ALGEBRAIC INTERLEAVINGS OF SPACES OVER THE CLASSIFYING SPACE OF THE CIRCLE

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Abstract. We bring spaces over the classifying space BS^1 of the circle group S^1 to persistence theory via the singular cohomology with coefficients in a field. Then, the *cohomology* interleaving distance (CohID) between spaces over BS^1 is introduced and considered in the category of persistent differential graded modules. In particular, we show that the distance coincides with the *interleaving distance in the homotopy category* in the sense of Lanari and Scoccola and the *homotopy interleaving distance* in the sense of Blumberg and Lesnick. Moreover, upper and lower bounds of the CohID are investigated with the cup-lengths of spaces over BS^1. As a computational example, we explicitly determine the CohID for complex projective spaces by utilizing the bottleneck distance of barcodes associated with the cohomology of the spaces.

Contents

1. Introduction

Persistence theory is developed rapidly in topological data analysis through the study of persistent homology and representations of algebras. Recently, persistence objects values in a category $\mathcal{C}$, namely, objects in the functor category $\mathcal{C}^{(\mathbb{R},\leq)}$, are investigated from the homological and homotopical points of view. As a consequence, for example, we have two-variable homotopy invariants. Indeed, Blumberg and Lesnick [8] introduce the *homotopy interleaving distance* d_{HI} and the *homotopy commutative interleaving distance* d_{HC} for persistence objects valued in a model

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category $\mathcal{M}$. In [35], Lanari and Scoccola define the *interleaving distance in the homotopy category*, denoted d_{IHC}, in the functor category $\mathcal{M}^{(\mathbb{R},\leq)}$ for a cofibrantly generated model category $\mathcal{M}$. Here the term *distance* means an extended pseudometric on a class. By a categorical consideration, for example, [10, Proposition 3.6], it is readily seen that

$$d_{\mathrm{HC}} \leq d_{\mathrm{IHC}} \leq d_{\mathrm{HI}} \tag{1.1}$$

on the class of objects in $\mathcal{M}^{(\mathbb{R},\leq)}$. Moreover, the result [35, Theorem A] asserts that $d_{\mathrm{HI}} \leq 2d_{\mathrm{HC}}$. Let Top be the category of topological spaces. Then, by [35, Proposition 3.12], we see that $c \geq \frac{3}{2}$ if $d_{\mathrm{HI}} \leq cd_{\mathrm{HC}}$ on $\mathsf{Top}^{(\mathbb{R},\leq)}$ and hence $d_{\mathrm{HI}} \neq d_{\mathrm{HC}}$ in general. Remarkably, this result is proved by using the notion of the Toda bracket; see [35, Section 3.2]. It is worthwhile to note that a positive answer to a version of the persistent Whitehead conjecture [8, Conjecture 8.6] is given by deeply considering interleavings in the homotopy category $\mathrm{Ho}(\mathsf{Top}^{(\mathbb{R},\leq)})$; see [35, Theorem B, Remark 5.14].

In this article, we introduce the *cohomological interleaving distance* $d_{\mathrm{CohI},\mathbb{K}}$ for persistence objects valued in $\mathsf{Ch}_{\mathbb{K}}$ the category of differential graded (dg) modules over a field $\mathbb{K}$. Then, we show the equalities

$$d_{\mathrm{CohI},\mathbb{K}} = d_{\mathrm{HC}} = d_{\mathrm{IHC}} = d_{\mathrm{HI}}$$

for objects in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$; see Theorem 3.3 for more details. Thus, we may compute the distances up to homotopy of persistence dg modules by using the *bottleneck distance* of barcodes associated with the cohomology groups of the given persistence objects.

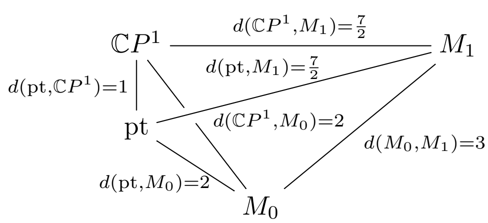
In persistence theory, the primary topological objects considered so far are simplicial complexes and the sublevel set of a map. The latter half of this article investigates spaces over the classifying space BS^1 of the circle group S^1 in persistence theory, specifically examining the cohomology interleaving distance $d_{\mathrm{CohI},\mathbb{K}}$ between such spaces. In fact, we bring a space over BS^1 to the category $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ of persistence dg modules via the singular cochain functor $C^*(\ ;\mathbb{K})$ with coefficients in a field $\mathbb{K}$; see Section 5. As Theorem 3.3 aforementioned, an algebraic result (Theorem 4.7) yields that the cohomology interleaving distance between spaces over BS^1 coincides with the homotopy interleaving distances d_{HC}, d_{IHC} and d_{HI} for the spaces. Moreover, Propositions 5.5 and 5.9 allow us to obtain upper and lower bounds of the cohomology interleaving distance of spaces over BS^1 with the cup-lengths of the spaces

We have many computational examples of the distance $d_{\mathrm{CohI},\mathbb{K}}$. Some of them enable us to realize the triangle inequality of the distance. For instance, let $S^3 \to \mathbb{C}P^1$ be the principal S^1-bundle and $M'_j \to M_j$ the S^1-bundle described in Proposition 6.3 and Remark 6.4 for $j = 0$ and 1. Observe that the total space M'_j has the rational homotopy type of $S^3 \times S^3 \times S^7$ for each $j = 0, 1$. We regard the base spaces of the bundles as spaces over BS^1 with the classifying maps. Then, by Propositions 4.12, 5.8, 6.3 and 6.9, we have the tetrahedron (1.2) below. It also follows that the distance $d_{\mathrm{CohI},\mathbb{Q}}$ of the spaces in (1.2) and the Borel construction of the free loop space of a simply-connected space are infinite; see Example 5.7.

We anticipate that the study of the cohomology interleaving of spaces developed in this article will bring new insights into persistence theory and topological comparison between spaces as the Gromov–Hausdorff distance is used in the study of Riemannian manifolds. Indeed, when we compare two spaces $p : X \to BS^1$ and

$q : Y \to BS^1$, it is natural to rely on an appropriate morphism between the spaces over BS^1, namely a continuous map $f : X \to Y$ with $q \circ f = p$. However, we may compare spaces over BS^1 with the cohomology interleaving distance even if there is no such morphism of the spaces; see Remark 6.7 and Proposition 6.9.

(1.2)



Here pt is the space over BS^1 with the trivial map from the one point to a base point and $d(X, Y)$ stands for the cohomology interleaving distance $d_{\mathrm{CohI},\mathbb{Q}}(X, Y)$ between spaces X and Y.

An outline of the manuscript is as follows. Section 2 recalls the interleaving distance of persistence objects and the bottleneck distance of persistence vector spaces. In Subsection 2.1, the homotopy interleaving distances d_{HC}, d_{IHC} and d_{HI} are defined. In Section 3, we show the formality of a persistence differential graded module over $(\mathbb{Z}, \leq)$; see Lemma 3.5. This fact allows us to prove Theorem 3.3. Section 4 addresses the interleaving distance of dg $\mathbb{K}[u]$-modules, where $\deg u = 2$. Moreover, we prove Theorem 4.7 mentioned above. We also consider the bigraded $\mathbb{K}[t]$-module $E^{*,*}$ associated with a filtered $\mathbb{K}[t]$-module H^*, where $\deg t = 1$. As a consequence, Lemma 4.9 enables us to recover the $\mathbb{K}[t]$-module structure of H^* from that of $E^{*,*}$ with no extension problem. Propositions 4.12 and 4.13 proved in Section 4 are helpful in computing the cohomology interleaving distance between a persistence dg module with a small barcode and a general one.

In Section 5, by applying the results described in the previous sections, we consider the cohomology interleaving distances in three classes consisting of spaces over BS^1. Example 5.7 mentioned above indeed asserts that the distance $d_{\mathrm{CohI},\mathbb{K}}$ between spaces, which belong to the different classes, is infinite. Section 6 is dedicated to explaining explicit calculations of the cohomology interleaving distances of the complex projective spaces and the orbit spaces M_0 and M_1 in (1.2).

Appendix A deals with some rational homotopy invariants, whereby we observe a difference between spaces having the positive cohomology interleaving distance. In particular, the rational toral ranks and the cup-lengths of the orbit spaces in Section 6 are considered. While as mentioned above, the cup-length is related to the cohomology interleaving distance, a relationship between the rational toral rank and the distance is currently unclear.

1.1. **Future work and perspective.** As mentioned above, Blumberg and Lesnick [8] have introduced the homotopy interleaving distance d_{HI} for $\mathbb{R}$-spaces, namely objects in $\mathsf{Top}^{(\mathbb{R},\leq)}$; see also [35]. In particular, the distance satisfies the condition $d_{\mathrm{HI}}(X, Y) = 0$ whenever $X \simeq Y$; see [8, Theorem 1.9].

By getting used to the *algebraic* interleaving distances in this article, we may introduce and consider the *rational* homotopy interleaving distance of $\mathbb{R}$-spaces. To this end, we deal with the interleaving distance in $(\mathsf{CDGA}^{\mathsf{op}})^{(\mathbb{R},\leq)}$ whose objects are persistent commutative differential graded algebras; see [26, 42]. We also refer the reader to [12] for the study of *tame* persistence objects values in a more general model category. It is worthwhile mentioning that homotopy invariants, which are obtained by applying the singular chain complex functor and the cohomology functor to a persistence space, give rise to more fruitful structures endowed with for example the cup products and Steenrod operations in persistent theory; see [6, 16, 22, 37, 38]. It may be possible to investigate each space but not a persistence space with the structures via the functor C introduced in Section 5.

It is also in our interest to consider multiparameter persistence theory in for example [35, 36]. In fact, spaces over the classifying space of a higher dimensional torus give rise to such objects in the theory via the singular cochain functor. Thus we may investigate the moment-angle complexes that appear in toric topology from the viewpoint of multiparameter persistence theory; see [3] for a related issue.

2. The interleaving distance and the bottleneck distance

We begin by reviewing the interleaving distance introduced in [13] and results in [10] related to the distance, which we use extensively in this article. Let $(\mathbb{R}, \leq)$ be the poset defined with the usual order. Considering the poset as a category, we have the functor category $\mathcal{C}^{(\mathbb{R},\leq)}$ for a category $\mathcal{C}$. For a real number $\varepsilon \geq 0$, define a functor $T_\varepsilon : (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$ by $T_\varepsilon(a) = a + \varepsilon$. Moreover, we define a natural transformation $\eta_\varepsilon : id_{(\mathbb{R},\leq)} \Rightarrow T_\varepsilon$ by $\eta_\varepsilon(a) : a \leq a + \varepsilon$.

Definition 2.1. ([13, Definition 4.2], [10, Definition 3.1]) Objects F and G in $\mathcal{C}^{(\mathbb{R},\leq)}$ are *ε-interleaved* if there exist natural transformations $\varphi : F \Rightarrow GT_\varepsilon$ and $\psi : G \Rightarrow FT_\varepsilon$, i.e.,

$$\begin{array}{ccccc} (\mathbb{R},\leq) & \xrightarrow{T_\varepsilon} & (\mathbb{R},\leq) & \xrightarrow{T_\varepsilon} & (\mathbb{R},\leq) \\ {\scriptstyle F}\big\downarrow & \overset{\varphi}{\Nearrow} & {\scriptstyle G}\big\downarrow & \overset{\psi}{\Nearrow} & {\scriptstyle F}\big\downarrow \\ \mathcal{C} & = & \mathcal{C} & = & \mathcal{C} \end{array}$$

such that $(\psi T_\varepsilon) \circ \varphi = F\eta_{2\varepsilon}$ and $(\varphi T_\varepsilon) \circ \psi = G\eta_{2\varepsilon}$, where $\circ$ denotes the vertical composition of natural transformations. Such a tuple (F, G, φ, ψ) is called an *ε-interleaving.*

Remark 2.2. The *shift functor* $(\ )^\varepsilon : \mathcal{C}^{(\mathbb{R},\leq)} \to \mathcal{C}^{(\mathbb{R},\leq)}$ is defined by $(\ )^\varepsilon(F) = F^\varepsilon := FT_\varepsilon$. Then, we see that (F, G, φ, ψ) is an ε-interleaving if and only if the tuple fits in the commutative diagrams

$$\begin{array}{ccccc} F & \longrightarrow & F^\varepsilon & \longrightarrow & F^{2\varepsilon} \\ & {\scriptstyle\psi}\nearrow \searrow{\scriptstyle\varphi} & & {\scriptstyle\psi^\varepsilon}\nearrow \searrow{\scriptstyle\varphi^\varepsilon} & \\ G & \longrightarrow & G^\varepsilon & \longrightarrow & G^{2\varepsilon} \end{array} \tag{2.1}$$

in which horizontal arrows are the natural transformations defined by the structure maps of F and G. The identities on the natural transformations in Definition 2.1 imply the commutativity of the diagrams

$$\begin{array}{ccc} F(i) & \xrightarrow{F(i \to i+2\varepsilon)} & F(i+2\varepsilon) \\ & {\scriptstyle\varphi(i)}\searrow \quad \nearrow{\scriptstyle\psi(i+\varepsilon)} & \\ & G(i+\varepsilon) & \end{array} \quad \text{and} \quad \begin{array}{ccc} & F(i+\varepsilon) & \\ & {\scriptstyle\psi(i)}\nearrow \quad \searrow{\scriptstyle\varphi(i+\varepsilon)} & \\ G(i) & \xrightarrow[G(i \to i+2\varepsilon)]{} & G(i+2\varepsilon) \end{array} \tag{2.2}$$

for all $i \in \mathbb{R}$. We note that F is isomorphic to G in $\mathcal{C}^{(\mathbb{R},\leq)}$ if and only if F and G are 0-interleaved.

Definition 2.3. For objects F and G in $\mathcal{C}^{(\mathbb{R},\leq)}$, the interleaving distance $d_{\mathrm{I}}(F,G)$ between F and G is defined by

$$d_{\mathrm{I}}(F,G) := \inf(\{\varepsilon \geq 0 \mid F \text{ and } G \text{ are } \varepsilon\text{-interleaved}\} \cup \{\infty\}).$$

Remark 2.4. Let $\mathcal{C}$ be an additive category and I a set. Suppose that for $i \in I$ $(F_i, G_i, \varphi_i, \psi_i)$ is an ε-interleaving in $\mathcal{C}^{(\mathbb{R},\leq)}$. Then, we see that $\oplus_{i\in I} F_i$ and $\oplus_{i\in I} G_i$ are ε-interleaved with $\oplus_{i\in I}\varphi_i$ and $\oplus_{i\in I}\psi_i$.

Theorem 2.5. [10, Theorem 3.3] *The function d_{I} defined above is an extended pseudometric on the class of objects in $\mathcal{C}^{(\mathbb{R},\leq)}$.*

Remark 2.6. The *closure theorem* implies that the interleaving distance is a metric on isomorphism classes of finitely presented (*multidimensional*) persistence modules; see [36, Section 6] for more details.

In what follows, let $\mathbb{K}$ be a field of arbitrary characteristic and $\mathsf{Mod}_{\mathbb{K}}$ the category of vector spaces over $\mathbb{K}$ unless otherwise specified. We refer to an object in $\mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ as a *persistence vector space*.

Example 2.7. Let F and G be persistence vector spaces. Suppose that there exists a real number δ such that $F(j) = 0$ for $j > \delta$. Moreover, we assume that there exist $i \in \mathbb{R}$ and an element $x \in G(i)$ such that $G(i \to i+\delta')(x) \neq 0$ for every $\delta' > 0$. Then, it follows that $d_{\mathrm{I}}(F,G) = \infty$. In fact, suppose that F and G are ε-interleaved. We choose a positive number ε' so that $\varepsilon' \geq \varepsilon$ and $i + \varepsilon' > \delta$. By virtue of [10, Lemma 3.4], we see that F and G are ε'-interleaved. Then, the commutativity of the right-hand side diagram in Remark 2.2 enables us to deduce that $G(i \to i + 2\varepsilon')(x) = 0$, which is a contradiction.

Let $\mathbb{K}[t]$ be the polynomial algebra generated by an element t with degree 1. For a graded $\mathbb{K}[t]$-module

$$K = \bigoplus_{i=1}^{n} \Sigma^{-a_i}\mathbb{K}[t] \oplus \bigoplus_{j=1}^{n'} \Sigma^{-b_j}(\mathbb{K}[t]/(t^{c_j})), \tag{2.3}$$

we define the *barcode $\mathcal{B}_K$ associated with* K by the multiset consisting of intervals $[a_i, \infty)$ and $[b_j, b_j+c_j)$. Here, Σ^l stands for the shift operator with degree $+l$; that is, $(\Sigma^l A)^i = A^{i+l}$. We also deal with the case where n and n' are infinite. The result [41, Theorem 1] implies that a bounded below graded $\mathbb{K}[t]$-module decomposes uniquely into the form such as (2.3)*. Furthermore, let J be an interval, namely, a subset of $\mathbb{R}$ which satisfies the condition that if $r < s < t$ with $r, t \in J$, then $s \in J$. We define an object χ_J in $\mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R},\leq)}$, called an *interval module*, by

$$\chi_J(x) = \begin{cases} \mathbb{K} & \text{if } x \in J, \\ 0 & \text{otherwise,} \end{cases} \qquad \chi_J(x \leq y) = \begin{cases} id_{\mathbb{K}} & \text{if } x, y \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the barcode $\mathcal{B}_K$ associated with a graded $\mathbb{K}[t]$-module K gives rise to the object $\chi(\mathcal{B}_K)$ in $\mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ defined by $\oplus_{J\in\mathcal{B}_K}\chi_J$.

We call a persistence vector space K *locally finite* (*pointwise finite-dimensional*) if $\dim K(t) < \infty$ for $t \in \mathbb{R}$. We observe that a locally finite persistence module can be decomposed uniquely as a direct sum of interval modules; see [9, 18, 43].

*The uniqueness of the decomposition follows from the Krull–Remak–Schmidt–Azumaya theorem.

Lemma 2.8. ([10, Proposition 4.12 (2)(3), Proposition 4.13(3)]) *Let J and J' be finite intervals.*

(1) *If $J' = \emptyset$ and J has endpoints a and b, then $d_{\mathrm{I}}(\chi_J, \chi_{J'}) = \frac{b-a}{2}$.*
(2) *If J and J' have endpoints a, b and a', b', respectively, then*

$$d_{\mathrm{I}}(\chi_J, \chi_{J'}) = \min\left\{\max\{|a-a'|, |b-b'|\}, \max\left\{\frac{b-a}{2}, \frac{b'-a'}{2}\right\}\right\}.$$

(3) *If $\sup(I) = \infty = \sup(I')$ and I and I' have left end points a and a', then $d_{\mathrm{I}}(\chi_I, \chi_{I'}) = |a-a'|$.*

For multisets A and B, define the multiset A_B by $A_B := A \amalg (\coprod_{|B|}\{\emptyset\})$. We write $f : A \leftrightarrow B$ for a bijection $f : A_B \to B_A$.

Definition 2.9. Let S and T be two barcodes. Define the bottleneck distance between S and T by

$$d_{\mathrm{B}}(S,T) := \inf_{f:S\leftrightarrow T} \sup_{I\in \mathrm{dom}(f)} d_{\mathrm{I}}(\chi_I, \chi_{f(I)}),$$

where $\chi_{\mathbb{R}}$ and $\chi_{\emptyset}$ denote the constant functors $\mathbb{K}$ and 0, respectively.

Theorem 2.10. (The isometry theorem) ([10, Theorem 4.16], [14, Theorem 4.11]) *For locally finite graded $\mathbb{K}[t]$-modules K and K', one has*

$$d_{\mathrm{I}}(\chi(\mathcal{B}_K), \chi(\mathcal{B}_{K'})) = d_{\mathrm{B}}(\mathcal{B}_K, \mathcal{B}_{K'}).$$

Observe that the bottleneck distance with the l^∞-metric introduced in [14] coincides with that in Definition 2.9; see [10, Section 4.3] for details.

We conclude this section by recalling interleaving distances up to homotopy introduced in [8, 35].

2.1. **Interleavings up to homotopy.** Let $\mathcal{M}$ be a cofibrantly generated model category and $\mathcal{M}^{(\mathbb{R},\leq)}$ the model category endowed with the *projective model structure*; see [28, Theorem 11.6.1].

(1) For objects X and Y in $\mathcal{M}^{(\mathbb{R},\leq)}$, we say that X and Y are *ε-homotopy interleaved* if there exist $X \simeq X'$ and $Y \simeq Y'$ such that X' and Y' are ε-interleaved in $\mathcal{M}^{(\mathbb{R},\leq)}$; see [8, Section 3.3]. Here $W \simeq W'$ means that there is a zigzag of weak equivalences connecting W and W'.
(2) We say that objects X and Y in $\mathcal{M}^{(\mathbb{R},\leq)}$ are *ε-interleaved in the homotopy category* if they are ε-interleaved in $\mathrm{Ho}(\mathcal{M}^{(\mathbb{R},\leq)})$. Observe that the shift functor $(\ )^\varepsilon : \mathcal{M}^{(\mathbb{R},\leq)} \to \mathcal{M}^{(\mathbb{R},\leq)}$ preserves weak equivalences. Thus, we can consider the commutative diagram (2.1) in $\mathrm{Ho}(\mathcal{M}^{(\mathbb{R},\leq)})$.
(3) Let $q_* : \mathcal{M}^{(\mathbb{R},\leq)} \to \mathrm{Ho}(\mathcal{M})^{(\mathbb{R},\leq)}$ be the functor induced by the localization functor $q : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$. We say that X and Y in $\mathcal{M}^{(\mathbb{R},\leq)}$ are *ε-homotopy commutative interleaved* if q_*X and q_*Y are ε-interleaved in $\mathrm{Ho}(\mathcal{M})^{(\mathbb{R},\leq)}$

Let X and Y be objects in $\mathcal{M}^{(\mathbb{R},\leq)}$. Blumberg and Lesnick [8] introduce the *homotopy interleaving distance* and the *homotopy commutative interleaving distance* defined by

$$d_{\mathrm{HI}}(X,Y) := \inf(\{\varepsilon \geq 0 \mid X,\, Y \text{ are } \varepsilon\text{-homotopy interleaved}\} \cup \{\infty\}) \text{ and}$$

$$d_{\mathrm{HC}}(X,Y) := \inf(\{\varepsilon \geq 0 \mid X, Y \text{ are } \varepsilon\text{-homotopy commutative interleaved}\} \cup \{\infty\}),$$

respectively. Moreover, Lanari and Scoccola [35] introduce the *interleaving distance in the homotopy category* define by

$$d_{\mathrm{IHC}}(X,Y) := \inf(\{\varepsilon \geq 0 \mid X, Y \text{ are } \varepsilon\text{-interleaved in the homotopy category}\} \cup \{\infty\}).$$

Observe that Berkouk has defined the distance in the derived category of multi-parameter persistence modules; see [2, Section 3] for more detail.

We exhibit relationships among the three distances. We first observe that the homotopy interleaving distance is also extended pseudometric on the class of objects in $\mathcal{M}^{(\mathbb{R},\leq)}$; see [8, Section 4] and [35, Proposition 2.3]. By applying the universal property of the homotopy category of $\mathrm{Ho}(\mathcal{M}^{(\mathbb{R},\leq)})$ to the functor q_* mentioned above, we have a functor $\theta : \mathrm{Ho}(\mathcal{M}^{(\mathbb{R},\leq)}) \to \mathrm{Ho}(\mathcal{M})^{(\mathbb{R},\leq)}$; see the diagram (4.2) below. Thus, we establish the inequalities (1.1).

3. Interleavings up to homotopy between persistence dg modules

Let $\mathsf{Ch}_{\mathbb{K}}$ be the category of differential graded (dg) modules, whose objects are not necessarily bounded. The differential of each object is assumed to be of degree $+1$. Let P be a poset. We view P as a category with the unique arrow $i \to j$ if $i \leq j$. Then, the functor category $\mathsf{Ch}_{\mathbb{K}}^P$ is the model category endowed with the projective model structure; see [4, Theorem 3.3] and [28, Theorem 11.6.1]. Thus, the three distances d_{HC}, d_{IHC} and d_{HI} are defined on the class of the objects in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$. We may call an object in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ a *persistence dg module*.

Let $\eta^k : \mathsf{grMod}_{\mathbb{K}}^{(\mathbb{R},\leq)} \to \mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ be the functor defined by $(\eta^k)(V)(i) = V(i)^k$ for each integer k and $(H)_* : \mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)} \to \mathsf{grMod}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ the homology functor.

Definition 3.1. The *cohomology interleaving distance* $d_{\mathrm{CohI}}(X,Y)$ of persistence dg modules X and Y is defined by

$$d_{\mathrm{CohI}}(X,Y) := \sup\{d_{\mathrm{I}}(\eta^k(H)_*(X), \eta^k(H)_*(Y)) \mid k \in \mathbb{Z}\}.$$

Remark 3.2. The composite $\eta^k(H_*)$ gives rise to a functor $\mathrm{Ho}(\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}) \to \mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ for each k. It is readily seen from [10, Proposition 3.6] that $d_{\mathrm{CohI}}(X,Y) \leq d_{\mathrm{HC}}(X,Y)$.

The main theorem in this short section is as follows.

Theorem 3.3. *One has the equalities $d_{\mathrm{HC}} = d_{\mathrm{IHC}} = d_{\mathrm{HI}} = d_{\mathrm{CohI}}$ on the class of the objects in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$.*

We prove the theorem by applying the following proposition.

Proposition 3.4. *Let X and Y be objects in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$ and m a non-negative integer. If $(H(X),0)$ and $(H(Y),0)$ are m-interleaved in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$, then X and Y are m-homotopy interleaved in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$.*

In order to prove Proposition 3.4, we regard a persistence dg-module X as a *differential bigraded (dbg)* $\mathbb{K}[t]$*-module* (a cochain complex of graded $\mathbb{K}[t]$-modules)

$$\left(\bigoplus_{(i,n)\in\mathbb{Z}^2} X(i)^n, d \right)$$

for which $(\bigoplus_n X(i)^n, d)$ is a dg module for each i and the module structure $\times t : X(i)^n \to X(i+1)^n$ is given by the structure map $X(i \to i+1)$. Observe that

$\times t\circ d = d\circ \times t$. We show that $\big(\bigoplus_{(i,n)\in\mathbb{Z}^2} X(i)^n, d\big) \simeq \big(\bigoplus_{(i,n)\in\mathbb{Z}^2} H^n(X(i)^*), 0\big)$ as a persistence dg module.

The following lemma is a generalization of the assertion of [1, Remark 3.7] to an unbounded case.

Lemma 3.5. *For a persistence dg module (X, d), there exist a dbg $\mathbb{K}[t]$-module Q and quasi-isomorphisms $X \overset{\simeq}{\leftarrow} Q \overset{\simeq}{\rightarrow} H(X)$ of dbg $\mathbb{K}[t]$-modules. As a consequence, $X \simeq H(X)$ in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$.*

As seen in Remark 3.6 below, Lemma 3.5 follows from a more general result. We here prove the lemma by a constructive approach.

Proof of Lemma 3.5. Let $\{[b_\lambda(i)^k]\}$ be a set of generators of $H(X)$ as a bigraded $\mathbb{K}[t]$-module, where $b_\lambda(i)^k$ is in $X(i)^k$. Observe that $H(X) = \bigoplus_{(i,n)} H^n(X(i))$.

Let F_0 be the free $\mathbb{K}[t]$-module generated by $\{b_\lambda(i)^k\}$. Since $\mathbb{K}[t]$ is a principal ideal domain as an ungraded ring, it follows that the kernel of the composite of the natural map $\varphi : F_0 \to \operatorname{Ker} d$ and the projection $p : \operatorname{Ker} d \to H(X)$ is a free ungraded $\mathbb{K}[t]$-module. Let $B = \{f_\mu(i)^k\}$ be the basis of the kernel. The proof of [29, Theorem IV 6.1] enables us to choose each element of the basis B to be homogeneous; see also [27, Theorem 5.1]. In fact, the proof is applicable to the free $\mathbb{K}[t]$-module generated by $\{b_\lambda(i)^k \mid k = N\}$ for each homological degree N. We assume that $f_\mu(i)^n$ is of bidegree (n, i).

Let F_1 be the free $\mathbb{K}[t]$-module generated by $\{\alpha_\mu(i)^n\}$. We define the differential $D : F_1 \to F_0$ by $D(\alpha_\mu(i)^n) = f_\mu(i)^n$. We observe that the element $\alpha_\mu(i)^n$ is of bidegree $(i, n-1)$. Since $(p\circ\varphi)(f_\mu(i)^n) = 0$ for each element $f_\mu(i)^n$ in the basis B, there exists an element $z_\mu(i)^n$ in X such that $\varphi(f_\mu(i)^n) = d(z_\mu(i)^n)$. We define a morphism $\psi : Q := (F_0 \oplus F_1, D) \to X$ of dg $\mathbb{K}[t]$-modules by $\psi(b_\lambda(i)^k) = b_\lambda(i)^k$ and $\psi(\alpha_\mu(i)^n) = z_\mu(i)^n$. Moreover, we have a quasi-isomorphism $g : Q \overset{\simeq}{\rightarrow} H(X)$ defined by $g(x + y) = [x]$, where $x \in F_0$ and $y \in F_1$. □

Remark 3.6. We may call an object M in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$ *formal* if there exists a sequence (zigzag) of quasi-isomorphisms which connects M and $H^*(M)$ in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$. Thus by Lemma 3.5, every persistence dg module is formal.

An object M in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$ is regarded as a chain complex of graded $\mathbb{K}[t]$-module and then an object in the derived category of graded $\mathbb{K}[t]$-modules. Thus, Lemma 3.5 follows from the more general result [31, Proposition 4.4.15] which asserts that every object in the derived category of an abelian category $\mathcal{A}$ is formal (*quasi-isomorphic to its cohomology* in the sense in [31]) if and only if $\mathcal{A}$ is *hereditary*; that is, the functor $\operatorname{Ext}^2_{\mathcal{A}}(\text{-},\text{-})$ vanishes. Therefore, the result on the derived category implies that Lemma 3.5 cannot be generalized to an assertion for multi-parameter persistence dg-modules. In fact, the second Ext functor does not vanish in the category of graded $\mathbb{K}[t]^{\otimes n}$-modules for $n \geq 2$.

Proof of Proposition 3.4. Let X and Y be persistence dg-modules. Then, the assumption and Lemma 3.5 imply that X and Y are m-homotopy interleaved. □

The following assertion and its proof are inspired by those of [35, Theorem A]; see the paragraph after (1.1) in Introduction.

Proposition 3.7. *Let X and Y be objects in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$. If $(H(X),0)$ and $(H(Y),0)$ are δ-interleaved, then X and Y are δ'-homotopy interleaved for each δ' greater than δ.*

Proof. We recall the self-functor (M_t) on $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ for each positive number $t \in \mathbb{R}$ defined by $M_t(r) = t \times r$; see [35, Section 3]. Let m be a positive integer. Then, by [35, Lemma 3.1], we see that $(M_{\delta/m})^*H(X)$ and $(M_{\delta/m})^*H(Y)$ are m-interleaved and hence $\iota^*(M_{\delta/m})^*H(X)$ and $\iota^*(M_{\delta/m})^*H(Y)$ are m-interleaved in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$, where $\iota^* : \mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)} \to \mathsf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$ is the functor induced by the inclusion $\iota : \mathbb{Z} \to \mathbb{R}$. The homology functor is compatible with the functor $\iota^*(M_{\delta/m})^*$. Thus, Proposition 3.4 enables us to deduce that $\iota^*(M_{\delta/m})^*X$ and $\iota^*(M_{\delta/m})^*Y$ are m-homotopy interleaved in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$.

It follows from [35, Lemma 3.2] that $(M_{\delta/m})^*X$ and $(M_{\delta/m})^*Y$ are $(m+2)$-homotopy interleaved in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$. Therefore, there exist objects X' and Y' with $(M_{\delta/m})^*X \simeq X'$ and $(M_{\delta/m})^*Y \simeq Y'$ such that X' and Y' are $(m+2)$-interleaved in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$. Then, it follows that $(M_{m/\delta})^*X'$ and $(M_{m/\delta})^*Y'$ are $(\delta + 2(\delta/m))$-interleaved. We observe that $X \simeq (M_{m/\delta})^*X'$ and $Y \simeq (M_{m/\delta})^*Y'$. It turns out that X and Y are $(\delta + 2(\delta/m))$-homotopy interleaved in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$. □

Proof of Theorem 3.3. We have the inequalities in (1.1) and Remark 3.2. Then, in order to prove the theorem, it suffices to show that $d_{\mathrm{HI}} \leq d_{\mathrm{CohI},\mathbb{K}}$. For any $\varepsilon > 0$, let δ be the positive number $d_{\mathrm{CohI}}(X,Y) + \varepsilon$. Since $H(Z) = \oplus_{k\geq 0}\eta^k(H_*)(Z)$ for a persistence dg-module Z, it follows from Remark 2.4 that $(H(X),0)$ and $(H(Y),0)$ are δ-interleaved in $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$. Proposition 3.7 enables us to deduce that $d_{\mathrm{HI}}(X,Y) \leq \delta + \varepsilon' = d_{\mathrm{CohI}}(X,Y) + \varepsilon + \varepsilon'$ for any ε'. We have the result. □

Remark 3.8. In [2], Berkouk has considered the homotopy interleaving distance in the derived category D^- of the category $\mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R}^d,\leq)}$ of multi-parameter persistence modules whose objects are bounded below. In particular, the result [2, Theorem 2] asserts that the canonical functor ι from $\mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R}^d,\leq)}$ to D^- is object-wise isometric with respect to the interleaving distance and the interleaving distance in the homotopy category described in Section 2.1, respectively. Theorem 3.3 implies that the functor ι factors through the category $(\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}, d_{\mathrm{CohI}})$ with object-wise isometric functors when $d = 1$ and $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ is restricted to cochain complexes bounded above in objects.

4. Interleavings of dg $\mathbb{K}[u]$-modules

Let $\mathbb{K}[u]$ be the polynomial algebra generated by an element u with degree 2.

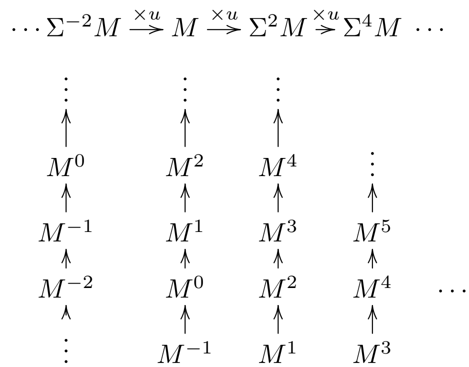
Definition 4.1. A differential graded module $\{M^l, \partial\}$ is a *differential graded (dg)* $\mathbb{K}[u]$*-module* if the complex has a $\mathbb{K}$-linear map $u : M^l \to M^{l+2}$ which satisfies the condition that $u \circ \partial = \partial \circ u$.

Let $\mathbb{K}[u]$-Ch denote the category of dg $\mathbb{K}[u]$-modules. In order to develop persistence theory for dg $\mathbb{K}[u]$-modules, we assign a persistence dg module to each dg $\mathbb{K}[u]$-module via a functor. For a dg $\mathbb{K}[u]$-module $M = \{M^l, \partial\}$, we define a functor $C : \mathbb{K}[u]\text{-}\mathsf{Ch} \to \mathsf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$ by $C(\{M^l, \partial\})(i) = \Sigma^{2i}M$ and

$$C(\{M^l, \partial\})(i \to i+1) : C(\{M^l, \partial\})(i) \xrightarrow{\times u} C(\{M^l, \partial\})(i+1)$$

with the multiplication by u.

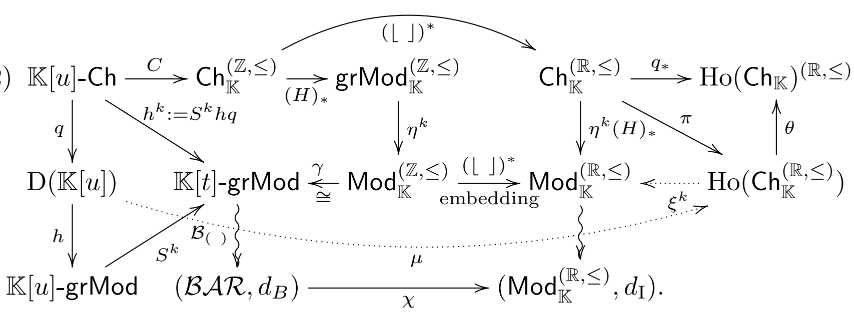
As seen in Section 5, the functor C allows us to bring topological spaces over BS^1 to persistence theory.

(4.1)


The diagram (4.1) explains the persistence module $C(M)$. The vertical arrows are differentials in the dg $\mathbb{K}[u]$-module M.

As for the functor h^k defined below, h^0M and h^1M are direct sums of the cohomology groups of the third and second rows, respectively.

Let $\mathbb{K}[t]$-grMod stand for the category of graded $\mathbb{K}[t]$-module. An object K in $\mathsf{Mod}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$ gives the graded $\mathbb{K}[t]$-module $\gamma(K) := \oplus_{i\in\mathbb{Z}}K(i)$ with the module structure defined by $t \cdot K(i) = K(i+1)$. It is readily seen that γ gives rise to an isomorphism $\gamma : \mathsf{Mod}_{\mathbb{K}}^{(\mathbb{Z},\leq)} \xrightarrow{\cong} \mathbb{K}[t]\text{-}\mathsf{grMod}$ of the categories. We recall the functors $\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)} \xrightarrow{(H)_*} \mathsf{grMod}_{\mathbb{K}}^{(\mathbb{R},\leq)} \xrightarrow{\eta^k} \mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ mentioned before Definition 3.1. Moreover, we have maps $(\mathbb{Z},\leq) \xrightarrow{\iota} (\mathbb{R},\leq) \xrightarrow{\lfloor\ \rfloor} (\mathbb{Z},\leq)$ of posets defined by the usual inclusion ι and the floor function $\lfloor\ \rfloor$, respectively. Observe that with $\lfloor\ \rfloor \circ \iota = 1$. Thus, we obtain a commutative diagram consisting of categories and functors

(4.2)


Here $\mathrm{D}(\mathbb{K}[u])$ denotes the derived category of dg $\mathbb{K}[u]$-modules; see [30, 32], q is the localization, h is the homology functor, S^k is the functor defined by $S^0(M) = \oplus_i M^{2i}$ and $S^1(M) = \oplus_i M^{2i+1}$. We remark that $(S^0(M))^i = M^{2i}$ and $(S^1(M))^i = M^{2i+1}$. The pair $(\mathcal{BAR}, d_B)$ stands for the set of barcodes (multisets of intervals) endowed with the bottleneck distance and $(\mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R},\leq)}, d_{\mathrm{I}})$ is the *class* of the diagrams endowed with the interleaving distance. The wave arrows denote the assignments of the objects, where $\mathcal{B}_{(\ )}$ is defined in the class of locally finite $\mathbb{K}[t]$-modules. The result [10, Theorem 4.16] asserts that χ gives an isometric embedding if the domain is restricted to the set of finite barcodes. Moreover, the functors μ and ξ^k in (4.2) are induced by the universality of the homotopy categories $\mathrm{D}(\mathbb{K}[u])$ and $\mathrm{Ho}(\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)})$, respectively.

Remark 4.2. (i) It follows from the definition of the functor h^k that $H(M) \cong H(N)$ for M and N in $\mathbb{K}[u]$-Ch if $h^k M \cong h^k N$ for $k = 0$ and 1. (ii) Every dg $\mathbb{K}[u]$-module M is formal in the sense that $M \cong (H(M), 0)$ in $\mathrm{D}(\mathbb{K}[u])$. This fact follows from the proof of Lemma 3.5.

Remark 4.3. We may regard the set of morphisms from F to G in $\mathrm{Ho}(\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)})$ as the homotopy set of maps from $\widetilde{F}$ to $\widetilde{G}$, where $\widetilde{F}$ and $\widetilde{G}$ denote the cofibrant replacements of F and G, respectively. In this manuscript, we do not use an explicit form of the cofibrant replacement; see [28, Section 11.6] for the form. Observe that all objects in $\mathrm{Ho}(\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)})$ are fibrant; see [4, Theorem 1.4].

Definition 4.4. Let M and N be dg $\mathbb{K}[u]$-modules. Then, the *even cohomology interleaving distance* $d^0_{\mathrm{CohI}}(M, N)$ and the *odd cohomology interleaving distance* $d^1_{\mathrm{CohI}}(M, N)$ are defined by

$$d_{\mathrm{I}}(\chi(\mathcal{B}_{S^0hq(M)}), \chi(\mathcal{B}_{S^0hq(N)})) \quad \text{and} \quad d_{\mathrm{I}}(\chi(\mathcal{B}_{S^1hq(M)}), \chi(\mathcal{B}_{S^1hq(N)})),$$

respectively; see the diagram (4.2) for the functors h, q and S^k .

By Theorem 2.10 and the commutativity of the diagram (4.2), we establish

Proposition 4.5. $d^k_{\mathrm{CohI}}(M, N) = d_{\mathrm{B}}(\mathcal{B}_{S^khqM}, \mathcal{B}_{S^khqN})$ *for* $k = 0$ *and* 1.

We have the natural functor $\theta := \overline{q^{\mathbb{R}}} : \mathrm{Ho}(\mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}) \to \mathrm{Ho}(\mathsf{Ch}_{\mathbb{K}})^{(\mathbb{R},\leq)}$, which is induced by the localization functor q. Let $\nu^k(\ ) : \mathbb{K}[u]$-$\mathsf{Ch} \to \mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ be the composite $(\lfloor\ \rfloor)^* \circ \eta^k \circ (H)_* \circ C$. Then, it turns out that

$$\begin{aligned} d^k_{\mathrm{CohI}}(M, N) &= d_{\mathrm{I}}(\nu^k M, \nu^k N) \\ &\leq d_{\mathrm{I}}((\theta \circ \mu \circ q)M, (\theta \circ \mu \circ q)N) \\ &\leq d_{\mathrm{I}}((\mu \circ q)M, (\mu \circ q)N) \end{aligned} \tag{4.3}$$

for $k = 0$, 1 and dg $\mathbb{K}[u]$-modules M and N. The inequalities follow from [10, Proposition 3.6].

Let $\alpha : \mathbb{K}[u]$-$\mathsf{Ch} \to \mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ be the functor $(\lfloor\ \rfloor)^* \circ C$. Then, we observe that $d_{\mathrm{HC}}(\alpha M, \alpha N) = d_{\mathrm{I}}((\theta \circ \mu \circ q)M, (\theta \circ \mu \circ q)N)$ and $d_{\mathrm{IHC}}(\alpha M, \alpha N) = d_{\mathrm{I}}((\mu \circ q)M, (\mu \circ q)N)$; see Section 2.1 for the distance d_{HC} and d_{IHC}.

The following proposition shows the reason why we consider the interleaving distances $d^k_{\mathrm{CohI}}(M, N)$ for $k = 0$ and 1 only.

Proposition 4.6. *For each* $l \in \mathbb{Z}$*, it holds that* $d^0_{\mathrm{CohI}}(M, N) = d_{\mathrm{I}}(\nu^{2l} M, \nu^{2l} N)$ *and* $d^1_{\mathrm{CohI}}(M, N) = d_{\mathrm{I}}(\nu^{2l+1} M, \nu^{2l+1} N)$.

Proof. We recall a translation functor $(l) : (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)$ defined by $(l)t = t + l$ and the functor $(l)^* : \mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R},\leq)} \to \mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ induced by (l). We see that $(\nu^{2l})M = (l)^*(\nu^0)M$ and $(\nu^{2l+1})M = (l)^*(\nu^1)M$. It follows that

$$\begin{aligned} d^0_{\mathrm{CohI}}(M, N) = d_{\mathrm{I}}((-l)^*(\nu^{2l})M, (-l)^*(\nu^{2l})N) &\leq d_{\mathrm{I}}((\nu^{2l})M, (\nu^{2l})N) \\ &\leq d^0_{\mathrm{CohI}}(M, N). \end{aligned}$$

By the same argument as above, we have the second equality. □

We describe our main theorem in this section.

Theorem 4.7. *The equalities*

$$d_{\mathrm{HC}}(\alpha M, \alpha N) = d_{\mathrm{IHC}}(\alpha M, \alpha N) = d_{\mathrm{HI}}(\alpha M, \alpha N) = \max\{d^k_{\mathrm{CohI}}(M, N) \mid k = 0, 1\}$$

hold for dg $\mathbb{K}[u]$*-modules* M *and* N*.*

In what follows, we may write $d_{\mathrm{CohI}}(M, N)$ for $\max\{d^k_{\mathrm{CohI}}(M, N) \mid k = 0, 1\}$ and call it the *cohomology interleaving distance* of dg $\mathbb{K}[u]$-modules M and N. By the commutativity of the diagram (4.2) and Proposition 4.6, we see that $d_{\mathrm{CohI}}(M, N) = d_{\mathrm{CohI}}(\alpha(M), \alpha(N))$ for dg $\mathbb{K}[u]$-modules M and N, where the right-hand side stands for the distance of persistence modules described in Definition 3.1.

Proof of Theorem 4.7. We recall the inequalities (4.3). In order to prove the assertion, it suffices to show that $d_{\mathrm{HI}}(\alpha M, \alpha N) \le d_{\mathrm{CohI}}(M, N) =: \varepsilon$. We observe that the funcor α is compatible with the homology functor. Then, Lemma 3.5 allows us to deduce that $\alpha L \simeq H(\alpha L) = \alpha H(L)$ for a dg $\mathbb{K}[u]$-module, where $\alpha = (\lfloor\ \rfloor)^* \circ C$ by definition. Therefore, we have that $d_{\mathrm{HI}}(\alpha M, \alpha N) \le d_{\mathrm{I}}(\alpha H(M), \alpha H(N))$. Moreover, with the same notation as in the proof of Proposition 4.6, we see that for each dg $\mathbb{K}[u]$-module L,

$$\alpha H(L) = \oplus_{l\in\mathbb{Z}}((l)^*\nu^0 H(L) \oplus (l)^*\nu^1 H(L)),$$

where $(l)^*\nu^k H(L)$ is regarded as a persistent dg module concentrated at degree $2l+k$ for $k = 0$ and 1. Since $\varepsilon \ge d^k_{\mathrm{CohI}}(M, N) = d^k_{\mathrm{CohI}}(HM, HN) = d_{\mathrm{I}}(\nu^k HM, \nu^k NH)$ for $k = 0$ and 1, it follows from Remark 2.4 that $d_{\mathrm{I}}(\alpha H(M), \alpha H(N)) \le \varepsilon$. □

Proposition 4.8. *Let* M *and* N *be dg* $\mathbb{K}[u]$*-modules.* (i) *If* $d^k_{\mathrm{CohI}}(M, N) < \frac{1}{2}$*, then* $h^k M \cong h^k N$ *as a* $\mathbb{K}[t]$*-module.*
(ii) *Suppose that* $d_{\mathrm{IHC}}(\alpha M, \alpha N) < \frac{1}{2}$*. Then* $\alpha M \cong \alpha N$ *in* $\mathrm{Ho}(\mathsf{Ch}^{(\mathbb{R},\le)}_{\mathbb{K}})$*. Thus, the distance* d_{IHC} *is an extended metric on objects in the image of* $\alpha : \mathbb{K}[u]\text{-}\mathsf{Ch} \to \mathsf{Ch}^{(\mathbb{R},\le)}_{\mathbb{K}}$*.*

Proof. (i) Let F and G be the persistence modules $\nu^k M$ and $\nu^k N$, respectively. By the assumption, there exists a positive real number ε less than $\frac{1}{2}$ such that F and G are ε-interleaved. For each integer i, we consider the commutative triangles in the diagram (2.2). In view of a property of the floor function, we see that $F(i \to i+2\varepsilon)$ and $G(i \to i+2\varepsilon)$ are isomorphism for each integer i. Therefore, the maps $\varphi(i)$ and $\varphi(i+\varepsilon)$ are injective and surjective, respectively. Moreover, we have a commutative diagram

$$\begin{array}{ccccc} F(i+\varepsilon) & \xleftarrow[\cong]{F(i\to i+\varepsilon)} & F(i) & & \\ & \searrow^{\varphi(i+\varepsilon)} & & \searrow^{\varphi(i)} & \\ & & G(i+2\varepsilon) & \xleftarrow[\cong]{G(i+\varepsilon\to i+2\varepsilon)} & G(i+\varepsilon) \end{array}$$

in which horizontal arrows are isomorphisms. Thus, it follows that $\varphi(i)$ is an isomorphism for each integer i. We observe that $G(i+\varepsilon) = G(i)$. It turns out that $h^0 M = \oplus_i H^{2i} M = \oplus_i \nu^0 M(i) \cong \oplus_i \nu^0 N(i) = \oplus_i H^{2i} N = h^0 N$ and $h^1 M = \oplus_i H^{2i+1} M = \oplus_i \nu^1 M(i) \cong \oplus_i \nu^1 N(i) = \oplus_i H^{2i+1} N = h^1 N$ as $\mathbb{K}[t]$-modules.
(ii) We see that M is formal and then $\alpha M \cong \alpha H(M)$ in $\mathrm{Ho}(\mathsf{Ch}^{(\mathbb{R},\le)}_{\mathbb{K}})$; see Remark 4.2 (ii). The same isomorphism holds for N. Then, the assertion (i), Theorem 4.7 and Remark 4.2 (i) yield the result. □

As mentioned in Introduction, we consider a filtered $\mathbb{K}[t]$-module, where $\deg t = 1$. Let H^* be a non-negatively graded $\mathbb{K}[t]$-module with a filtration

$$H^k = F^0 \supset F^1 \supset \cdots \supset F^i \supset \cdots \supset F^{k+1} = 0$$

of $\mathbb{K}[t]$-submodules for $k \geq 0$. Suppose that $tF^i \subset F^{i+1}$. Then, we have a bigraded $\mathbb{K}[t]$-module $E^{*,*}$ defined by $E^{p,q} := F^pH^{p+q}/F^{p+1}H^{p+q}$. Observe that $t \cdot E^{p,q} = E^{p+1,q}$.

For a bigraded module $E^{*,*}$, we define a graded module $\operatorname{Tot} E^{*,*}$, which is called the *total complex of* $E^{*,*}$, by $(\operatorname{Tot} E^{*,*})^i := \oplus_{p+q=i} E^{p,q}$.

Lemma 4.9. *As a graded* $\mathbb{K}[t]$*-module,* $\operatorname{Tot} E^{*,*} \cong H^*$ *provided* $\dim H^i < \infty$ *for each* i.

Proof. We say that an element $a \in H^*$ has the *filtration degree* p, denoted fil-deg $a = p$, if $a \in F^p$ and $a \notin F^{p+1}$. We prove the lemma by the induction on the degrees and filtration degrees of a basis $\{a^k_\lambda\}_{k\geq 0,\lambda}$ of $H^*/(t)H^*$, where $\deg a^k_\lambda = k$.

Let S_k be the subset $\{a^k_{\lambda_1}, \dots, a^k_{\lambda_{s_k}}\}$ consisting of linearly independent elements of $E^{*,*}/(t)E^{*,*}$ with degree k. We may view S_k as a subset of H^* with fil-deg $a^k_{\lambda_i} \geq$ fil-deg $a^k_{\lambda_j}$ for $i < j$. Let $[S_k]$ be the subset $\{[a^k_{\lambda_1}], \dots, [a^k_{\lambda_{s_k}}]\}$ of $E^{*,*}$, where $[a]$ denotes the image of a by the projection of $F^{\text{fil-deg}a} \to E^{\text{fil-deg}a,*}$.

Since $F^iH^0 = 0$ for $i > 0$, it follows that the $\mathbb{K}[t]$-submodule of H^* generated by S_0 coincides with that of $E^{*,*}$ generated by $[S_0]$. Assume that the map φ_k defined by $\varphi_k([a_\mu]) = a_\mu$ is an isomorphism from the $\mathbb{K}[t]$-submodule E_k of $\operatorname{Tot} E^{*,*}$ generated by $[S_0] \cup \cdots \cup [S_k]$ to the $\mathbb{K}[t]$-submodule H_k of H^* generated by $S_0 \cup \cdots \cup S_k$. We may replace elements in S_i and $[S_i]$ to construct the isomorphism preserving the linear independence of the elements in each set if necessary†.

Suppose that $[a^{k+1}_{\lambda_1}]t^n \neq 0$ in $(E_k + \mathbb{K}[t] \cdot [a^{k+1}_{\lambda_1}])/E_k$ for each $n \geq 0$. Then, it follows that $E_k + \mathbb{K}[t] \cdot [a^{k+1}_{\lambda_1}] = E_k \oplus \mathbb{K}[t] \cdot [a^{k+1}_{\lambda_1}]$. Thus, we have an isomorphism $E_k \oplus \mathbb{K}[t] \cdot [a^{k+1}_{\lambda_1}] \xrightarrow{\cong} H_k \oplus \mathbb{K}[t] \cdot a^{k+1}_{\lambda_1}$ extending φ_k.

Suppose that $[a^{k+1}_{\lambda_1}]t^m = 0$ and $[a^{k+1}_{\lambda_1}]t^{m-1} \neq 0$ in $(E_k + \mathbb{K}[t] \cdot [a^{k+1}_{\lambda_1}])/E_k$ for some $m \geq 0$. Then, we have

$$a^{k+1}_{\lambda_1} t^m - \sum_{i\leq k,j} \beta_{j,i} a^i_{\lambda_j} t^{l_{j,i}} = 0 \tag{4.4}$$

for some nonzero elements $\beta_{j,i} \in \mathbb{K}$. By degree reasons, we see that $l_{j,i} \geq m$. We rewrite $a^{k+1}_{\lambda_1}$ for $a^{k+1}_{\lambda_1} - \sum_{i\leq k,j} \beta_{j,i} a^i_{\lambda_j} t^{l_{j,i}-m}$. Then, we have an isomorphism

$$\begin{aligned}\varphi_{(1)} : E_k \oplus \Sigma^{\deg[a^{k+1}_{\lambda_1}]}(\mathbb{K}[t]/(t^m)) &\cong E_k + \mathbb{K}[t] \cdot [a^{k+1}_{\lambda_1}] \\ &\longrightarrow H_k + \mathbb{K}[t] \cdot a^{k+1}_{\lambda_1} \cong H_k \oplus \Sigma^{\deg a^{k+1}_{\lambda_1}}(\mathbb{K}[t]/(t^m))\end{aligned}$$

defined by $\varphi_{(1)}([a^{k+1}_{\lambda_1}]) = a^{k+1}_{\lambda_1}$ and $\varphi_{(1)}|_{E_k} = \varphi_k$. Observe that $a^{k+1}_{\lambda_1} \cdot t^{m-1} \neq 0$. In fact, if $a^{k+1}_{\lambda_1} \cdot t^{m-1} = 0$, then for the original $a^{k+1}_{\lambda_1}$, $[a^{k+1}_{\lambda_1}] \cdot t^{m-1} = 0$ in $(E_k + \mathbb{K}[t] \cdot [a^{k+1}_{\lambda_1}])/E_k$, which is a contradiction.

Moreover, by applying the same construction of the isomorphism as above to elements $[a^{k+1}_{\lambda_2}], \dots, [a^{k+1}_{\lambda_{s_{k+1}}}]$, we obtain an isomorphism $\varphi_{k+1} : E_{k+1} \xrightarrow{\cong} H_{k+1}$. As a consequence, we have an isomorphism $\varphi : \operatorname{Tot} E^{*,*} \to H^*$ of $\mathbb{K}[t]$-modules. □

†This procedure is clarified by considering the induction step described below.

Remark 4.10. Let M^* be a non-negatively graded $\mathbb{K}[u]$-module with a filtration $M^k = F^0 \supset F^1 \supset \cdots \supset F^i \supset \cdots \supset F^{k+1} = 0$ of $\mathbb{K}[u]$-submodules for $k \geq 0$. Then, we may apply the same argument as in the proof of Lemma 4.9 to M^* provided $uF^i \subset F^{i+1}$ for $i \geq 0$. As a consequence, we see that $\operatorname{Tot} E^{*,*} \cong M^*$ as a graded $\mathbb{K}[u]$-module.

Remark 4.11. So far we consider dg $\mathbb{K}[v]$-modules with $\deg v = 1$ or 2. Even if the degree of u is a positive integer, the same arguments as in this section are applicable to the case and the results remain true with an appropriate degree shift. For example, the functor C in (4.2) is replaced with one defined by $C(\{M^l, \partial\})(i) = \Sigma^{(\deg u)i} M$ for a dg $\mathbb{K}[u]$-module $\{M^l, \partial\}$.

Let M be a dg $\mathbb{K}[u]$-module. In the rest of this section, we compute the cohomology interleaving distances $d^k_{\mathrm{CohI},\mathbb{K}}(M, \mathbb{K})$ and $d^k_{\mathrm{CohI},\mathbb{K}}(M, \mathbb{K}[u]/(u^2))$. Here, we regard $\mathbb{K}$ and $\mathbb{K}[u]/(u^2)$ as dg $\mathbb{K}[u]$-modules with zero differentials. Suppose that the cohomology of M is of finite dimension. For the graded $\mathbb{K}[u]$-module $S^k H^*(M)$, we denote $\mathrm{cup}^k(M)$ by the greatest non-negative integer n such that $(u\times)^n \neq 0$ on $S^k H^*(M)$; see Section 4 for the functor S^k. Observe that $\mathrm{cup}^0(M)$ concerns the cup-length of spaces over BS^1; see the paragraph before Proposition 5.8 for details.

Let $\mathcal{B}^k_M = \{[b_\lambda, b_\lambda + c_\lambda) \mid \lambda \in \Lambda\}$ be the barcode associated with $S^k H^*(M)$, where the index set Λ is finite. Then, it is readily seen that $\mathrm{cup}^k(M) + 1 = \max\{c_\lambda \mid \lambda \in \Lambda\}$. In order to state and prove results of the computations of the distances mentioned above, let $\Lambda_i = \{\lambda \in \Lambda \mid c_\lambda = i\}$ and $\Lambda_{0,i} = \{\lambda \in \Lambda_i \mid b_\lambda = 0\}$. Furthermore, put $l^k := \mathrm{cup}^k(M)$ for short.

Proposition 4.12. *Let M be a dg $\mathbb{K}[u]$-module concentrated in non-negative degrees whose cohomology is of finite dimension. Then, for $k = 0, 1$ it holds that*

$$d^k_{\mathrm{CohI},\mathbb{K}}(M, \mathbb{K}) = \begin{cases} 0 & (H^*(M) \cong \mathbb{K}) \\ \frac{1}{2}(l^k + 1) & (\textit{otherwise}). \end{cases}$$

Proof. When $H^*(M) \cong \mathbb{K}$, we see that $d^k_{\mathrm{CohI},\mathbb{K}}(M, \mathbb{K}) = 0$ immediately. Assume that $H^*(M) \ncong \mathbb{K}$. Let $\mathcal{B}^k_1$ be the barcode associated with $S^k(\mathbb{K})$, namely, $\mathcal{B}^0_1 = \{[0, 1)\}$ and $\mathcal{B}^1_1 = \emptyset$. By virtue of Theorem 2.10, it suffices to compute the bottleneck distance $d_{\mathrm{B}}(\mathcal{B}^k_M, \mathcal{B}^k_1)$ instead of $d^k_{\mathrm{CohI},\mathbb{K}}(M, \mathbb{K})$. For $k = 1$, it is readily seen that

$$d_{\mathrm{B}}(\mathcal{B}^1_M, \mathcal{B}^1_1) = \inf_{h:\mathcal{B}^1_M \leftrightarrow \mathcal{B}^1_1} \sup_{J \in \mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J, \chi_\emptyset) = \inf_{h:\mathcal{B}^1_M \leftrightarrow \mathcal{B}^1_1} \left\{ \frac{1}{2}(l^1 + 1) \right\} = \frac{1}{2}(l^1 + 1).$$

We consider the case $k = 0$. Let $I_\lambda := [b_\lambda, b_\lambda + c_\lambda)$ in $\mathcal{B}^0_M$. Then, Lemma 2.8 enables us to deduce that

$$\begin{aligned} d_{\mathrm{I}}(\chi_{I_\lambda}, \chi_{[0,1)}) &= \min\left\{ \max\{b_\lambda, b_\lambda + c_\lambda - 1\}, \max\left\{\frac{1}{2}, \frac{c_\lambda}{2}\right\} \right\} \\ &= \min\left\{ b_\lambda + c_\lambda - 1, \frac{c_\lambda}{2} \right\} = \begin{cases} c_\lambda - 1 & (b_\lambda = 0, c_\lambda = 1, 2) \\ \frac{c_\lambda}{2} & (\text{otherwise}). \end{cases} \end{aligned}$$

Given a bijection $h : \mathcal{B}^0_M \leftrightarrow \mathcal{B}^0_1$ with $h(I_\lambda) = [0, 1)$. First, consider the case $\lambda \in \Lambda_{0,1}$. Since $H^*(M) \ncong \mathbb{K}$, it follows that $\Lambda \setminus \{\lambda\} \neq \emptyset$. Thus, we have

$$\begin{aligned} \sup_{J \in \mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J, \chi_{h(J)}) &= \max\left\{ d_{\mathrm{I}}(\chi_{I_\lambda}, \chi_{[0,1)}),\ d_{\mathrm{I}}(\chi_{I_\mu}, \chi_\emptyset) \mid \mu \in \Lambda \setminus \{\lambda\} \right\} \\ &= \max\left\{ 0, \frac{c_\mu}{2} \;\middle|\; \mu \in \Lambda \setminus \{\lambda\} \right\} = \frac{1}{2}(l^0 + 1). \end{aligned}$$

If $\lambda \in \Lambda_{0,2}$, then

$$\begin{aligned}\sup_{J\in \mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J, \chi_{h(J)}) &= \max\left\{ d_{\mathrm{I}}(\chi_{I_\lambda}, \chi_{[0,1)}),\ d_{\mathrm{I}}(\chi_{I_\mu}, \chi_{\emptyset}) \;\middle|\; \mu \in \Lambda\setminus\{\lambda\}\right\} \\ &= \max\left\{1, \frac{c_\mu}{2} \;\middle|\; \mu\in\Lambda\setminus\{\lambda\}\right\} \\ &= \begin{cases} 1 & (l^0=1) \\ \frac{1}{2}(l^0+1) & (l^0\geq 2)\end{cases} = \frac{1}{2}(l^0+1).\end{aligned}$$

Observe that $l^0 \geq 1$ in the case where $\Lambda_{0,2} \neq \emptyset$. Furthermore, if $\lambda \in \Lambda \setminus (\Lambda_{0,1}\cup\Lambda_{0,2})$,

$$\sup_{J\in \mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J, \chi_{h(J)}) = \max\left\{\frac{c_\lambda}{2}, \frac{c_\mu}{2} \;\middle|\; \mu\in\Lambda\setminus\{\lambda\}\right\} = \frac{1}{2}(l^0+1).$$

The computations of suprema enables us to obtain the equality

$$d_{\mathrm{B}}(\mathcal{B}^0_M, \mathcal{B}^0_1) = \inf_{h:\mathcal{B}^0_M\leftrightarrow\mathcal{B}^0_1}\ \sup_{J\in\mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J,\chi_{h(J)}) = \frac{1}{2}(l^0+1).$$

We have the result. □

With the same notation as above, we have the following result.

Proposition 4.13. *Let M be a dg $\mathbb{K}[u]$-module concentrated in non-negative degrees whose cohomology is of finite dimension. Then, one gets*

(1) $d^0_{\mathrm{CohI},\mathbb{K}}(M, \mathbb{K}[u]/(u^2)) = \begin{cases} 1 & (l^0=0) \\ l^0-1 & (\#\Lambda=1,\ l^0=1,2) \\ \frac{1}{2}l^0 & (\#\Lambda\geq 2,\ l^0=i,\ \#\Lambda_i=1,\ i=1,2) \\ \frac{1}{2}(l^0+1) & (\textit{otherwise}),\end{cases}$

(2) $d^1_{\mathrm{CohI},\mathbb{K}}(M, \mathbb{K}[u]/(u^2)) = \frac{1}{2}(l^1+1)$.

Here $\#S$ denotes the cardinal of a set S.

In order to prove Proposition 4.13, we set up more notation. Let $\mathcal{B}^k_2$ denote the barcode associated with $S^k\left(\mathbb{K}[u]/(u^2)\right)$. Since $\mathbb{K}[u]/(u^2)$ is concentrated in even degrees, $\mathcal{B}^0_2 = \{[0,2)\}$ and $\mathcal{B}^1_2 = \emptyset$. Let Π be the set of all bijections between $\mathcal{B}^0_M$ and $\mathcal{B}^0_2$, and $\Pi_{0,i}$ the subset of Π consisting of bijections $h : \mathcal{B}^0_M \leftrightarrow \mathcal{B}^0_2$ such that $h([b_\lambda, b_\lambda + c_\lambda)) = [0,2)$ with $\lambda\in\Lambda_{0,i}$. We write Π_+ for the complement of the union $\cup_i \Pi_{0,i}$ in Π.

Proof of Proposition 4.13. Since $\mathcal{B}^1_2 = \emptyset$, the assertion (2) follows from Theorem 2.10 and Lemma 2.8 (1).

By applying Theorem 2.10, we see that the right-hand side of the equality in (1) coincides with the bottleneck distance $d_{\mathrm{B}}(\mathcal{B}^0_M, \mathcal{B}^0_2)$, which is the smallest value of the infima

$$\mathcal{I}_+ := \inf_{h\in\Pi_+}\ \sup_{J\in\mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J,\chi_{h(J)}) \quad\text{and}\quad \mathcal{I}_{0,i} := \inf_{h\in\Pi_{0,i}}\ \sup_{J\in\mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J,\chi_{h(J)})$$

for $i = 1, 2, \ldots, l^0+1$. To obtain the equality in (1), we determine the values of these infima.

For any barcode $I_\lambda := [b_\lambda, b_\lambda + c_\lambda)$ in $\mathcal{B}^0_M$, the assumption implies $b_\lambda \geq 0$. It follows from Lemma 2.8 that

$$d_{\mathrm{I}}(\chi_{I_\lambda}, \chi_{[0,2)}) = \begin{cases} 1 & (c_\lambda = 1) \\ \frac{1}{2}c_\lambda & (c_\lambda\geq 2)\end{cases} \tag{4.5}$$

for the case $b_\lambda \geq 1$, and

$$d_{\mathrm{I}}(\chi_{I_\lambda}, \chi_{[0,2)}) = \begin{cases} 1 & (c_\lambda = 1) \\ c_\lambda - 2 & (c_\lambda = 2, 3) \\ \frac{1}{2} c_\lambda & (c_\lambda \geq 4) \end{cases} \tag{4.6}$$

for the case $b_\lambda = 0$. Given a bijection $h : \mathcal{B}^0_M \leftrightarrow \mathcal{B}^0_2$. If $h \in \Pi_+$, then (4.5) yields

$$\sup_{J \in \mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J, \chi_{h(J)}) = \begin{cases} 1 & (l^0 = 0) \\ \frac{1}{2}(l^0 + 1) & (l^0 \geq 1). \end{cases}$$

Hence, $\mathcal{I}_+ = 1$ if $l^0 = 0$, and $\mathcal{I}_+ = (l^0 + 1)/2$ if $l^0 \geq 1$. We next consider the case $h \in \Pi_{0,i}$ with $h(I_\lambda) = [0, 2)$ for some $\lambda \in \Lambda_{0,i}$. Then, the equality (4.6) enables us to deduce that

$$\begin{aligned} \sup_{J \in \mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J, \chi_{h(J)}) &= \max \left\{ d_{\mathrm{I}}(\chi_{I_\lambda}, \chi_{[0,2)}),\ d_{\mathrm{I}}(\chi_{I_\mu}, \chi_{\emptyset}) \ \middle|\ \mu \in \Lambda \setminus \{\lambda\} \right\} \\ &= \max \left\{ d_{\mathrm{I}}(\chi_{I_\lambda}, \chi_{[0,2)}),\ \frac{1}{2} c_\mu \ \middle|\ \mu \in \Lambda \setminus \{\lambda\} \right\} \\ &= \begin{cases} \max \left\{ 1, \frac{1}{2} c_\mu \ \middle|\ \mu \in \Lambda \setminus \{\lambda\} \right\} & (c_\lambda = 1) \\ \max \left\{ c_\lambda - 2, \frac{1}{2} c_\mu \ \middle|\ \mu \in \Lambda \setminus \{\lambda\} \right\} & (c_\lambda = 2, 3) \\ \frac{1}{2}(l^0 + 1) & (c_\lambda \geq 4). \end{cases} \end{aligned} \tag{4.7}$$

Consider the case $\#\Lambda = 1$ with $\Lambda = \{\lambda\}$. We observe that $\Pi_{0,i} = \emptyset$ for $i = 1, 2, \ldots, l^0$. Since $c_\lambda = l^0 + 1$ and $\Lambda \setminus \{\lambda\} = \emptyset$, it follows from (4.7) that

$$\mathcal{I}_{0,l^0+1} = \begin{cases} 1 & (l^0 = 0) \\ l^0 - 1 & (l^0 = 1, 2) \\ \frac{1}{2}(l^0 + 1) & (l^0 \geq 3). \end{cases}$$

In the case $\#\Lambda \geq 2$, we see that $\mathcal{I}_{0,i} = (l^0 + 1)/2$ for $l^0 \geq 1$ and $i = 1, 2, \ldots, l^0$. Indeed, for any $h \in \Pi_{0,i}$ associated with I_λ; that is, $h(I_\lambda) = [0, 2)$ and $I_\lambda = [0, i)$, the equality (4.7) gives

$$\sup_{J \in \mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J, \chi_{h(J)}) = \frac{1}{2}(l^0 + 1)$$

since there exists $\mu \in \Lambda \setminus \{\lambda\}$ such that $c_\mu = l^0 + 1$. Furthermore, it follows from (4.7) that

$$\begin{aligned} \mathcal{I}_{0,l^0+1} &= \begin{cases} \min \left[\max \left\{ 1,\ \frac{1}{2} c_\mu \ \middle|\ \mu \in \Lambda \setminus \{\lambda\} \right\} \ \middle|\ \lambda \in \Lambda_{0,1} \right] & (l^0 = 0) \\ \min \left[\max \left\{ l^0 - 1,\ \frac{1}{2} c_\mu \ \middle|\ \mu \in \Lambda \setminus \{\lambda\} \right\} \ \middle|\ \lambda \in \Lambda_{0,l^0+1} \right] & (l^0 = 1, 2) \\ \frac{1}{2}(l^0 + 1) & (l^0 \geq 3) \end{cases} \\ &= \begin{cases} 1 & (l^0 = 0) \\ \frac{1}{2} l^0 & (l^0 = 1, 2,\ \#\Lambda_{l^0+1} = 1) \\ \frac{1}{2}(l^0 + 1) & (\text{otherwise}). \end{cases} \end{aligned}$$

We remark that $c_\lambda = 1$ for any $\lambda \in \Lambda$ in the case $l^0 = 0$. The condition $\#\Lambda_{l^0+1} = 1$ implies $\Lambda_{l^0+1} = \Lambda_{0,l^0+1}$ and the inequality $c_\mu < l^0 + 1$ for any $\mu \in \Lambda \setminus \Lambda_{0,l^0+1}$. On the other hand, if $\#\Lambda_{l^0+1} \geq 2$, then for any $\lambda \in \Lambda_{0,l^0+1}$, there exists $\mu \in \Lambda$ such that $c_\mu = l^0 + 1$ and $\mu \neq \lambda$. Therefore, by taking the smallest value among $\mathcal{I}_+$ and $\mathcal{I}_{0,i}$ for $i = 1, 2, \ldots, l^0 + 1$ computed above, we have the assertion (1). $\square$

5. The cohomology interleaving of spaces over BS^1

Let $\mathbb{K}$ be a field. Unless otherwise explicitly stated, it is assumed that a space X is connected and the singular cohomology of X with coefficients $\mathbb{K}$ is locally finite; that is, the ith cohomology of X is of finite dimension for $i \geq 0$.

Let $p : X \to BS^1$ be a space over BS^1. We have a quasi-isomorphism $\kappa : \mathbb{K}[u] \to C^*(BS^1; \mathbb{K})$ and the morphism $p^* : C^*(BS^1; \mathbb{K}) \to C^*(X; \mathbb{K})$ of differential graded algebras (DGAs). Then, the singular cochain complex $C^*(X; \mathbb{K})$ is regarded as a $\mathbb{K}[u]$-module via the maps $p^* \circ \kappa$.

The even and odd cohomology interleaving distances (Definition 4.4) give the *cohomology interleaving distances* $d^k_{\mathrm{CohI},\mathbb{K}}(X, Y)$ between the spaces X and Y over BS^1 defined by $d^k_{\mathrm{CohI}}(C^*(X; \mathbb{K}), C^*(Y; \mathbb{K}))$ for $k = 0$ and 1, respectively. We write $d_{\mathrm{CohI},\mathbb{K}}(X, Y)$ for $\max\{d^k_{\mathrm{CohI}}(M, N) \mid k = 0, 1\}$. By Theorem 4.7, we see that the distance $d_{\mathrm{CohI},\mathbb{K}}(X, Y)$ determines d_{HC}, d_{IHC} and d_{HI} between $\alpha C^*(X; \mathbb{K})$ and $\alpha C^*(Y; \mathbb{K})$.

Let Y be an S^1-space. We consider the Borel construction $Y_{hS^1} := ES^1 \times_{S^1} Y$ which fits into the Borel fibration $Y \to Y_{hS^1} \xrightarrow{p} BS^1$. Let LX be the free loop space, namely, the space of continuous maps from S^1 to X endowed with the compact-open topology. The rotation on S^1 induces the action $\mu : S^1 \times LX \to LX$ on the free loop space. Thus, we have the Borel fibration $p : LX_{hS^1} \to BS^1$. For a space X, we denote by $l(X)_{\mathbb{K}}$ the integer $\max\{i \mid H^i(X; \mathbb{K}) \neq 0, i \geq 0\}$. We investigate the cohomology interleaving distance between spaces, which are in the classes defined below.

- Class (I) consists of the Borel constructions $(LX)_{hS^1}$ of the free loop spaces LX of simply-connected spaces X.
- Class (II) consists of the spaces X for each of which X fits in a fibration $\mathcal{F} : F \to X \to BS^1$ with $l(F)_{\mathbb{K}} < \infty$.
- Class (III) consists of the spaces $X \to BS^1$ over BS^1 with $l(X)_{\mathbb{K}} < \infty$. As a consecuence, the local finiteness condition of the cohomology implies that $H^*(X; \mathbb{K})$ is of finite dimension.

In order to exhibit our result on the cohomology interleaving distance between spaces in Class (I), we here introduce the *BV-exactness* of a simply-connected space X; see [34, Definition 2.9]. By definition, the BV-operator Δ on $H^*(LX; \mathbb{Q})$ is the composite

$$\Delta : H^*(LX; \mathbb{Q}) \xrightarrow{\mu^*} H^*(S^1 \times LX; \mathbb{Q}) \xrightarrow{\int_{S^1}} H^{*-1}(LX; \mathbb{Q}),$$

where $\int_{S^1}$ stands for the integration along the fundamental class of S^1.

Definition 5.1. A simply-connected space X is *Batalin-Vilkovisky exact (BV-exact)* if $\operatorname{Im} \widetilde{\Delta} = \operatorname{Ker} \widetilde{\Delta}$, where $\widetilde{\Delta} \colon \widetilde{H}^*(LX; \mathbb{Q}) \to \widetilde{H}^*(LX; \mathbb{Q})$ is the restriction of the BV-operator to the reduced cohomology groups.

We also recall the *S-action* on $H^*(LX_{hS^1}; \mathbb{Q})$ which is the multiplication $S := \times u : H^*((LX)_{hS^1}; \mathbb{Q}) \to H^*((LX)_{hS^1}; \mathbb{Q})$ defined by $S(x) := p^*(u)x$ for $x \in H^*((LX)_{hS^1}; \mathbb{Q})$, where $p : (LX)_{hS^1} \to BS^1$ is the projection. We view the one-point space pt as the S^1-space with the trivial action.

Theorem 5.2. [34, Theorem 2.11] *A simply-connected space X is BV-exact if and only if the reduced S-action on $\widetilde{H}^*(LX_{hS^1}; \mathbb{Q})$ is trivial, where $\widetilde{H}^*(LX_{hS^1}; \mathbb{Q})$*

denotes the cokernel of the map $H^*(\mathrm{pt}_{hS^1};\mathbb{Q}) \to H^*(LX_{hS^1};\mathbb{Q})$ *induced by the trivial map.*

We call a simply-connected space X *formal* if there exists a zig-zag of quasi-isomorphisms of differential graded algebras between the singular cochain algebra $C^*(X,\mathbb{Q})$ and the cohomology algebra $H^*(X;\mathbb{Q})$ with the trivial differential.

Corollary 5.3. ([34, Corollary 2.13]) *If a simply-connected space X is formal, then it is BV-exact.*

The cohomology interleaving distance between BV-exact spaces in Class (I) is determined explicitly.

Proposition 5.4. *Let X and Y be formal spaces, more general BV-exact spaces. Then, it holds that for $k=0$ and 1,*

$$d^k_{\mathrm{CohI},\mathbb{Q}}((LX)_{hS^1},(LY)_{hS^1}) = \begin{cases} 0 & \text{if } h^kC^*((LX)_{hS^1},\mathbb{Q}) \cong h^kC^*((LY)_{hS^1},\mathbb{Q}) \\ & \text{as a } \mathbb{Q}[t]\text{-module,} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

In particular, $d_{\mathrm{CohI},\mathbb{Q}}((LX)_{hS^1},(LY)_{hS^1}) = 0$ *if and only if* $C^*((LX)_{hS^1};\mathbb{Q}) \cong C^*((LY)_{hS^1};\mathbb{Q})$ *in* $\mathrm{D}(\mathbb{Q}[\mathrm{u}])$.

Proof. We first prove that the cohomology interleaving distance $d^k_{\mathrm{CohI},\mathbb{Q}}$ is less than or equal to $\frac{1}{2}$. For a simply-connected space X, we observe that $1\cdot t^s \neq 0$ for each $s\geq 0$ and the unit $1\in H^0(LX_{hS^1};\mathbb{Q})$; see [40, Theorem A] for a Sullivan model for LX_{hS^1}. Moreover, it follows from the BV-exactness that $x\cdot t = 0$ for each element $x\in H^i(LX_{hS^1};\mathbb{Q})$ with $i>0$. Then, the barcode associated with $S^kC^*(LX_{hS^1};\mathbb{Q})$ for each $k=0$ and 1 consists of one interval $[0,\infty)$ and intervals of the form $[i,i+1)$. Observe that the interval $[0,\infty)$ appears in the barcode only if $k=0$. By [10, Propositions 4.13] and Lemma 2.8, we see that $d_{\mathrm{I}}(\chi_{[0,\infty)},\chi_{[0,\infty)}) = 0$, $d_{\mathrm{I}}(\chi_{[0,\infty)},\chi_{[i,i+1)}) = \infty$, $d_{\mathrm{I}}(\chi_{\emptyset},\chi_{[0,\infty)}) = \infty$, $d_{\mathrm{I}}(\chi_{\emptyset},\chi_{[0,\infty)}) = \infty$, $d_{\mathrm{I}}(\chi_{\emptyset},\chi_{[i,i+1)}) = \frac{1}{2}$ and $d_{\mathrm{I}}(\chi_{[j,j+1)},\chi_{[i,i+1)}) \leq \frac{1}{2}$. We consider the bottleneck distance between barcodes $\mathcal{B}_{H^*(LX_{hS^1};\mathbb{Q})}$ and $\mathcal{B}_{H^*(LY_{hS^1};\mathbb{Q})}$. If a bijection f in Definition 2.9 assigns $[0,\infty)$ to $[0,\infty)$, then the supremum $\sup_{I\in\mathrm{dom}(f)} d_{\mathrm{I}}(\chi_I,\chi_{f(I)})$ is less than or equal to $\frac{1}{2}$. On the otherwise, the supremum is infinite. Thus, Proposition 4.5 enables us to deduce that $d^k_{\mathrm{CohI},\mathbb{Q}}((LX)_{hS^1},(LY)_{hS^1}) \leq \frac{1}{2}$ for $k=0$ and 1.

Assume further that $M := h^kC^*((LX)_{hS^1},\mathbb{Q})$ and $N := h^kC^*((LY)_{hS^1},\mathbb{Q})$ are ε-interleaved for some $\varepsilon < \frac{1}{2}$. By Proposition 4.8 (i), we see that $M\cong N$ as a $\mathbb{Q}[t]$-module.

The latter half of the assertions follows from the first half and Remark 4.2. □

Before describing upper and lower bounds of the cohomology interleaving distance of spaces, we recall the *cup-length* $\mathrm{cup}(f)_R$ of a map $f : X\to Y$ with the coefficient in a commutative ring R. By definition, the integer $\mathrm{cup}(f)_R$ is the length of the longest non-zero product in the image of the homomorphism $f^* : \widetilde{H}^*(Y;R)\to \widetilde{H}^*(X;R)$ between the reduced cohomology groups. We observe that $\mathrm{cup}(f)\leq \mathrm{cat}(f)$, where $\mathrm{cat}(f)$ denotes the category of the map f, namely the least integer n such that X can be covered by $n+1$ open subsets U_i, for which the restriction of f to each U_i is nullhomotopic, see [7, Proposition 1.10].

The following proposition gives a rough evaluation of the interleaving distance between spaces over BS^1.

Proposition 5.5. *Let $v_1 : X \to BS^1$ and $v_2 : Y \to BS^1$ be spaces over BS^1. Then, it holds that for $k = 0$ and 1,*

$$d^k_{\mathrm{CohI},\mathbb{K}}(X,Y) \leq \frac{1}{2}\max\{\mathrm{cup}(v_1)_{\mathbb{K}} + 1, \mathrm{cup}(v_2)_{\mathbb{K}} + 1\}.$$

In particular, the cohomology interleaving distances between spaces in Class (III) *are finite.*

Lemma 5.6. *Let $v : X \to BS^1$ be a space over BS^1 and $\mathbb{K}$ be a field. Then, the length of the longest bar J in $\mathcal{B}_{h^0H^*(X;\mathbb{K})}$ and $\mathcal{B}_{h^1H^*(X;\mathbb{K})}$ is less than or equal to $\mathrm{cup}(v)_{\mathbb{K}} + 1$.*

Proof. Let s be the integer $\mathrm{cup}(v)_{\mathbb{K}} + 1$. Then, it follows from the definition of the cup-length that $v^*(u)^s = 0$ in $H^*(X;\mathbb{K})$. Therefore, we see that $m_i v^*(u)^s = 0$ for each element m_i of a basis $\{m_i\}_{i\in\Lambda}$ of $H^*(X;\mathbb{K})/(v^*(u))H^*(X;\mathbb{K})$. This fact enables us to deduce that the length of J is less than or equal to s. □

Proof of Proposition 5.5. The result follows from Lemmas 2.8 and 5.6. □

Example 5.7. Let $(LM)_{hS^1}$ and Y be in Classes (I) and (III), respectively.

(1) It follows that $d^0_{\mathrm{CohI},\mathbb{K}}((LM)_{hS^1}, Y) = \infty$. In fact, we see that $1 \cdot t^l \neq 0$ for each $l \geq 1$ and the unit $1 \in H^0(LM_{hS^1};\mathbb{Q})$. The argument in Example 2.7 allows us to obtain the result.

(2) Let F be the fiber of a fibration $\mathcal{F} : X \to BS^1$ in Class (II). Assume that the dimension of the cohomology $H^*(F;\mathbb{K})$ is greater than or equal to 2 and the Leray–Serre spectral sequence for $\mathcal{F}$ with coefficients in $\mathbb{K}$ collapses at the E_2-term. Then, we see that $d_{\mathrm{CohI},\mathbb{K}}(X,Y) = \infty$ and $d_{\mathrm{CohI},\mathbb{Q}}(X,(LM)_{hS^1}) = \infty$ if M is BV-exact. These facts follow from Example 2.7, Remark 4.10 and Theorem 2.10.

Let $f : X \to BS^1$ be a space over BS^1. We will denote by $\mathrm{cup}^k(f)_{\mathbb{K}}$ the largest positive integer n such that the action of u^n on $S^kH^*(X;\mathbb{K})$ is nontrivial; see the diagram (4.2) for the functor S^k. Observe that the integer $\mathrm{cup}^0(f)_{\mathbb{K}}$ coincides with the cup-length of f mentioned above: $\mathrm{cup}^0(f)_{\mathbb{K}} = \mathrm{cup}(f)_{\mathbb{K}}$. Recall the notation $\mathrm{cup}^k(C^*(X;\mathbb{K}))$ stated before Proposition 4.12. It follows from the definition that

$$\mathrm{cup}^k(C^*(X;\mathbb{K})) = \mathrm{cup}^k(f)_{\mathbb{K}}. \tag{5.1}$$

Proposition 5.8. *Let $v : X \to BS^1$ a space over BS^1 in Class (III). Then, the cohomology interleaving distance between $v : X \to BS^1$ and $\mathrm{pt} \to BS^1$ is computed as follows.*

$$d^k_{\mathrm{CohI},\mathbb{K}}(X,\mathrm{pt}) = \begin{cases} 0 & (H^*(X;\mathbb{K}) \cong \mathbb{K}) \\ \frac{1}{2}(\mathrm{cup}^k(v)_{\mathbb{K}} + 1) & (\textit{otherwise}). \end{cases}$$

Proof. Proposition 4.12 and (5.1) yield the result. □

Proposition 5.9. *Let $v_1 : X \to BS^1$ and $v_2 : Y \to BS^1$ be spaces over BS^1 in Class (III). Assume further that $H^*(X;\mathbb{K}) \not\cong \mathbb{K}$ and $H^*(Y;\mathbb{K}) \not\cong \mathbb{K}$. Then, it holds that*

$$d^k_{\mathrm{CohI},\mathbb{K}}(X,Y) \geq \frac{1}{2}|\mathrm{cup}^k(v_1)_{\mathbb{K}} - \mathrm{cup}^k(v_2)_{\mathbb{K}}|.$$

Proof. The triangle inequality of the interleaving distance, we have

$$d^k_{\mathrm{CohI},\mathbb{K}}(X,Y) \quad \geq \quad |d^k_{\mathrm{CohI},\mathbb{K}}(X,\mathrm{pt}) - d^k_{\mathrm{CohI},\mathbb{K}}(Y,\mathrm{pt})|.$$

Thus, Proposition 5.8 allows us to deduce the result. □

An argument on a spectral sequence is helpful to consider the cohomology interleaving distance between given spaces over BS^1.

Proposition 5.10. *Let $\mathcal{F}_i : F_i \to X_i \to BS^1$ be a fibration for $i = 1$ and 2. Assume that F_i is a connected and $H^*(F_i;\mathbb{K})$ is locally finite for each i. Let $\{{}_iE_r^{*,*}, d_r\}$ be the Leray–Serre spectral sequence for $\mathcal{F}_i$ with coefficients in $\mathbb{K}$. Suppose that the spectral sequences collapse at the E_{r+1}-term. Then,*

$$d_{\mathrm{CohI},\mathbb{K}}(X,Y) = d_{\mathrm{IHC}}(\alpha(\mathrm{Tot}({}_1E_r^{*,*}, d_r)), \alpha(\mathrm{Tot}({}_2E_r^{*,*}, d_r))).$$

In particular, $d_{\mathrm{CohI},\mathbb{K}}(X,Y) = d_{\mathrm{IHC}}(\alpha(\mathrm{Tot}({}_1E_2^{*,*}, 0)), \alpha(\mathrm{Tot}({}_2E_2^{*,*}, 0)))$ if the spectral sequences collapse at the E_2-term.

Proof. Since the spectral sequences collapse at the E_{r+1}-term, it follows that

$$\begin{aligned} d_{\mathrm{CohI},\mathbb{K}}(\mathrm{Tot}\,{}_1E_\infty^{*,*}, \mathrm{Tot}\,{}_2E_\infty^{*,*}) &= d_{\mathrm{CohI},\mathbb{K}}((\mathrm{Tot}\,{}_1E_{r+1}^{*,*}, 0), (\mathrm{Tot}\,{}_2E_{r+1}^{*,*}, 0)) \\ &= d_{\mathrm{IHC}}(\alpha(\mathrm{Tot}({}_1E_r^{*,*}, d_r)), \alpha(\mathrm{Tot}({}_2E_r^{*,*}, d_r))). \end{aligned}$$

Observe that Theorem 4.7 gives the second equality. The result follows from Lemma 4.9; see Remark 4.10. □

Remark 5.11. The same result as above holds for the cobar type Eilenberg–Moore spectral sequence converging to $H^*(X_{hS^1};\mathbb{K})$ for an S^1-space X; see, for example, [19], [33, Theorem 2.2, ii)] for the spectral sequence. In fact, let $\{E_r^{*,*}, d_r\}$ and $\{{}'E_r^{*,*}, {}'d_r\}$ be the Eilenberg–Moore spectral sequences converging to $H^*(X_{hS^1};\mathbb{K})$ and $H^*(\mathrm{pt}_{hS^1};\mathbb{K})$, respectively. We have the S^1-equivariant map $f : X \to \mathrm{pt}$. Then, the naturality of the multiplicative spectral sequence gives a morphism $\{f_r\} : \{{}'E_r^{*,*}, {}'d_r\} \to \{E_r^{*,*}, d_r\}$ of spectral sequences with

$$f_2 : \mathbb{K}[u] \cong {}'E_2^{p,q} \cong \mathrm{Cotor}^{*,*}_{H^*(S^1)}(\mathbb{K},\mathbb{K}) \to E_2^{*,*} \cong \mathrm{Cotor}^{p,q}_{H^*(S^1)}(\mathbb{K}, H^*(X)),$$

where $\mathrm{bideg}\, u = (1,1)$. Thus, the spectral sequence $\{E_r^{*,*}, d_r\}$ has a dg $\mathbb{K}[u]$-module structure which is compatible with the $\mathbb{K}[u]$-module structure on $H^*(X_{hS^1};\mathbb{K})$.

The following corollary provides an approach for computing the interleaving distance between spaces in Class (II).

Corollary 5.12. *Let $\mathcal{F}_i : F_i \to X_i \to BS^1$ be a fibration with connected fiber for $i = 1$ and 2. Suppose further that for each i, the spectral sequence for $\mathcal{F}_i$ collapses at the E_2-term and $l(F_i)_\mathbb{K} < \infty$. Then, the equality*

$$d^k_{\mathrm{CohI},\mathbb{K}}(X_1, X_2) = \inf_{f : J^k_{F_1} \leftrightarrow J^k_{F_2}} \sup_{j \in \mathrm{dom}(f)} \{|j - f(j)|\}$$

holds for $k = 0$ and 1, where $J^k_{F_i}$ denotes the multiset defined by

$$\coprod_{l = 2m+k \ \text{with} \ H^l(F_i;\mathbb{K}) \neq 0} \left(\coprod_{\dim H^l(F_i;\mathbb{K})} \{\lfloor \frac{l}{2} \rfloor\} \right).$$

Proof. The collapsing of the spectral sequence for $\mathcal{F}_i$ yields that the barcode B_i associated with $H^*(X_i;\mathbb{K})$ consists of infinite intervals $[\lfloor \frac{l}{2} \rfloor, \infty)$ with $\dim H^l(F_i;\mathbb{K}) \neq 0$. We observe that each barcode B_i is finite. Then, the result follows from Theorem 2.10 and Lemma 2.8 (3). □

We conclude this section with a result which describes an upper bound of the cohomology interleaving distance between manifolds. It is worthwhile that a map between the manifolds gives rise to one of the interleavings which induce the upper bound.

Proposition 5.13. *Let $u : X \to BS^1$ and $v : Y \to BS^1$ be connected closed oriented manifolds over BS^1. Suppose that there exists a continuous map $f : X \to Y$ with $v \circ f = u$. Then*

(i) $d_{\mathrm{CohI},\mathbb{K}}(X, Y) \leq \frac{1}{2}(\dim Y - \dim X)$ *if* $\dim X$ *and* $\dim Y$ *are even and* $\dim Y \geq 2 \dim X$, *and*

(ii) $d_{\mathrm{CohI},\mathbb{K}}(X, Y) < \frac{1}{2}(\dim Y - \dim X)$ *if* $\dim X$ *and* $\dim Y$ *are odd and* $\dim Y > 2 \dim X$.

Before proving the result, we recall a *δ-trivial* persistence module M which satisfies the condition that $M(i \to i + \delta) : M(i) \to M(i + \delta)$ is trivial for any i.

Proof of Proposition 5.13. Let m be the non-negative integer $\dim Y - \dim X$. The shriek map $f^!$ is an element of $\mathrm{Ext}^m_{C^*(Y;\mathbb{K})}(C^*(X; \mathbb{K}), C^*(Y; \mathbb{K}))$ which assigns the volume form of Y to that of X, where the Ext group is defined in the derived category of $C^*(Y; \mathbb{K})$-modules; see, for example, [24]. We have the composite map $\mathbb{K}[u] \stackrel{\kappa}{\to} C^*(BS^1; \mathbb{K}) \stackrel{v^*}{\to} C^*(Y; \mathbb{K})$ of morphisms of dg algebras, where $H(\kappa)(u)$ is the generator of $H^*(BS^1; \mathbb{K})$. Then, the map $H(f^!) : H^*(X; \mathbb{K}) \to \Sigma^m H^*(Y; \mathbb{K})$ induced by shriek map $f^!$ is a morphism of $\mathbb{K}[u]$-modules. Observe that the map $H(f^!)$ gives rise to map $H(f^!) : h^k C^*(X, \mathbb{K}) \to (h^k C^*(Y, \mathbb{K}))^{\frac{m}{2}}$ for each $k = 0, 1$ because m is even. Here $(\ )^{\frac{m}{2}}$ denotes the shift functor defined in Remark 2.2 and we regard the codomain of the functor h^k as $\mathsf{Mod}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ suppressing the isomorphism γ and the embedding $(\lfloor\ \rfloor)^*$ in the diagram (4.2).

(i) In view of [5, Corollary 6.6], in order to prove that $h^k C^*(X, \mathbb{K})$ and $h^k C^*(Y, \mathbb{K})$ for $k = 0$ and 1 are $\frac{m}{2}$-interleaved, it suffices to show that the kernel and the cokernel of $H(f^!)$ are $2(\frac{m}{2})$-trivial. Since the shriek map $f^!$ preserve the volume forms, it follows that $\mathrm{Ker}\, H(f^!)$ is $2(\frac{m}{2})$-trivial if $2(\frac{m}{2}) \geq \frac{\dim Y}{2}$. Moreover, we see that $\mathrm{Coker}\, H(f^!)$ is $2(\frac{m}{2})$-trivial if $2(\frac{m}{2}) \geq \frac{\dim X}{2}$. Thus, the result follows from the assumption that $\dim Y \geq 2 \dim X$.

(ii) The same argument as in the proof of (i) enables us to obtain the result (ii). We observe that the maps $H(f^!)|_{h^k C^*(X,\mathbb{K})}$ for $k = 0$ and 1 are $2(\frac{m}{2})$-trivial if $\dim Y > 2 \dim X$. □

6. Toy examples

By applying Proposition 4.5, we give computational examples of the cohomology interleaving distances.

Proposition 6.1. *Let $f_{n,j} : \mathbb{C}P^n \to BS^1$ be a map which represents an integer j under the identifications $[\mathbb{C}P^n, BS^1] \cong H^2(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$. Then, it holds that*

(1) $d^0_{\mathrm{CohI},\mathbb{Q}}((\mathbb{C}P^n, f_{n,1}), (\mathbb{C}P^m, f_{m,1})) = \min\left\{|n - m|, \max\left\{\frac{m+1}{2}, \frac{n+1}{2}\right\}\right\}$,

(2) $d^0_{\mathrm{CohI},\mathbb{Q}}((\mathbb{C}P^n, f_{n,0}), (\mathbb{C}P^n, f_{n,1})) = \lceil \frac{n}{2} \rceil$,

(3) $d^0_{\mathrm{CohI},\mathbb{Q}}((\mathbb{C}P^n, f_{n,0}), (\mathbb{C}P^m, f_{m,0})) = \begin{cases} 0 & (n = m), \\ \frac{1}{2} & (n \neq m). \end{cases}$

Here $\lceil\ \rceil$ denotes the ceiling function.

Remark 6.2. Since the cohomology of $\mathbb{C}P^n$ is concentrated in even degrees, it follows that $d^1_{\mathrm{CohI},\mathbb{Q}}((\mathbb{C}P^n, f_{n,j}), (\mathbb{C}P^m, f_{m,j'})) = 0$.

To prove Proposition 6.1, we now set up some notations. Observe that the algebra map $f^*_{n,j} : \mathbb{Q}[u] \cong H^*(BS^1;\mathbb{Q}) \to H^*(\mathbb{C}P^n;\mathbb{Q}) \cong \mathbb{Q}[x]/(x^{n+1})$ induced by $f_{n,j}$ in rational cohomology satisfies the condition that $f^*_{n,0}(u) = 0$ and $f^*_{n,1}(u) = x$, where $\deg x = 2$. These $\mathbb{Q}[u]$-module structures give the $\mathbb{Q}[t]$-module structures on $S^0H^*(\mathbb{C}P^n;\mathbb{Q})$ and then the barcodes associated with the modules as in Section 5. Let $\mathcal{B}_{n,j}$ denote the barcode obtained by $f^*_{n,j}$. Then, it is readily seen that

$$\mathcal{B}_{n,j} = \begin{cases} \{[0,1),[1,2),\dots,[n,n+1)\} & (j=0), \\ \{[0,n+1)\} & (j=1). \end{cases}$$

For simplicity, we put $\chi_{n,j} = \chi(\mathcal{B}_{n,j})$.

Proof of Proposition 6.1. The assertion (1) follows immediately from Lemma 2.8 (2). In view of Proposition 4.5, in order to show (2), it suffices to determine the bottleneck distance $d_B(\mathcal{B}_{n,0}, \mathcal{B}_{n,1})$. Given a bijection $h : \mathcal{B}_{n,0} \leftrightarrow \mathcal{B}_{n,1}$, if $h^{-1}([0,n+1)) = [i,i+1)$ for some $i = 1,2,\dots,n$, then we have

$$\sup_{I\in\mathrm{dom}(h)} d_{\mathrm{I}}(\chi_I, \chi_{h(I)}) = d_{\mathrm{I}}(\chi_{[i,i+1)}, \chi_{[0,n+1)}) = \min\left\{\max\{i, n-i\}, \frac{n+1}{2}\right\}$$

by Lemma 2.8 (1) and (2). If $h^{-1}([0,n+1)) = \emptyset$, then Lemma 2.8 (1) shows that

$$\sup_{I\in\mathrm{dom}(h)} d_{\mathrm{I}}(\chi_I, \chi_{h(I)}) = d_{\mathrm{I}}(\chi_\emptyset, \chi_{[0,n+1)}) = \frac{n+1}{2}.$$

Hence, we have

$$d_B(\mathcal{B}_{n,0}, \mathcal{B}_{n,1}) = \inf_{h:\mathcal{B}_{n,0}\leftrightarrow\mathcal{B}_{n,1}} \sup_{I\in\mathrm{dom}(h)} d_I(\chi_I, \chi_{h(I)}) = \min_{1\le i\le n}\left\{\max\{i, n-i\}, \frac{n+1}{2}\right\}.$$

Observe that the right-hand side integer coincides with $\lceil n/2 \rceil$, which completes the proof for (2).

The assertion (3) for $n = m$ is trivial. We consider the case where $n \neq m$. Since $d_{\mathrm{I}}(\chi_{[i,i+1)}, \chi_{[j,j+1)}) \le 1/2$ and $d_I(\chi_\emptyset, \chi_{[j,j+1)}) = 1/2$ from Lemma 2.8, we see that

$$\sup_{I\in\mathrm{dom}(h)} d_{\mathrm{I}}(\chi_I, \chi_{h(I)}) = \frac{1}{2}$$

for every bijection $h : \mathcal{B}_{n,0} \leftrightarrow \mathcal{B}_{m,0}$. Therefore, we have $d_B(\mathcal{B}_{n,0}, \mathcal{B}_{m,0}) = \frac{1}{2}$. Theorem 2.10 yields the result (3). $\square$

In the rest of this section, we use terminology in rational homotopy theory; see Appendix A for (relative) Sullivan models for spaces.

Proposition 6.3. *For each $j = 0,1$, let $v_j : M_j \to BS^1$ be a space over BS^1 whose relative Sullivan model has the form $(\wedge u, 0) \to (\wedge(x,y,z,u), d)$ with $dz = jxyu + u^4$ and $dx = 0 = dy$, where $\deg x = \deg y = 3$, $\deg z = 7$ and $\deg u = 2$. Then, one has*

$$d^0_{\mathrm{CohI},\mathbb{Q}}(M_0, M_1) = 3 \quad \textit{and} \quad d^1_{\mathrm{CohI},\mathbb{Q}}(M_0, M_1) = 0.$$

In order to prove Proposition 6.3, we first determine the $\mathbb{Q}$-cohomology of M_j as a $\mathbb{Q}[u]$-module. It is readily seen that $H^*(M_0;\mathbb{Q}) \cong \wedge(x,y)\otimes\mathbb{Q}[u]/(u^4)$ as an algebra. In order to compute the cohomology of M_1, we define the *weights* of elements x, y, z and u by $\mathrm{weight}(x) = \mathrm{weight}(y) = \mathrm{weight}(z) = 0$ and $\mathrm{weight}(u) = 2$.

The weight of a monomial is defined by the sum of the weights of elements above constructing the monomial. We define a filtration F^* of the model $\mathscr{M}$ for M_1 by $F^i := \{w \in \mathscr{M} \mid \text{weight}(w) \geq i\}$. Then, the filtration gives rise to the first quadrant multiplicative spectral sequence $\{E_r^{*,*}, d_r\}$ conversing to $H^*(\mathscr{M}) = H^*(M_1;\mathbb{Q})$. We see that

$$E_2^{*,*} \cong \wedge(x,y,z) \otimes \mathbb{Q}[u]$$

and $d_2(z) = xyu$, $d_2(x) = 0 = d_2(y)$. It follows that as a $\mathbb{Q}[u]$-module,

$$E_3^{*,*} \cong \mathbb{Q}[u]\{1,x,y,xz,yz,xyz\} \oplus (\mathbb{Q}[u]/(u))\{xy\}.$$

The next nontrivial differentials d_r are given by $d_8(xz) = xu^4$ and $d_8(yz) = yu^4$. The element xyz in the E_8-term represents the element $xyz - u^3z$ in $\mathscr{M}$. Therefore, we have $d_8(xyz) = d_8(xyz - u^3z) = 0$. Thus, we see that as a $\mathbb{Q}[u]$-module,

$$E_9^{*,*} \cong \mathbb{Q}[u]/(u^4)\{x,y\} \oplus \mathbb{Q}[u]\{1,xyz\} \oplus (\mathbb{Q}[u]/(u))\{xy\}.$$

Since $d_{14}(xyz) = d_{14}(xyz - u^3z) = u^7$, it follows that as $\mathbb{Q}[u]$-modules,

$$E_\infty \cong E_{15}^{*,*} \cong \mathbb{Q}[u]/(u^4)\{x,y\} \oplus \mathbb{Q}[u]/(u^7)\{1\} \oplus (\mathbb{Q}[u]/(u))\{xy\}.$$

Thus, Lemma 4.9 implies that $H^*(M_1;\mathbb{Q}) \cong \mathrm{Tot}E_\infty^{*,*}$ as a $\mathbb{Q}[u]$-module.

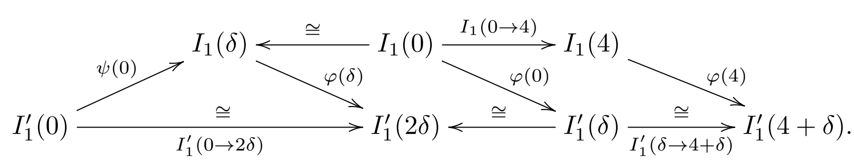
Proof of Proposition 6.3. By applying the functors S^0 and S^1 in the diagram (4.2), we see that

$$\begin{aligned} S^1(C^*(M_0;\mathbb{Q}) &\cong \Sigma^{-1}(\mathbb{Q}[t]/(t^4))^{\oplus 2} \cong S^1(C^*(M_1;\mathbb{Q})), \\ C_0 := S^0(C^*(M_0;\mathbb{Q})) &\cong \Sigma^0(\mathbb{Q}[t]/(t^4)) \oplus \Sigma^{-3}(\mathbb{Q}[t]/(t^4)) \quad \text{and} \\ C_1 := S^0(C^*(M_1;\mathbb{Q})) &\cong \Sigma^0(\mathbb{Q}[t]/(t^7)) \oplus \Sigma^{-3}(\mathbb{Q}[t]/(t)). \end{aligned}$$

The results follow from the computation of the cohomology mentioned above. Thus, we have the assertion on $d^1_{\mathrm{CohI},\mathbb{Q}}$

We prove the first equality. Let I_1, I_2, I_1' and I_2' be the interval modules in C_0 and C_1 corresponding the intervals $[0,4)$, $[3,7)$, $[0,7)$ and $[3,4)$, respectively. It follows from Lemma 2.8 and Remark 2.4 that $d^0_{\mathrm{CohI},\mathbb{Q}}(M_0,M_1) \leq 3$.

Suppose that C_0 and C_1 are δ-interleaved, where $\delta < 3$. Then, there exist natural transformations $\varphi : C_0 \longleftrightarrow C_1 : \psi$ which give the δ-interleaving. Since $I_2(i) = 0$ for $i < 3$, it follows that the nontrivial image of restriction $\psi : I_1' \longrightarrow I_1 \oplus I_2$ is in I_1. By the same reason for I_2' as that for I_2, we see that the nontrivial image of the restriction $\varphi : I_1 \longrightarrow I_1' \oplus I_2'$ is in I_1'. Thus, the restrictions of φ and ψ induce a δ-interleaving $\varphi : I_1 \longleftrightarrow I_1' : \psi$. Therefore, we have a commutative diagram



Observe that the horizontal arrows are isomorphisms except for $I_1(0 \to 4)$ and $I_1(0 \to 4) = 0$. Therefore, the map $\varphi(\delta)$ is nontrivial and hence $\varphi(0)$ is. This yields that $I_1'(\delta \to 4+\delta) \circ \varphi(0)$ is nontrivial, which is a contradiction. We have $d^0_{\mathrm{CohI},\mathbb{Q}}(M_0,M_1) = 3$. □

One might be interested in a relationship between M_j in Proposition 6.3 and an S^1-action and a higher dimensional torus action on a space. The issue is dealt with in the following remark.

Remark 6.4. In general, for a given relative Sullivan algebra of the form $\iota : (\wedge(u), 0) \to (\wedge W \otimes \wedge(u), d)$, there exists a fibration $M \to X \to BS^1$ whose model is the given Sullivan algebra. In fact, by [21, Proposition 17.9], we have a fibration $|\iota| : |(\wedge W \otimes \wedge(u), d)| \to |(\wedge(u), 0)|$. The pullback of the fibration along the rationalization map $l : BS^1 \to |(\wedge(u), 0)|$ gives rise to a commutative diagram

$$\begin{array}{ccccc}
M & \xrightarrow{\simeq} & X' & \longrightarrow & ES^1 \\
\| & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
M & \longrightarrow & X & \longrightarrow & BS^1 \\
\| & & \downarrow & & \downarrow{\scriptstyle l} \\
|(\wedge W, \overline{d})| & \longrightarrow & |(\wedge W \otimes \wedge(u), d)| & \xrightarrow[|\iota|]{} & |(\wedge(u), 0)|
\end{array}$$

in which p is the universal S^1-bundle and the right-hand upper squares is also pullback. The result [21, Proposition 15.6] yields that the map $\iota : (\wedge(u), 0) \to (\wedge W \otimes \wedge(u), d)$ is the relative Sullivan model for $X \to BS^1$. Since ES^1 is contractible, it follows that X' is weak homotopy equivalent to the fiber M. Moreover, we see that X is the orbit space of the S^1-space X' with a free action.

For example, it follows that each space M_j in Proposition 6.3 is the orbit space of an S^1-space M'_j which is rationally homotopy equivalent to $S^3 \times S^3 \times S^7$. In particular, the bundle $p = 1 \times \pi : M'_0 = (S^3 \times S^3) \times S^7 \to M_0 = (S^3 \times S^3) \times \mathbb{C}P^3$ is given by the usual principal S^1-bundle $\pi : S^7 \to \mathbb{C}P^3$. Moreover, we see that the free S^1-action on M'_0 does not extend to any free $S^1 \times S^1$-action. This follows from Proposition A.2 which computes the rational toral ranks of M_0 and M_1.

Remark 6.5. While the computation before the proof of Proposition 6.3 yields that $H^*(M_0; \mathbb{Q}) \cong H^*(M_1; \mathbb{Q})$ as a graded vector space, Proposition A.2 in particular implies that the rational homotopy types of M_0 and M_1 are different from each other. Moreover, Theorem 4.7 and Proposition 6.3 enable us to deduce that $\alpha C^*(M_0; \mathbb{Q})$ is not isomorphic to $\alpha C^*(M_1; \mathbb{Q})$ in the category $\mathrm{Ho}(\mathsf{Ch}_{\mathbb{K}})^{(\mathbb{R}, \leq)}$.

It may hold that $d_{\mathrm{CohI}}(X, Y) = 0$ for spaces X and Y over BS^1 even if $H^*(X; \mathbb{Q})$ is not isomorphic to $H^*(Y; \mathbb{Q})$ as an algebra. We describe such an example.

Remark 6.6. For $a \in \mathbb{Q}\backslash\{0\}$, let $p_a : X_a \to BS^1$ be a space over BS^1 whose relative Sullivan minimal model is given by

$$\iota : (\wedge(u), 0) \to \mathscr{M}(X_a) := (\wedge(u, x, y, z), d_a)$$

with $|x| = |u| = 2$, $|y| = |z| = 3$, $d_a u = d_a x = 0$, $d_a y = ux$, $d_a z = x^2 + au^2$ and $\iota(u) = u$. We observe that $A_a := H^*(X_a; \mathbb{Q}) \cong \mathbb{Q}[u, x]/(ux, x^2 + au^2)$ as an algebra. Moreover, it follows that $A_a \cong A_b$ as an algebra if and only if ab^{-1} is in $\mathbb{Q}^2$; see [39, Proposition 3.2]. On the other hand, it is readily seen that $A_a \cong A_b$ as a $\mathbb{Q}[u]$-module for $a, b \in \mathbb{Q}\backslash\{0\}$ and hence $C^*(X_a; \mathbb{Q}) \cong C^*(X_b; \mathbb{Q})$ in $\mathrm{D}(\mathbb{Q}[\mathrm{u}])$. Thus, there exist spaces over BS^1 with infinitely many different rational homotopy types one another such that their cohomology interleaving distances are zero.

The spaces X_{-1} and X_1 are realized as spaces $\mathbb{C}P^2 \sharp \mathbb{C}P^2 \to BS^1$ and $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} \to BS^1$ over BS^1, respectively, for each which the map from the connected sum is defined by the composite of the pinching map, the projection in the first factor and the map $f_{2,1}$ in Proposition 6.1; see [23, Example 3.7] for the Sullivan model of such a connected sum.

Remark 6.7. We consider a map between $(\mathbb{C}P^n, f_{n,1})$ and the space M_j over BS^1 in Proposition 6.3. The minimal model of $\mathbb{C}P^n$ is given by $\mathscr{M}(\mathbb{C}P^n) = (\wedge(u,w), d)$ where $dw = u^{n+1}$. Therefore, if there is a map between the two spaces, it is one of the cases.

(1) $f : \mathbb{C}P^n \to M_j$ $(j = 0, 1)$ whose Sullivan representative is given by $\mathscr{M}(f)(x) = \mathscr{M}(f)(y) = 0$ and $\mathscr{M}(f)(z) = u^{3-n}w$ for $1 \le n \le 3$.
(2) $f : M_0 \to \mathbb{C}P^n$ whose Sullivan representative is given by $\mathscr{M}(f)(w) = u^{n-3}z + au^{n-1}x + bu^{n-1}y$ $(a, b \in \mathbb{Q})$ for $n \ge 3$.

We refer the reader to [21, Section 12 (c)] for a Sullivan representative for a map.

Assertion 6.8. There is no morphism between $\mathbb{C}P^n$ $(n > 3)$ and M_1 over BS^1.

Proof. Suppose that there is a morphism $f : \mathbb{C}P^n \to M_1$ of spaces over BS^1 for $n > 3$. Then, since $|w| = 2n + 1 > 7$, it follows that $\mathscr{M}(f)(z) = 0$. However, $\mathscr{M}(f)$ is a morphism of DGAs with $\mathscr{M}(f)(u) = u$, which is a contradiction.

If there is a morphism $f : M_1 \to \mathbb{C}P^n$ of spaces over BS^1, then we have $\mathscr{M}(f)(w) = u^{n-3}z + g(u, x, y)$ for some $g \in \mathbb{Q}[u] \otimes \wedge^+(x, y)$. It follows that $d(g) = 0$. This contradicts that $\mathscr{M}(f)$ is a morphism of DGAs. □

Proposition 6.9. *Let $f_{n,1} : \mathbb{C}P^n \to BS^1$ and $v_j : M_j \to BS^1$ be the spaces over BS^1 described in Proposition 6.1 and 6.3, respectively. Then,*

$$d^0_{\mathrm{CohI},\mathbb{Q}}((M_0, v_0), (\mathbb{C}P^n, f_{n,1})) = \begin{cases} 2 & (1 \le n \le 5) \\ 3 & (6 \le n \le 9) \\ n-6 & (10 \le n \le 13) \\ \frac{n+1}{2} & (14 \le n), \end{cases}$$

$$d^0_{\mathrm{CohI},\mathbb{Q}}((M_1, v_1), (\mathbb{C}P^n, f_{n,1})) = \begin{cases} \frac{7}{2} & (1 \le n \le 2) \\ -n+6 & (3 \le n \le 5) \\ \frac{1}{2} & (n = 6) \\ n-6 & (7 \le n \le 13) \\ \frac{n+1}{2} & (14 \le n) \end{cases}$$

$$\textit{and} \quad d^1_{\mathrm{CohI},\mathbb{Q}}((M_j, v_j), (\mathbb{C}P^n, f_{n,1})) = 2.$$

Proof. First, we prove the first two equalities by computing the bottleneck distances. Recall the barcode $\mathcal{B}_{n,1} = \{[0, n+1)\}$ associated with $h^0H^*(\mathbb{C}P^n;\mathbb{Q})$ described above. We also recall the barcodes associated with $h^0H^*(M_j;\mathbb{Q})$ in the proof of Proposition 6.3 which are given by

$$\mathcal{B}_{h^0H^*(M_0;\mathbb{Q})} = \{[0,4), [3,7)\} \quad \text{and} \quad \mathcal{B}_{h^0H^*(M_1;\mathbb{Q})} = \{[0,7), [3,4)\},$$

respectively. Given a bijection $h : \mathcal{B}_{h^0H^*(M_0;\mathbb{Q})} \leftrightarrow \mathcal{B}_{n,1}$, if $h([0,4)) = [0, n+1)$, then Lemma 2.8 (1) and (2) allow us to deduce that

$$(6.1) \qquad \sup_{J \in \mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J, \chi_{h(J)}) = \max\{d_{\mathrm{I}}(\chi_{[0,4)}, \chi_{[0,n+1)}), d_{\mathrm{I}}(\chi_{[3,7)}, \chi_{\emptyset})\} = \begin{cases} 2 & (1 \le n \le 5) \\ n-3 & (6 \le n \le 7) \\ \frac{n+1}{2} & (8 \le n). \end{cases}$$

Similarly, it is readily seen that

$$\sup_{J\in \mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J, \chi_{h(J)}) = \begin{cases} 2 & (1 \le n \le 3) \\ \frac{n+1}{2} & (4 \le n \le 5) \\ 3 & (6 \le n \le 9) \\ n-6 & (10 \le n \le 13) \\ \frac{n+1}{2} & (14 \le n) \end{cases} \tag{6.2}$$

in the case where $h([3,7)) = [0, n+1)$, and

$$\sup_{J\in \mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J, \chi_{h(J)}) = \begin{cases} 2 & (1 \le n \le 3) \\ \frac{n+1}{2} & (4 \le n) \end{cases} \tag{6.3}$$

in the case where $h(\emptyset) = [0, n+1)$. Since the distance $d_{\mathrm{B}}(\mathcal{B}_{h^0H^*(M_0;\mathbb{Q})}, \mathcal{B}_{n,1})$ is the smaller value of (6.1), (6.2) and (6.3), the result for $d^0_{\mathrm{CohI},\mathbb{Q}}((M_0, v_0), (\mathbb{C}P^n, f_{n,1}))$ is shown from Theorem 2.10. By the same argument above, we compute the bottleneck distance between $\mathcal{B}_{H^{\mathrm{even}}(M_1;\mathbb{Q})}$ and $\mathcal{B}_{n,1}$, which completes the proof of (1).

More precisely, let $h : \mathcal{B}_{H^{\mathrm{even}}(M_1;\mathbb{Q})} \leftrightarrow \mathcal{B}_{n,1}$ be a bijection satisfying $h([0,7)) = [0, n+1)$. Then, we have

$$\sup_{J\in \mathrm{dom}(h)} d_{\mathrm{I}}(\chi_J, \chi_{h(J)}) = \begin{cases} \frac{7}{2} & (1 \le n \le 2) \\ -n+6 & (3 \le n \le 5) \\ \frac{1}{2} & (n = 6) \\ n-6 & (7 \le n \le 13) \\ \frac{n+1}{2} & (14 \le n). \end{cases} \tag{6.4}$$

Similarly, we have

$$\sup_{J\in \mathrm{dom}(h')} d_{\mathrm{I}}(\chi_J, \chi_{h'(J)}) = \begin{cases} \frac{7}{2} & (1 \le n \le 6) \\ \frac{n+1}{2} & (7 \le n) \end{cases} \tag{6.5}$$

for a bijection $h' : \mathcal{B}_{h^0H^*(M_1;\mathbb{Q})} \leftrightarrow \mathcal{B}_{n,1}$ satisfying $h'([3,4)) = [0, n+1)$, and

$$\sup_{J\in \mathrm{dom}(h'')} d_{\mathrm{I}}(\chi_J, \chi_{h''(J)}) = \begin{cases} \frac{7}{2} & (1 \le n \le 6) \\ \frac{n+1}{2} & (7 \le n) \end{cases} \tag{6.6}$$

for a bijection $h'' : \mathcal{B}_{h^0H^*(M_1;\mathbb{Q})} \leftrightarrow \mathcal{B}_{n,1}$ satisfying $h''(\emptyset) = [0, n+1)$. Since the bottleneck distance $d_{\mathrm{B}}(\mathcal{B}_{h^0H^*(M_1;\mathbb{Q})}, \mathcal{B}_{n,1})$ is the smaller value of (6.4) and (6.5), Theorem 2.10 shows the assertion on $d^0_{\mathrm{CohI},\mathbb{Q}}((M_1, v_1), (\mathbb{C}P^n, f_{n,1}))$.

It follows from the $\mathbb{Q}[t]$-module structure of $S^1(C^*(M_j;\mathbb{Q}))$ described in the proof of Proposition 6.3 that the barcode associated with $h^1H^*(M_j;\mathbb{Q})$ is given by $\{[1,5), [1,5)\}$ for $j = 0$ and 1. On the other hand, the barcode associated with $h^1H^*(\mathbb{C}P^n;\mathbb{Q})$ is the empty set since the cohomology of $\mathbb{C}P^n$ is concentrated in even degrees. Therefore, we have

$$d^1_{\mathrm{CohI},\mathbb{Q}}((M_j, v_j), (\mathbb{C}P^n, f_{n,1})) = d_{\mathrm{I}}(\chi_{[1,5)}, \chi_\emptyset) = 2$$

This follows from Lemma 2.8 (2). □

Remark 6.10. Let X and Y be spaces over BS^1. Then, as seen in the proof of Proposition 5.9, the triangle inequality of the interleaving distance allows us to deduce an inequality

$$\left|\, d^k_{\mathrm{CohI},\mathbb{K}}(X, \mathbb{C}P^n) - d^k_{\mathrm{CohI},\mathbb{K}}(Y, \mathbb{C}P^n) \,\right| \le d^k_{\mathrm{CohI},\mathbb{K}}(X, Y)$$

for each $n \ge 1$, $k = 0, 1$ and an arbitrary field $\mathbb{K}$. The computation of the distance $d^k_{\mathrm{CohI},\mathbb{K}}(X, \mathbb{C}P^n)$ in the proof of Proposition 6.9 is comprehensively easy with the bottleneck distance because the barcode associated with $\mathbb{C}P^n$ consists of one bar. This is an advantage of the lower bound of the interleaving distance mentioned above.

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Appendix A. Some rational homotopy invariants and the CohID

We begin by briefly reviewing a (relative) Sullivan algebra. Let $\mathscr{M}(X) = (\wedge V, d)$ be the Sullivan minimal model of a simply-connected space X; see [21, Section 12]. It is a free commutative differential graded algebra (DGA) over $\mathbb{Q}$ with a locally finite $\mathbb{Q}$-graded vector space $V = \bigoplus_{i\geq 1} V^i$ and a decomposable differential; that is,

$$\dim V^i < \infty, d(V^i) \subset (\wedge^+ V \cdot \wedge^+ V)^{i+1} \text{ and } d \circ d = 0.$$

Here $\wedge^+ V$ denotes the ideal of $\wedge V$ generated by elements of positive degree. The degree of a homogeneous element x of the graded algebra is denoted by $|x|$. By definition, the commutativity of the model gives the formula $xy = (-1)^{|x||y|} yx$ and the differential d satisfies the condition that $d(xy) = d(x)y + (-1)^{|x|} xd(y)$ for homogeneous elements x and y in $\wedge V$. Note that $\mathscr{M}(X)$ determines the rational homotopy type of X. In particular, we see that $V^* \cong \mathrm{Hom}(\pi_*(X), \mathbb{Q})$ and $H^*(\wedge V, d) \cong H^*(X; \mathbb{Q})$.

Let $f : X \to Y$ be a map between simply-connected spaces. Then, the relative Sullivan model of f is given by

$$\mathscr{M}(Y) = (\wedge W, d_Y) \to (\wedge W \otimes \wedge V, D) \to (\wedge V, \overline{D}),$$

where $D\mid_W = d_Y$ and $(\wedge W \otimes \wedge V, D)$ is quasi-isomorphic to $\mathscr{M}(X)$ [21, Section 14].

We also recall a spectral sequence introduced in [21, Section 32 (b)]. Let $(\wedge V, d)$ be a Sullivan algebra for which V is finite dimensional. We give the Sullivan algebra a bigrading $(\wedge V, d)^{*,*}$ defined by $(\wedge V^{\mathrm{even}} \otimes \wedge^k V^{\mathrm{odd}})^n = (\wedge V)^{k+n,k}$. Then, a generator x with odd degree and a generator y with even degree have the bidegrees $(\deg x + 1, -1)$ and $(\deg y, 0)$, respectively. The filtration $F^*(\wedge V)$ of $\wedge V$ defined by $F^p(\wedge V) = (\wedge V)^{\geq p,*}$ gives rise to the forth quadrant spectral sequence converging to $H(\wedge V, d)$, which is called the *odd spectral sequence* of the Sullivan algebra $(\wedge V, d)$. Observe that the E_0-term is a DGA of the form $(\wedge V, d_\sigma)$ with the differential of bidegree $(0, +1)$ characterized by

$$d_\sigma(V^{\mathrm{even}}) = 0, d_\sigma : V^{\mathrm{odd}} \to \wedge V^{\mathrm{even}} \text{ and } d - d_\sigma : V^{\mathrm{odd}} \to \wedge V^{\mathrm{even}} \otimes \wedge^+ V^{\mathrm{odd}}.$$

Proposition A.1. [21, Proposition 32.4] *Let $(\wedge V, d)$ be a minimal Sullivan algebra in which V is of finite dimension and $V = V^{\geq 2}$. Then the following three conditions are equivalent.* (i) $\dim E_1 = \dim H(\wedge V, d_\sigma) < \infty$, (ii) $\dim H(\wedge V, d) < \infty$ *and* (iii) *the LS category* $\mathrm{cat}(\wedge V, d)$ *is finite; see* [21, Section 29] *for the definition of* $\mathrm{cat}(\wedge V, d)$.

Let $r_0(X)$ be the *rational toral rank* of a simply-connected CW complex X of $\dim H^*(X;\mathbb{Q}) < \infty$; that is, the largest integer r such that an r-torus $T^r = S^1 \times \cdots \times S^1$ (r-factors) can act continuously on a CW-complex Y having the rational homotopy type of X with all its isotropy subgroups finite (almost free action); see [23, 7.3] and [25]. If an r-torus T^r acts on X by $\mu : T^r \times X \to X$, then the Borel fibration

$$X \to ET^r \times^{\mu}_{T^r} X \to BT^r$$

is constructed. Thus, we have a relative Sullivan model

$$(\mathbb{Q}[u_1, \dots, u_r], 0) \to (\mathbb{Q}[u_1, \dots, u_r] \otimes \wedge V, D) \to (\wedge V, d) \qquad (*)_r$$

for the fibration, where $\deg u_i = 2$ for $i = 1, \dots, r$, $Du_i = 0$ and $Dv \equiv dv$ modulo the ideal $(u_1, \dots, u_r)$ for $v \in V$. According to [25, Proposition 4.2], $r_0(X) \geq r$ if and only if there exists a relative Sullivan algebra of the form $(*)_r$ such that $(\wedge V, d)$ is the minimal model for X and $\dim H(\mathbb{Q}[u_1, \cdots, u_r] \otimes \wedge V, D) < \infty$.

We recall the spaces M_0 and M_1 over BS^1 in Proposition 6.3.

Proposition A.2. *$r_0(M_0) = 2$ and $r_0(M_1) = 0$.*

Proof. It follows that $r_0(M_0) \geq 2$. In fact, we define $Dx = u_1^2$, $Dy = u_2^2$ and $Dz = dz = u^4$ in $(*)_2$. Then, we have $\dim H(\mathbb{Q}[u_1, u_2] \otimes \wedge V, D) < \infty$. If $r_0(M_0) \geq 3$, then there is a relative Sullivan model

$$(\mathbb{Q}[u_1, u_2, u_3], 0) \to (\mathbb{Q}[u_1, u_2, u_3] \otimes \wedge(u, x, y, z), D) \to (\wedge(u, x, y, z), d) \qquad (*)_3$$

such that $\dim H^*(\mathbb{Q}[u_1, u_2, u_3] \otimes \wedge(u, x, y, z), D) < \infty$. We write $(\wedge W, D)$ for $(\mathbb{Q}[u_1, u_2, u_3] \otimes \wedge(u, x, y, z), D)$. Then, the result [21, Proposition 32.10] implies that $\dim W^{\text{odd}} - \dim W^{\text{even}} \geq 0$. However, it follows from $(*)_3$ that $\dim W^{\text{odd}} - \dim W^{\text{even}} = 3 - 4 = -1$, which is a contradiction.

Suppose that $r_0(M_1) \geq 1$. Then, the DGA $(\wedge W, D) := (\wedge(x, y) \otimes \wedge(u_1, u, z), D)$ in $(*)_1$ for M_1 satisfies the condition that $Dx = Dy = 0$. Indeed, let $Dz = xyu + u^4 + f + axyu_1$ for some $f = f(u, u_1) \in \mathbb{Q}[u, u_1]$ and $a \in \mathbb{Q}$. We have

$$DDz = gyu - hxu + agyu_1 - ahxu_1 \neq 0$$

if $Dx = g(u, u_1) \neq 0$ or $Dy = h(u, u_1) \neq 0$ in $\mathbb{Q}[u, u_1]$. Thus the differential D is trivial on $\wedge(x, y)$. We consider the odd spectral sequence converging to $H(\wedge W, D)$. The E_0-term is a DGA of the form $(\wedge(x, y), 0) \otimes \wedge(u_1, u, z), D_\sigma)$ with $D_\sigma z = u^4 + f$. Thus, by applying [21, Proposition 32.10] again, we see that the E_1-term is of infinite dimension. Proposition A.1 implies that $\dim H(\wedge W, D) = \infty$, which is a contradiction. We have $r_0(M_1) = 0$. □

We conclude this section with comments on upper and lower bounds of the cohomology interleaving distance. The proof of Proposition 6.3 enables us to deduce the following result.

Proposition A.3. *Let $v_j : M_j \to BS^1$ be the space over BS^1 in Proposition 6.3 for each $j = 0$ and 1. Then,* $\mathrm{cup}(v_0)_{\mathbb{Q}} = 3$ *and* $\mathrm{cup}(v_1)_{\mathbb{Q}} = 6$.

It follows that the equalities in the inequalities in Proposition 5.5 and Remark 6.10 do not hold in general. In fact, we have

$$\begin{aligned} & \left| \, d^k_{\mathrm{CohI},\mathbb{Q}}(M_0, \mathbb{C}P^6) - d^k_{\mathrm{CohI},\mathbb{Q}}(M_1, \mathbb{C}P^6) \right| \\ = \;\; & 3 - \frac{1}{2} < d^0_{\mathrm{CohI},\mathbb{Q}}(M_0, M_1) = 3 < \frac{7}{2} = \frac{1}{2}\max\{\mathrm{cup}(v_0)_{\mathbb{Q}} + 1, \mathrm{cup}(v_1)_{\mathbb{Q}} + 1\}. \end{aligned}$$

Observe that the first equality follows from Proposition 6.9.

We have $\mathrm{cup}_{\mathbb{Q}}(\mathbb{C}P^3) = \mathrm{cup}_{\mathbb{Q}}(f_{3,1}) = 3$. Then, Proposition 6.9 (1) and Proposition 5.5 allow us to deduce that

$$d^0_{\mathrm{CohI},\mathbb{Q}}((\mathbb{C}P^3, f_{3,1}), M_0) = 2 = \frac{4}{2} = \frac{1}{2}\max\{3+1, \mathrm{cup}(v_0)_{\mathbb{Q}} + 1\}.$$

On the other hand, the inclusion $\mathbb{C}P^3 \to (S^3 \times S^3) \times \mathbb{C}P^3 = M_0$ defined with a base point in $S^3 \times S^3$ satisfies the assumption in Proposition 5.13 (i); see also Remark 6.4. Thus, the evaluation in the proposition gives the inequality $d^0_{\mathrm{CohI},\mathbb{Q}}(\mathbb{C}P^3, M_0) \leq 3$. The equality in Proposition 5.13 does not hold either in general.

List of Symbols

symbol	meaning	page
α	the functor $\alpha : \mathbb{K}[u]\text{-}\mathsf{Ch} \to \mathsf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$	11
cup^k	the cup-length	14 18
d_{CohI}	the cohomology interleaving distance of persistence dg modules	7
d^k_{CohI}	the (even, odd) cohomology interleaving distance of dg $\mathbb{K}[u]$-modules	11 12
$d^k_{\mathrm{CohI},\mathbb{K}}$	the (even, odd) cohomology interleaving distance of spaces	17
d_{IHC}	the interleaving distance in the homotopy category	7
$\lfloor\ \rfloor$, $\lceil\ \rceil$	the floor function, the ceiling function	10 21
S^k	the functor $S^k : \mathbb{K}[u]\text{-}\mathsf{grMod} \to \mathbb{K}[t]\text{-}\mathsf{grMod}$	10
$\mathscr{M}(X)$	the Sullivan minimal model for a space X	27

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