On the Félix and Tanré rational model for a polyhedral product and its application to partial quotients

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§1. Definition of a polyhedral product

Definition 1.1 (Kishimoto and Levi '22)

Let $(\underline{X}, \underline{A}) := ((X_1, A_1), \ldots, (X_m, A_m))$ be a tuple of spaces with $A_i \subset X_i$ for each i and K a simplicial complex with m vertices. The *polyhedral product* $(\underline{X}, \underline{A})^K$ of the tuple $(\underline{X}, \underline{A})$ corresponding to K is defined by

$$(\underline{X}, \underline{A})^K := \mathrm{hocolim}_{\sigma \in K} (\underline{X}, \underline{A})^\sigma,$$

where $(\underline{X}, \underline{A})^\sigma = Y_1 \times \cdots \times Y_m$ with $Y_i = \begin{cases} A_i & i \notin \sigma \\ X_i & i \in \sigma. \end{cases}$

We write $(X, A)^K$ for $(\underline{X}, \underline{A})^K$ if there are a space X and a subspace A such that $X_i = X$ and $A_i = A$ for each i.

Remark 1.2

If each $\iota_i : A_i \to X_i$ is a cofibration, then there is an equivalence

$$(\underline{X}, \underline{A})^K \overset{\simeq_w}{\to} \mathrm{colim}_{\sigma \in K} (\underline{X}, \underline{A})^\sigma =: \mathcal{Z}_K(\underline{X}, \underline{A}).$$

$DJ(K) := \mathcal{Z}_K(BS^1, *)$: Davis–Januszkiewicz spaces,

$\mathcal{Z}_K(D^2, S^1)$: moment-angle complexes,

For a simplicial complex K with m vertices, $(\mathbb{C}^*)^m \curvearrowright \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)$.
$\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/(\mathbb{C}^*)^l$: toric manifolds, partial quotients (mentioned below), ...

Facts:

- The cohomology of $DJ(K)$ is isomorphic to the *Stanley–Reisner algebra* (Davis and Januszkiewicz '91): For a simplicial complex K with m vertices,

$$H^*(\mathcal{Z}_K(BS^1, *); \mathbb{K}) \cong \mathbb{K}[t_1, \cdots, t_m]/I(K) =: SR(K),$$

  where $\mathbb{K}$ is a field and $\deg t_i = 2$ and $I(K)$ denotes the ideal generated by monomials $t_{i_1} \cdots t_{i_s}$ for $\{i_1, ..., i_s\} \notin K$.
- $\mathcal{Z}_K(BS^1, *)$ is formal for arbitrary simplicial complex K. (Notbohm and Ray '05)
- $\mathcal{Z}_K(D^2, S^1)$ is not formal in general. (Denham and Suciu '07)
- A toric manifold $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/(\mathbb{C}^*)^l$ is formal. (Panov and Ray '08)

By applying the Félix–Tanré rational models for polyhedral products, we have

Theorem 1.3 (The main theorem)

Let $X_\Sigma = \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H$ *be a toric manifold*[a] *and* H' *a subtorus of* H. *For the partial quotient* $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H'$, *the following conditions are equivalent.*

(i) $H' = H$.

(ii) $H^{\mathrm{odd}}(\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H'; \mathbb{Q}) = 0$.

(iii) *The projection* $\pi_{H'} : \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H' \to DJ(K)$ *in the fibration (1) is 'formalizable'.*

[a] via Cox's construction

$$L \longrightarrow EG \times_H \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*) \xrightarrow{\pi_H} EG \times_G \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*) \cong DJ(K), \qquad (1)$$

where $K \subset 2^{[m]}$, $G = (\mathbb{C}^*)^m$ and $L = (\mathbb{C}^*)^m/H$. Observe that $\dim L =$ the dimension of the fan Σ.

The rest of the talk

§2. An overview of rational homotopy theory (RHT)

§3. The Félix–Tanré model for a polyhedral product and examples

- a fibre inclusion
- a pair of Lie groups
- the pair of the the free loop space of a space and its Borel construction

§4. A characterization of toric manifolds with formalizability of maps
– Sketch of the proof of the main theorem

§2 An overview of rational homotopy theory (RHT)

Let $A_{PL}(X)$ be the polynomial differential forms on a space X, nemely, a CDGA $A_{PL}(X) := \mathbf{Set}^{\Delta^{\mathrm{op}}}(S(X), A_{PL_\bullet})$, where $S(X)$ denotes the singular simplex and $A_{PL_\bullet}$ is the simplicial CDGA of polynomial differential forms.

We have an equivalence between the homotopy categories

$$\mathrm{fN}\mathbb{Q}\text{-}\mathrm{Ho}(\mathbf{Top}) \underset{\langle\ \rangle := |\mathbf{CDGA}(\ ,A_{PL_\bullet})|}{\overset{Q\circ A_{PL}(\ )}{\rightleftarrows}} \mathrm{f}\mathbb{Q}\text{-}\mathrm{Ho}(\mathbf{CDGA}^{op}) \quad (\simeq)$$

nilpotent rational connected spaces of finite $\mathbb{Q}$-type

cofibrant connected commutative differential graded algebras (CDGAs) of finite $\mathbb{Q}$-type

Here Q denotes the cofibrant replacement; that is, one has a quasi-iso.

$$QA_{PL}(X) = \wedge V_X = ((\text{poly. alg} \otimes \text{exterior alg}), d) \xrightarrow{\simeq} A_{PL}(X)$$

for a space X. $\wedge V_X$ is called a *Sullivan model* for X.

Fact: $^\exists$ a natural iso. $H^*(A_{PL}(X)) \cong H^*(X;\mathbb{Q})$ of algebras.

Some advantages of RHT

- In particular, if $\wedge V_X \xrightarrow{\simeq} A_{PL}(X)$ is a *minimal* Sullivan model for a simply-connected space X (that is, $d(v)$ is decomposable for $v \in V_X$), then

$$V_X \cong \mathrm{Hom}(\pi_*(X), \mathbb{Q}).$$

- An algebraic model for a fibration $F \xrightarrow{i} X \xrightarrow{\pi} Y$ is obtained by so-called the KS(Koszul–Sullivan)-extension.

$$\begin{array}{ccccc} \wedge V & \overset{j}{\rightarrowtail} & \wedge V \otimes \wedge W & \xrightarrow{\rho} & \wedge W \\ \simeq \downarrow v & & \simeq \downarrow \alpha & & \downarrow u \\ A_{PL}(Y) & \xrightarrow[\pi^*]{} & A_{PL}(X) & \xrightarrow[i^*]{} & A_{PL}(F) \end{array} \tag{2}$$

Theorem 2.1 (Halperin '83. A model of the fibre)

If $\pi_1(Y)$ acts on $H^*(F;\mathbb{Q})$ nilpotently, then u is a quasi-isomorphism; that is, $\wedge W$ is a Sullivan model for the fibre F.

Definition 2.2

For a connected space X not necessarily nilpotent, we call a DGA C a *(rational or commutative) model* for X if there is a zigzag of quasi-isomorphisms connecting C with $A_{PL}(X)$: $A_{PL}(X) \xrightarrow{\simeq} \bullet \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} C$

Theorem 2.3 (A model for an adjunction space)

Let (X, A) be a CW-pair and $(\wedge V, d) \xrightarrow{\varphi} (\wedge U, d) \xleftarrow{\psi} (\wedge W, d)$ models for maps $Y \xleftarrow{f} A \xrightarrow{i} X$. Suppose that one of φ and ψ is surjective, then there exists a zigzag of quasi-isomorphisms between $A_{PL}(Y \cup_f X)$ and the fibre product

$$(\wedge V \times_{\wedge U} \wedge W, D)$$

that is, the product is a commutative model for the adjunction space $Y \cup_f X$.

§3. The Félix–Tanré model for a polyhedral product

Let $\iota_j : A_j \to X_j$ be the inclusion and $\varphi_j : \mathcal{M}_j \to \mathcal{M}'_j$ a surjective model for ι_j, namely, an epimorphism of CDGA's which fits in a commutative diagram

$$\begin{array}{ccc} \wedge W_j & \xrightarrow{\varphi_j} & \wedge V_j \\ u \downarrow \simeq & & \simeq \downarrow v \\ A_{PL}(X_j) & \xrightarrow[\iota_j^*]{} & A_{PL}(A_j) \end{array} \qquad (3)$$

with quasi-isomorphisms u and v. For each $\tau \notin K$, let I_τ denote the ideal of $\bigotimes_{i=1}^m \mathcal{M}_i$ defined by $I_\tau = E_1 \otimes \cdots \otimes E_m$, where $E_i = \begin{cases} \mathcal{M}_i & i \notin \tau \\ \text{Ker } \varphi_i & i \in \tau. \end{cases}$

Theorem 3.1 (Félix and Tanré '09)

There is a zigzag of quasi-isomorphisms connecting the CDGA $A_{PL}((\underline{X}, \underline{A})^K)$ and the quotient

$$(\bigotimes_{i=1}^m \mathcal{M}_i)/J(K),$$

where $J(K) := \sum_{\tau \notin K} I_\tau$.

Proposition 3.2 (Félix and Tanré '09)

The Félix–Tanré rational models are natural with respect to surjective models used in the construction of the models of polyhedral products and an inclusion of simplicial complexes.

▷ This model is complicated!

For a simplicial complex K, we say that a CDGA (A, d) is of *Stanley–Reisner (SR) K-type* if the underlying algebra A is of the form

$$\bigotimes_{j=1}^{m}(\wedge V_j \otimes B_j))/(b_{j_1}\cdots b_{j_s} \mid b_j \in B_j^+, \{j_1, ..., j_s\} \notin K),$$

where B_j is a free commutative algebra.

Proposition 3.3 (K. '23)

Each polyhedral product $(\underline{X}, \underline{A})^K$ has a Stanley–Reisner K-type CDGA model C; that is, there is a zigzag of quasi-isomorphisms of CDGA's connecting C and $A_{PL}((\underline{X}, \underline{A})^K)$.

For $1 \leq j \leq m$, let $F_j \xrightarrow{\iota_j} X_j \xrightarrow{p_j} Y_j$ be a fibration with simply-connected base. Assume that $H^*(Y_j;\mathbb{Q})$ is locally finite for each j. Then, a relative Sullivan model $\widetilde{p_j}$ for the map p_j gives a commutative diagram of CDGA's

$$\begin{array}{ccccc} \wedge W_j & \xrightarrow{\widetilde{p_j}} & (\wedge V_j \otimes \wedge W_j, d_j) & \xrightarrow{\widetilde{\iota_j}} & (\wedge V_j, \overline{d_j}) \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ A_{PL}(Y_j) & \xrightarrow{p_j^*} & A_{PL}(X_j) & \xrightarrow{\iota_j^*} & A_{PL}(F_j). \end{array}$$

Proposition 3.4

The polyhedral product $(\underline{X}, \underline{F})^K$ for the tuple of fibre inclusions ι_j has a SR type CDGA model $\mathcal{M}((\underline{X}, \underline{F})^K)$ of the form

$$\Big(\bigotimes_{j=1}^{m}(\wedge V_j \otimes \wedge W_j)/(b_{j_1}\cdots b_{j_s} \mid b_j \in W_j, \{j_1, ..., j_s\} \notin K), d\Big)$$

for which $d(W_j) \subset \wedge W_j$ and $d(V_j) \subset (\wedge^{\geq 2} V_j) \otimes \wedge W_j + \wedge V_j \otimes \wedge^+ W_j$.

By using the model mentioned above, we have a spectral sequence.

Corollary 3.5

Let K be a simplicial complex with m vertices. For $1 \leq j \leq m$, let $F_j \to X_j \to Y_j$ be a fibration with simply-connected base Y_j. Then there is a first quadrant spectral sequence converging to $H^*((\underline{X}, \underline{F})^K; \mathbb{Q})$ as an algebra with

$$E_2^{*,*} \cong \big(\bigotimes_{j=1}^{m} H^*(F_j; \mathbb{Q})\big) \otimes H^*((\underline{Y}, \underline{*})^K; \mathbb{Q})$$

as a bigraded algebra, where

$$E_2^{p,q} \cong ((\bigotimes_{j=1}^{m} H^*(F_j; \mathbb{Q})\big) \otimes H^p((\underline{Y}, \underline{*})^K; \mathbb{Q}))^{p+q}.$$

Example 3.6 (A model for the moment-angle complex)

(i) Let $S^1 \to ES^1 \to BS^1$ be the universal S^1-bundle and K be a simplicial complex with m vertices. Then, we have a model for the bundle of the form $\wedge(t) \to \wedge(t) \otimes \wedge(x) \stackrel{\widetilde{\iota}}{\to} \wedge(x)$, where $\widetilde{\iota}$ is the canonical projection, $\deg x = 1$ and $\deg t = 2$. It follows from Proposition 3.4 that

$$\mathcal{M}((ES^1, S^1)^K) \cong (\wedge(x_1, ..., x_m) \otimes SR(K), d)$$

where $d(x_j) = t_j$: the generator of $SR(K) = \mathbb{K}[t_1, \cdots, t_m]/I(K)$. Observe that

$$\mathcal{Z}_K(D^2, S^1) \simeq \mathcal{Z}_K(ES^1, S^1) \simeq_w (ES^1, S^1)^K.$$

Example 3.7

Let X be a simply-connected space and $LX = \mathrm{map}(S^1, X)$ the free loop space. With the rotation action $S^1 \curvearrowright LX$, we have the Borel fibration

$$LX \xrightarrow{i} ES^1 \times_{S^1} LX = (LX)_h \xrightarrow{p} BS^1.$$

Let $(\wedge V, d)$ be the Sullivan minimal model for X. Then, the sequence

$$\wedge(t) \xrightarrow{\widetilde{p}} (\wedge(t) \otimes \wedge(V \oplus \overline{V}), \delta) \xrightarrow{\widetilde{i}} (\wedge(V \oplus \overline{V}), \delta')$$

is a model for the Borel fibration, where $\overline{V}^i = V^{i+1}$, $\delta'(v) = d(v)$ for $v \in V$, $s : \wedge V \to \wedge \overline{V}$ is the derivation with degree -1, $\delta'(\overline{v}) = -sd(v)$ and

$$\delta u = \delta'(u) + ts(u)$$

for $u \in V \oplus \overline{V}$. Thus Proposition 3.4 enables us to obtain a Félix–Tanré model for the polyhedral product $((LX)_h, LX)^K$ of the form

$$(\bigotimes_{i=1}^{m}(\wedge(V_i \oplus \overline{V}_i) \otimes SR(K), \bigotimes_i \delta_i).$$

Let $HH_*(A_{PL}(X))$ denote the Hochschild homology of $A_{PL}(X)$. There exists an isomorphism

$$HH_*(A_{PL}(X)) \cong H^*(LX;\mathbb{Q})$$

of algebras (Vigué-Poirrier and Burghelea '85). Therefore, Example 3.7 allows us to obtain the following result.

Corollary 3.8

Let X be a simply-connected space. Then, there exists a first quadrant spectral sequence converging to the cohomology $H^*(((LX)_h, LX)^K;\mathbb{Q})$ as an algebra with

$$E_2^{*,*} \cong HH_*(A_{PL}(X))^{\otimes m} \otimes SR(K)$$

as a bigraded algebra, where bideg $x = (0, \deg x)$ *for* $x \in HH_*(A_{PL}(X))$ *and* bideg $t_i = (2,0)$ *for the generator* $t_i \in SR(K)$.

A model for $(G, H)^K$ for a pair of Lie groups

Let $H \stackrel{i}{\to} G \stackrel{\pi}{\to} G/H$ be the principal H-bundle. rank $G = l$, rank $H = k$.

$$(Bi)^* : \underset{=: \wedge V_{BG}}{H^*(BG;\mathbb{Q}) \cong \mathbb{Q}[y_1, ...y_l]} \longrightarrow \underset{=: \wedge V_{BH}}{H^*(BH;\mathbb{Q}) \cong \mathbb{Q}[t_1, ...t_k]}$$

$H^*(G;\mathbb{Q}) \cong \wedge(x_1, .., x_l) =: \wedge P_G$, $H^*(H;\mathbb{Q}) \cong \wedge(u_1, ...u_k) =: \wedge P_H$.

We have a model (γ the projection) for the inclusion i:

$$\begin{array}{ccc} (\wedge V_{BH} \otimes \wedge P_G \otimes \wedge P_H, \partial) & \xrightarrow{\gamma} & (\wedge P_H, 0) \\ \simeq\downarrow & & \simeq\downarrow \\ A_{PL}(G) & \xrightarrow{A_{PL}(i)} & A_{PL}(H), \end{array}$$

where $\partial(u_i) = t_i$ for $u_i \in P_H$ and $\partial(x_i) = (Bi)^*(y_i)$ for $x_i \in P_G$.

Proposition 3.9

One has a rational model of the form

$$(C, d) := ((\wedge P_H)^{\otimes m} \otimes \left((\wedge V_{BH} \otimes \wedge P_G)^{\otimes m}/I(K)\right), \partial) \tag{4}$$

for the polyhedral product $(G, H)^K$, where $I(K)$ denotes the Stanley–Reisner ideal generated by elements in $(\wedge V_{BH} \otimes \wedge P_G)^{\otimes m}$.

§4 A characterization of toric manifolds with formalizability of maps

Definition 4.1 (J.-C. Thomas '82)

A map $p : E \to B$ is *formalizable* if there exists a commutative diagram up to homotopy

$$\begin{array}{ccc} A_{PL}(B) & \xrightarrow{A_{PL}(p)} & A_{PL}(E) \\ \simeq\uparrow & & \uparrow\simeq \\ (\wedge W, d) & \xrightarrow{l} & (\wedge Z, d') \\ \simeq\downarrow & & \downarrow\simeq \\ H^*(B;\mathbb{Q}) & \xrightarrow[p^*]{} & H^*(E;\mathbb{Q}) \end{array}$$

in which $(\wedge W, d)$ and $(\wedge Z, d')$ are minimal Sullivan algebras and vertical arrows are quasi-isomorphisms.

Proposition 4.2

The projection $\pi_H : X_\Sigma = \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H \to DJ(K)$ in the fibration (1) is formalizable, where X_Σ is a toric manifold.

Theorem 4.3 (The main theorem)

Let $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H$ be a toric manifold and H' a subtorus of H. For the partial quotient $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H'$, the following conditions are equivalent.

(i) $H' = H$.

(ii) $H^{\mathrm{odd}}(\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H'; \mathbb{Q}) = 0$.

(iii) *The map $\pi_{H'} : \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H' \to DJ(K)$ in the diagram (5) is formalizable.*

We have a commutative diagram consisting of two pullbacks

$$\begin{array}{ccccc} \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H' = EG \times_{H'} \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*) & \xrightarrow{p} & (EG)/H' & \longrightarrow & EL' \\ \downarrow{\scriptstyle \pi_{H'}} & & \downarrow & & \downarrow \\ DJ(K) = EG \times_G \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*) & \xrightarrow[q]{} & BG & \xrightarrow[B\rho]{} & BL', \end{array} \tag{5}$$

where $K \subset 2^{[m]}$, $G = (\mathbb{C}^*)^m$ and $L' = (\mathbb{C}^*)^m/H'$. (Franz '21)

Sketch of the proof of the main theorem.

Construct a model for $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H'$ with a Félix–Tanré model for $DJ(K)$ and the pushout construction:

$$C' := (\wedge(x_1, ..., x_l) \otimes SR(K), d(x_i) = q^*(B\rho)^*(t'_i)),$$

where $l = \dim L'$.
▷ '(i) $\Longrightarrow$ (ii)' follows from a general theory of toric varieties.
▷ (ii) $\Longrightarrow$ (iii) : The sequence $d(x_1), ..., d(x_l)$ is regular.

Proposition 4.4 (P.F. Baum '68)

Let A be a connected commutative algebra and $a_1, ..., a_t$ elements of $A^{>0}$. Set $\Lambda = \mathbb{K}[x_1, ..., x_t]$ with $\deg x_i = \deg a_i$ and consider A to be a Λ-module by means of the map $f : \Lambda \to A$ defined by $f(x_i) = a_i$. Then the following are equivalent:

(i) $a_1, ..., a_t$ *is a regular sequence.*

(ii) $\mathrm{Tor}_\Lambda^{-j,*}(\mathbb{K}, A) = 0$ *for all* $j \geq 1$.

(iii) *As a Λ-module A is isomorphic to* $\Lambda \otimes (A/(a_1, ..., a_t))$.

$\triangleright$ (iii) $\Longrightarrow$ (i)

Let H' be a proper subtorus. Then $d(x_1), \dots, d(x_l)$ is not regular[1]. Assume that $\pi_{H'}$ is formalizable. We have a commutative diagram

$$\begin{array}{ccc} A_{PL}((DJ(K)) & \xrightarrow{A_{PL}(\pi_{H'})} & A_{PL}(\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H') \\ \phi' \uparrow \simeq & & \uparrow \simeq \\ (\wedge W, d) & \xrightarrow{i} & (\wedge(x_1, \dots, x_l) \otimes \wedge W, d') \\ \phi \downarrow \simeq & & \eta \downarrow \simeq \\ H^*(DJ(K); \mathbb{Q}) & \xrightarrow[(\pi_{H'})^*]{} & H^*(\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H'; \mathbb{Q}). \end{array} \quad (6)$$

Lemma 4.5

The map $B\rho \circ q : DJ(K) \to BL'$ in the diagram (5) is formalizable.

Then, the algebraic spectral sequence converging to the torsion group $\mathrm{Tor}^*_{\mathbb{Q}[t_1,\dots,t_l]}(\mathbb{Q}, \wedge W) = H(\wedge(x_1, \dots, x_l) \otimes \wedge W, d')$ with

$$E_2^{*,*} \cong \mathrm{Tor}^{*,*}_{\mathbb{Q}[t_1,\dots,t_l]}(\mathbb{Q}, SR(K))$$

collapses at the E_2-term.

[1] considering the Poincaré series of $DJ(K)$

Then, the fact allows us to obtain a sequence

$$\mathrm{Tor}_P^{-1,*}(\mathbb{Q}, SR(K)) \cong E_0^{-1,*} \xleftarrow{p} F^{-1}\mathrm{Tor}_P^{*-1}(\mathbb{Q}, \wedge W) \xrightarrow{i} \mathrm{Tor}_P^{*-1}(\mathbb{Q}, \wedge W).$$

Here $\{F^j\}$: the filtration associated to the spectral sequence, $P := \mathbb{Q}[t_1, ..., t_l]$. Since $d(x_1), ..., d(x_l)$ is not a regular sequence, it follows form Proposition 4.4 that there is a non-exact cocycle

$$w = \sum_{j=1}^{l} u_j x_j - z$$

in $F^{-1}\mathrm{Tor}^{*-1}_{\mathbb{Q}[t_1,...,t_l]}(\mathbb{Q}, \wedge W)$, where u_i and z are in $\wedge W$.

The element z is of odd degree (because $*$ above is even) and in the image of the map i; see (6). Moreover, $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H'$ is 1-connected. Thus, since $H^{\mathrm{odd}}(DJ(K); \mathbb{Q}) = 0$, it follows that

$$H^*(\eta)([w]) = \sum_{j=1}^{l} \eta(u_j)\eta(x_j) - (\pi_{H'})^*\phi(z) = 0,$$

which is a contradiction.

A generalization

Let G be a compact connected Lie group. We have (homotopy) pullback diagrams

$$\begin{array}{ccccc} X_{K,(G,H)} := E(\Pi^m G) \times_H (EG,G)^K & \xrightarrow{p} & E(\Pi^m G)/H & \longrightarrow & EL \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow \\ E(\Pi^m G) \times_{\Pi^m G} (EG,G)^K & \xrightarrow[q]{} & B(\Pi^m G) & \xrightarrow[B\rho]{} & BL, \end{array} \tag{7}$$

where H is a normal subgroup of $(\Pi^m G)$ and $L = (\Pi^m G)/H$.

Theorem 4.6

Suppose that $H^{\text{odd}}(X_{K,(G,H)};\mathbb{Q}) = 0$*. Then* $X_{K,(G,H)}$ *is formal.*

Corollary 4.7

With the same notation as above, suppose that $H^{\text{odd}}(X_{K,(G,H)};\mathbb{Q}) = 0$ *and* $H^*(X_{K,(G,H)};\mathbb{Q}) \cong H^*(X_{K',(G',H')};\mathbb{Q})$*. Then* $X_{K,(G,H)} \simeq_{\mathbb{Q}} X_{K',(G',H')}$ *if the spaces are nilpotent. In particular. if* $H^*(X_\Sigma;\mathbb{Q}) \cong H^*(X_{\Sigma'};\mathbb{Q})$*, then* $X_\Sigma \simeq_{\mathbb{Q}} X_{\Sigma'}$*.*