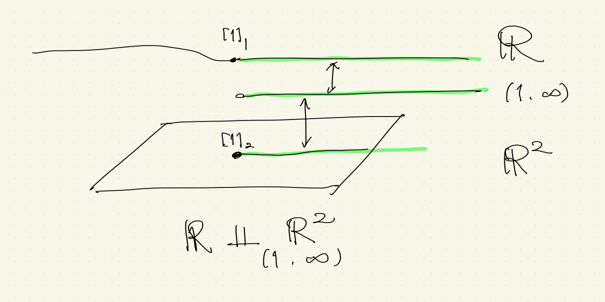
$$g((s, v_1, v_2)) := g_{U_P}((s, v_1, v_2)) = g_X((s, v_1, v_2))$$

for $(s, v_1, v_2) \in T_2(U_{P_X})$ *is a well-defined weak Riemannian metric on the diffeological adjunction space* $X \coprod_A Y$ *endowed with the quotient diffeology. Moreover, if* g_X *and* g_Y *are definite, then so is* g*.*

Theorem 2.5 allows us to conclude that the adjunction space

$$\boldsymbol{M := \mathbb{R} \coprod_{(1,\infty)} \mathbb{R}^2}$$

is a definite weak Riemannian diffeological space which is not a manifold. Observe that $\boldsymbol{D(M)}$ is non-Hausdorff.



A Riemannian metric on a diffeological mapping space

Let X and Y be diffeological spaces and $\mathcal{G}$ a generating family of $\mathcal{D}^Y$. Then, the set $C^\infty(X, Y)$ of smooth maps is endowed with the *functional diffeology* $\mathcal{D}_{\mathrm{func}}$; that is,

$(P : U_P \to C^\infty(X, Y)) \in \mathcal{D}_{\mathrm{func}} \overset{\mathrm{def}}{\iff} ad(P) : U_P \times X \to Y$ is smooth.

We introduce a subdiffeology of $\mathcal{D}_{\mathrm{func}}$. To this end, we consider the following condition (E) for a plot $P \in \mathcal{D}_{\mathrm{func}}$.

(E) For $r \in U_P$ and $m \in X$, there exists an open neighborhood $W_{r,m}$ of r in U_P such that the composite

$$W_{r,m} \hookrightarrow U_P \xrightarrow{P} C^\infty(X, Y) \xrightarrow{\mathbf{ev}_m} Y$$

is in $\mathcal{G}$, where $\mathbf{ev}_m$ denotes the evaluation map at m.

Let $\mathcal{F}_{\mathcal{G}}^{XY}$ be the subset of plots in $\mathcal{D}_{\mathrm{func}}$ each of which satisfies the condition (E).

Let M be a closed orientable manifold and N a definite weak Riemannian diffeological space with a generating family $\mathcal{G}$ which gives the definiteness. We define $g : T_2(C^\infty(M, N)) \to \mathbb{R}$ by

$$g(P)_r(v, w) = \int_M g_N(\mathrm{ev}_m \circ P)_r(v, w)\mathrm{vol}_M \tag{5}$$

for $r \in U_P$, where vol_M denotes a volume form on M.

Theorem 2.6 (KSS '25)

Let g be the weak Riemannian metric on $C^\infty(M, N)$ defined as above, where the diffeology is restricted to $\mathcal{D}' := \langle \mathcal{F}_{\mathcal{G}}^{MN} \rangle$. Then the metric g is definite with respect to the generating family $\mathcal{F}_{\mathcal{G}}^{MN}$ in the sense of Definition 2.4.

Proposition 2.7 (KSS '25)

For the weak Riemannian metric g on $(C^\infty(M, N), \mathcal{D}')$ in (5), the pseudodistance d defined by g is indeed a distance provided the pseudodistance defined by g_N is distance on N.

Mixing the constructions mentioned above, we have

Example 2.8

Let M be a closed orientable manifold and (N, g_N) a Remannian manifold. The diffeological adjunction space

$$C^\infty(M,N)\coprod_N C^\infty(M,N)$$

obtained by the section $N \to C^\infty(M,N)$ admits a definite weak Riemannian metric $\widetilde{g}$ for which

$$\iota^*(\widetilde{g}) = (\int_M \mathrm{vol}_M) \times g_N,$$

where the left-hand side is the pullback of the metric $\widetilde{g}$ by the canonical injection $\iota : N \to C^\infty(M,N)\coprod_N C^\infty(M,N)$ and vol_M denotes the volume form of M.

A metric on the free loop space and the concatenation map

We consider the subduction $\pi : S^1 \coprod S^1 \to S^1 \vee S^1$ and the *smooth* pinching map $p : S^1 \to S^1 \vee S^1$. Let M be a Riemannian manifold and LM the (smooth) free loop space $C^\infty(S^1, M)$.

$$\begin{array}{c}
(C^\infty(S^1 \coprod S^1, M) \cong LM \times LM, g \oplus g) \\
\uparrow \pi^* \\
(C^\infty(S^1 \vee S^1, M), g_\vee := (\pi^*)^*(g \oplus g)) \xrightarrow[c:=p^*]{} (LM := C^\infty(S^1, M), g)
\end{array}$$

Theorem 2.9 (KSS '25)

The concatenation map c preserves the metrics; that is,

$$c^* g = g_\vee.$$