A DISTANCE BETWEEN MAPS VIA INTERLEAVINGS OF RELATIVE SULLIVAN ALGEBRAS

KATSUHIKO KURIBAYASHI, TAKAHITO NAITO, KENGO SEKIZUKA, SHUN WAKATSUKI, AND TOSHIHIRO YAMAGUCHI

Abstract. In this article, we consider extended tame persistence commutative differential graded algebras (CDGAs) associated with relative Sullivan algebras. In particular, if the relative Sullivan algebra is a model for a map between spaces, then the persistence CDGA is isomorphic to the persistence object obtained by a Postnikov tower for the map with the polynomial de Rham functor in the homotopy category of extended tame persistence CDGAs.

Moreover, the interleaving distance in the homotopy category (IHC) in the sense of Lanari and Scoccola enables us to introduce a pseudodistance on the homotopy set of maps via the persistence CDGA models for maps. In contrast to persistence cochain complexes, the IHC of persistence CDGAs does not coincide with the cohomology interleaving distance in general. Due to the reason, we also discuss formalities of a persistence CDGA with interleavings. Computational examples of the pseudodistances between maps are showcased.

Contents

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1. Introduction

Recently, fascinating persistence invariants appear in algebra, geometry and topology which are obtained by swing-backs from topological data analysis; see, for example, [4, 6, 9, 20, 26, 34].

In [4], the *interleavings* for persistence objects with values in a model category are considered. Being based on the consideration due to Blumberg and Lesnick [4], Lanari and Scoccola [24] introduce the *interleaving distance in the homotopy category* d_{IHC} for persistence objects. In the considerations of d_{IHC} and other distance, which are essentially given in [4], Lanari and Scoccola use the projective model structure of the category of persistence objects with values in a cofibrantly generated model category. In [9], Chachólski, Giunti and Landi have given the category of *tame* persistence objects a model structure.

In this article, by applying the distance d_{IHC} to tame persistence objects, we construct a two parameter homotopy invariant for continuous maps; see Sections 2 and 3. More precisely, the interleaving distance between maps is introduced and investigated by making use of the model structure on the category $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$ of *extended* tame persistence commutative differential graded algebras (CDGAs) over the rational field $\mathbb{Q}$. The key to the discussion is the functor Θ constructed in Theorem 3.3, which assigns an object in the homotopy category of $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$ to a map between path-connected spaces via a relative Sullivan model for the map.

The functor Θ is indeed defined algebraically with the dimension of the extended part of a relative Sullivan algebra for a map. However, we remark that the extended tame persistence CDGA $\Theta(f)$ associated with a map f between simply-connected spaces is isomorphic to the image of the Moore–Postnikov tower of f by the polynomial de Rham functor in the homotopy category of $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$; see Theorem 4.9. Thus, by formulating the result with a partial Quillen equivalence in the sense of Moreno-Fernández and B. Stonek [28], we see that the distance d_{IHC} between $\Theta(f)$ and $\Theta(g)$ is equal to the distance d_{IHC} between the pointwise rationalizations of the Moore–Postnikov towers of f and g in the category of extended tame *copersistence* simplicial sets; see Corollary 4.11. Furthermore, we have a pseudodistance

$$d_{\mathrm{H}} : [X, Y] \times [X, Y] \to \mathbb{R}_{\geq 0} \cup \{\infty\}$$

on the homotopy set by making use of the functor Θ; see Theorem 5.2.

The result [23, Theorem 3.3] yields that the distance d_{IHC} coincides with the *cohomology interleaving distance* d_{CohI} in the category of persistence cochain complexes over a field even though the homology functor induces the inequality $d_{\mathrm{CohI}} \leq d_{\mathrm{IHC}}$ in general. On the other hand, we have examples which give the strict inequality $d_{\mathrm{CohI}} < d_{\mathrm{IHC}}$ if the two distances are dealt with in the category $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$. Indeed, the Hopf map $S^3 \to S^2$ and the trivial map with the same domain and codomain is such an example. Another example is given by considering the distance d_{H} between appropriate two elements in the homotopy group $\pi_3\left(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}\right) \otimes \mathbb{Q}$; see Section 6. We stress that the strict inequality is depend on the choice of the underlying field. In fact, in the second example above, the two distances between the elements coincides if we choose the complex field $\mathbb{C}$ in the consideration instead of $\mathbb{Q}$; see Remark 6.6.

A key to proving the equality $d_{\mathrm{CohI}} = d_{\mathrm{IHC}}$ in [23, Theorem 3.3] is that every persistence cochain complexes is d_{HI}-*formal*; see [23, Proposition 3.5]. In general, the formality is defined by using the *homotopy interleaving distance* on a model category; see [4, 24]. Thus, we are interested in considering formalities for persistence CDGAs. The topic is discussed in Section 7 relating to the notion of formalizability of a map in the sense of Thomas [33]; see Propositions 7.5 and 7.9.

It is worthwhile mentioning that a transferred model structure on the category of persistence CDGAs, which are not necessarily tame, is introduced in [20] by using the *interval sphere* model structure on the category of persistence cochain complexes over $\mathbb{Q}$. In this article, our interest is restricted to extended tame persistence CDGAs and their invariants and then we invoke the model structure on $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$ introduced in [9]. The novelty is that we assign a persistence CDGA to a single continuous map.

The rest of this manuscript is arranged as follows. Section 2 recalls the interleavings up to homotopy. Moreover, interleaving distances introduced in [4, 24] are reconsidered in the category of extended tame persistence objects. To this end, we use the model structure of the category of tame persistence objects with values in a model category, which is due to Chachólski, Giunti and Landi [9]. In Section 3, after recalling the notion of a relative Sullivan algebra, we introduce the functor Θ mentioned above and investigate its properties. In Section 4, we relate the persistence CDGA associated with a map f via the functor Θ to the Moore–Postnikov tower for f. Section 5 considers the interleaving distance in the homotopy category for maps. Section 6 is devoted to producing examples of persistence CDGAs associated with maps for each of which the distance d_{IHC} is greater than d_{CohI}. In Section 7, we show that formalities of a persistence CDGA defined by interleaving distances are equivalent to one another. Moreover, the formalities is related to the formalizability of a map. Section 8 gives more computational examples of the distances d_{IHC} of maps. Section 9 describes perspective of our work.

2. The interleaving distances between extended tame objects

Let $\mathcal{C}$ be a category and $\mathcal{C}^{(\mathbb{R},\leq)}$ the functor category, where $(\mathbb{R},\leq)$ is the poset defined with the usual order which is regarded as a category. Originally, the interleavings *up to homotopy* are defined in the category $\mathcal{M}^{(\mathbb{R},\leq)}$ endowed with the projective model structure for a cofibrantly generated model category $\mathcal{M}$. In order to define interleavings up to homotopy in a full subcategory of $\mathcal{M}^{(\mathbb{R},\leq)}$, we need to reconsider results in [24].

We begin by recalling strict interleavings in the functor category $\mathcal{C}^{(\mathbb{R},\leq)}$ for a general category $\mathcal{C}$. For a real number $\varepsilon \geq 0$, define a functor $T_\varepsilon : (\mathbb{R},\leq) \to (\mathbb{R},\leq)$ by $T_\varepsilon(a) = a+\varepsilon$. The *$\varepsilon$-shift functor* $(\ )^\varepsilon : \mathcal{C}^{(\mathbb{R},\leq)} \to \mathcal{C}^{(\mathbb{R},\leq)}$ is defined by $(\ )^\varepsilon(F) = F^\varepsilon := FT_\varepsilon$.

Definition 2.1. ([10, Definition 4.2], [7, Definition 3.1]) Objects F and G in $\mathcal{C}^{(\mathbb{R},\leq)}$ are *ε-interleaved* if there exists a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[dr, "\varphi"] & F^\varepsilon \arrow[r] \arrow[dr, "\varphi^\varepsilon"] & F^{2\varepsilon} \\ G \arrow[r] \arrow[ur, "\psi"] & G^\varepsilon \arrow[r] \arrow[ur, "\psi^\varepsilon"] & G^{2\varepsilon} \end{tikzcd} \tag{2.1}$$

in which horizontal arrows are the natural transformations defined by the structure maps of F and G. The pair (φ,ψ) of the natural transformations is called an *ε-interleaving* between F and G.

Remark 2.2. The commutative diagram in Definition 2.1 yields the commutativity of the diagrams

$$\begin{tikzcd} F(i) \arrow[rr, "F(i\le i+2\varepsilon)"] \arrow[dr, "\varphi(i)"'] & & F(i+2\varepsilon) \\ & G(i+\varepsilon) \arrow[ur, "\psi(i+\varepsilon)"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & F(i+\varepsilon) \arrow[dr, "\varphi(i+\varepsilon)"] & \\ G(i) \arrow[ur, "\psi(i)"] \arrow[rr, "G(i\le i+2\varepsilon)"'] & & G(i+2\varepsilon) \end{tikzcd} \tag{2.2}$$

for all $i \in \mathbb{R}$. We note that F is isomorphic to G in $\mathcal{C}^{(\mathbb{R},\le)}$ if and only if F and G are 0-interleaved.

Definition 2.3. For objects F and G in $\mathcal{C}^{(\mathbb{R},\le)}$, the interleaving distance $d_{\mathrm{I}}(F,G)$ between F and G is defined by

$$d_{\mathrm{I}}(F,G) := \inf(\{\varepsilon \ge 0 \mid F \text{ and } G \text{ are } \varepsilon\text{-interleaved}\} \cup \{\infty\}).$$

Definition 2.4. Let $\mathcal{C}$ be a category and $\mathbb{R}_+$ the full subcategory of $(\mathbb{R},\le)$ whose objects are non-negative real numbers. A sequence $\tau_0 < \tau_1 < \cdots < \tau_n < \cdots$ in $[0,\infty)$, which is divergent or finite, *discretises* a functor $X : \mathbb{R}_+ \to \mathcal{C}$ if $X(s \le t) : X(s) \to X(t)$ may fail to be an isomorphism only if there is $a \in \mathbb{N}$ such that $s < \tau_a \le t$. A functor $X : \mathbb{R}_+ \to \mathcal{C}$ is called an *extended tame functor* if there is a sequence that discretises it. Let $\mathsf{et}(\mathcal{C}^{\mathbb{R}_+})$ denote the full subcategory of the functor category $\mathcal{C}^{\mathbb{R}_+}$ whose objects are extended tame functors. An object X in $\mathsf{et}(\mathcal{C}^{\mathbb{R}_+})$ is called *tame* if the sequence which discretises X is finite; see [9].

Following [9, Section 2.2], we introduce a factorization of a morphism $g : X \to Y$ in $\mathsf{et}(\mathcal{C}^{\mathbb{R}_+})$. Let $\tau_0 < \tau_1 < \cdots < \tau_n < \cdots$ be a sequence discretising both X and Y. By induction on $\{0,1,2,...,n,...\}$, we define morphisms $\bar{g}(\tau_a) : X(\tau_a) \to Q(\tau_a)$ and $\hat{g}(\tau_a) : Q(\tau_a) \to Y(\tau_a)$ in $\mathcal{C}$ as follows: $(\bar{g}(0) : X(0) \to Q(0)) := (1 : X(0) \to X(0))$ and $(\hat{g}(0) : Q(0) \to Y(0)) := (g(0) : X(0) \to Y(0))$. For $a > 0$, the object $Q(\tau_a)$ is defined by $\mathrm{colim}(\ Y(\tau_{a-1}) \xleftarrow{g(\tau_{a-1})} X(\tau_{a-1}) \xrightarrow{X(\tau_{a-1}<\tau_a)} X(\tau_a)\ )$ with $\bar{g}(\tau_a) : X(\tau_a) \to Q(\tau_a)$ and $\hat{g}(\tau_a) : Q(\tau_a) \to Y(\tau_a)$ which fit in the commutative diagram

$$\begin{tikzcd} X(\tau_{a-1}) \arrow[r, "X(\tau_{a-1}<\tau_a)"] \arrow[d, "g(\tau_{a-1})"'] & X(\tau_a) \arrow[d, "\bar{g}(\tau_a)"] \arrow[ddr, bend left, "g(\tau_a)"] & \\ Y(\tau_{a-1}) \arrow[r] \arrow[drr, bend right, "Y(\tau_{a-1}<\tau_a)"'] & Q(\tau_a) \arrow[dr, "\hat{g}(\tau_a)"] & \\ & & Y(\tau_a) \end{tikzcd} \tag{2.3}$$

consisting of the pushout square. For $a > 0$, define $Q(\tau_{a-1} < \tau_a) : Q(\tau_{a-1}) \to Q(\tau_a)$ to be the composite of the morphism $Y(\tau_{a-1}) \to Q(\tau_a)$ represented by the bottom horizontal arrow in (2.3) and $\hat{g}(\tau_{a-1}) : Q(\tau_{a-1}) \to Y(\tau_{a-1})$. The construction above gives an extended persistence object Q via the left Kan extension along the inclusion $\{\tau_0 < \cdots < \tau_n < \cdots\} \to \mathbb{R}$ of poset*; see [9, Section 2.1]. Thus, we have a factorization $g = \hat{g}\bar{g}$ of g.

Throughout this manuscript, we use the same terminology as that in [11] for model categories. The following two results are due to Chachólski, Giunti and Landi [9].

*We observe that the functor $(\lfloor\ \rfloor)^*$ mentioned in Section 3.1 below is nothing but the left Kan extension along the inclusion $j : \mathbb{Z} \to \mathbb{R}$.

Theorem 2.5. [9, Theorem 2.2] *Let $\mathcal{M}$ be a model category. The following choices of weak equivalences, fibrations and cofibrations form a model structure on $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$. A morphism $g : X \to Y$ in $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$ is a*

- *weak equivalence if $g(t) : X(t) \to Y(t)$ is a weak equivalence for all t,*
- *fibration if $g(t) : X(t) \to Y(t)$ is a fibration for all t,*
- *cofibration if $\hat{g}(t) : Q(t) \to Y(t)$ is a cofibration for all t.*

In the definition above, it is not necessarily assumed that the model category admits functorial factorizations.

Proposition 2.6. [9, Proposition 2.3] *Let $\mathcal{M}$ be a model category.*
(i) *If $g : X \to Y$ is a cofibration in $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$, then $g(t) : X(t) \to Y(t)$ is a cofibration in $\mathcal{M}$ for any t in $\mathbb{R}_+$.*
(ii) *An object X in $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$ is cofibrant if and only if $X(0)$ is cofibrant and, for any $s < t$ in $\mathbb{R}_+$, the transition morphism $X(s < t) : X(s) \to X(t)$ is cofibration in $\mathcal{M}$.*

Originally, the results [9, Theorem 2.2 and Proposition 2.3] are proved for the full subcategory of $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$ consisting of tame objects. The induction argument in the original proofs of [9, Theorem 2.2 and Proposition 2.3] are valid to obtain Theorem 2.5 and Proposition 2.6. For an extended tame object F, a sequence $\{\tau_n\}_{n\geq 0}$ which discretises F is unbounded if the sequence is not finite. Then, in particular, we may construct a cofibrant replacement of F by taking inductively a cofibrant replacement of each structure map $F(\tau_n) \to F(\tau_{n+1})$. Here, we use the condition that the sequence is divergent.

Remark 2.7. The model structure in Proposition 2.6 is closely related to that of the category of towers in a category described in [18, Section VI, Definition 1.1]; see Remark 4.10 (iii).

Let $\mathcal{M}$ be a model category. Since the δ-shift functor $(\ )^\delta : \mathsf{et}(\mathcal{M}^{\mathbb{R}_+}) \to \mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$ preserves weak equivalences, it follows that the functor $(\ )^\delta$ induces the self functor $(\ )^\delta$ on the homotopy category $\mathrm{Ho}(\mathsf{et}(\mathcal{M}^{\mathbb{R}_+}))$. Then, we can consider the commutative diagram (2.1) in $\mathrm{Ho}(\mathsf{et}(\mathcal{M}^{\mathbb{R}_+}))$. Moreover, we say that objects F and G in $\mathrm{Ho}(\mathsf{et}(\mathcal{M}^{\mathbb{R}_+}))$ are *ε-interleaved in the homotopy category* if they are ε-interleaved in the sense in Definition 2.1; see [24, Section 2.2.2]. The *interleaving distance in the homotopy category* between objects F and G in $\mathrm{Ho}(\mathsf{et}(\mathcal{M}^{\mathbb{R}_+}))$ is define by

$$d_{\mathrm{IHC}}(F, G) := \inf(\{\varepsilon \geq 0 \mid F, G \text{ are } \varepsilon\text{-interleaved in the homotopy category}\} \cup \{\infty\}).$$

For objects X and Y in $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$, we say that X and Y are *ε-homotopy interleaved* if there exist $X \simeq X'$ and $Y \simeq Y'$ such that X' and Y' are ε-interleaved in $\mathcal{M}^{(\mathbb{R},\leq)}$; see [4, Section 3.3]. Here $W \simeq W'$ means that there is a zigzag of weak equivalences connecting W and W'.

Let $j : \mathsf{et}(\mathcal{M}^{\mathbb{R}_+}) \to \mathcal{M}^{(\mathbb{R},\leq)}$ be the inclusion functor and $q_* : \mathcal{M}^{(\mathbb{R},\leq)} \to \mathrm{Ho}(\mathcal{M})^{(\mathbb{R},\leq)}$ the the functor induced by the localization functor $q : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$. We say that X and Y in $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$ are *ε-homotopy commutative interleaved* if q_*jX and q_*jY are ε-interleaved in $\mathrm{Ho}(\mathcal{M})^{(\mathbb{R},\leq)}$.

Let X and Y be objects in $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$. Following Blumberg and Lesnick [4], and Lanari and Scoccola [24], we introduce the *homotopy interleaving distance* and the *homotopy commutative interleaving distance* defined by

$$d_{\mathrm{HI}}(X, Y) := \inf(\{\varepsilon \geq 0 \mid X,\, Y \text{ are } \varepsilon\text{-homotopy interleaved}\} \cup \{\infty\}) \text{ and}$$

$d_{\mathrm{HC}}(X,Y) := \inf(\{\varepsilon \geq 0 \mid X, Y \text{ are } \varepsilon\text{-homotopy commutative interleaved}\} \cup \{\infty\})$, respectively. We refer the reader to Proposition 7.2 for inequalities which hold for the interleaving distances mentioned above.

Remark 2.8. Interleavings in the homotopy category and homotopy commutative interleavings can be composed, respectively. Thus, we see that the distances d_{IHC} and d_{HC} satisfy the triangle inequality and then d_{IHC} and d_{HC} are pseudodistances on the class of objects in $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$.

Let $\mathcal{M}^{(\mathbb{R},\leq)}$ be the category endowed with the projective model structure for a cofibrantly generated model category $\mathcal{M}$. In [24, 2.2.3], it is proved that the homotopy interleavings on the category $\mathcal{M}^{(\mathbb{R},\leq)}$ can be composed by using the functorial fibrant replacement. However, it is not immediate that the homotopy interleavings on $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$ for a general model category $\mathcal{M}$ are composable.

Proposition 2.9. *Suppose that each object in $\mathcal{M}$ is fibrant. Then, homotopy interleavings can be composed in $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$. As a consequence, the distance d_{HI} is a pseudodistance on the class of objects in $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$.*

Proof. It follows from the assumption that each object in $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$ is also fibrant; see the model category structure of $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$ described in Definition 2.5. Then, the result follows from the proof of [24, Proposition 2.3] with [24, Lemma 2.2]. We may apply [30, Lemma 6.1.4] when making trivial fibrations with a common domain in the argument [24, Proposition 2.3]. Observe that a trivial fibration is stable under pullback. □

Remark 2.10. Let CDGA denote the category of commutative differential graded algebras (CDGAs) over $\mathbb{Q}$. In the rest of the manuscript except for Section 7, we mainly focus on considering the interleaving distance in the homotopy category of $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$ the category of extended tame persistence CDGAs. We observe that the category CDGA is endowed with the model structure introduced in [5, 4.2 Definition]. In particular, each object is fibrant. Then, by Proposition 2.9, the distance d_{HI} is a pseudodistance on the class of objects in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$.

3. Extended tame commutative differential graded algebras

We begin by recalling relative Sullivan algebras and explain how to relate the algebra to a persistence object; see [19] for relative Sullivan algebras (KS-extensions) and their homotopy theory.

3.1. Relative Sullivan models for maps and tame persistence CDGAs.

Let $\iota : \wedge V_A \to \wedge V_A \otimes \wedge W$ be a minimal relative Sullivan algebra. Then, by definition, there exists a filtration $\{W^p(r)\}_{r\geq 0}$ for each W^p such that

$$d : 1 \otimes W^p(r) \to \wedge V_A \otimes \wedge (W^{<p} \oplus W^p(r-1)). \tag{3.1}$$

Observe that the minimal relative Sullivan model for a map is unique up to isomorphism; see [14, Theorem 14.12]. In what follows, we assume further that $\wedge V_A$ is minimal. Then, the model gives rise to a sequence $\theta(\iota)$ of CDGAs defined by $\theta(\iota)(n) := \wedge V_A \otimes \wedge (W^{\leq n})$ together with the inclusions

$$\theta(\iota)(n < n+1) : \wedge V_A \otimes \wedge (W^{\leq n}) \to \wedge V_A \otimes \wedge (W^{\leq n+1}) = \wedge V_A \otimes \wedge (W^{\leq n}) \otimes \wedge (W^{n+1})$$

as maps connecting the CDGAs. By the definition of the minimality of a relative Sullivan algebra, we have

Proposition 3.1. *Each inclusion $\theta(\iota)(n < n+1)$ is a minimal relative Sullivan algebra with the filtration $\{W^{n+1}(r)\}_{r\geq 0}$.*

Let $f : X \to A$ be a continuous map between path-connected spaces and $\iota_f : \wedge V_A \to \wedge V_A \otimes \wedge W$ a minimal relative Sullivan model for the map f. Thus, we have a commutative diagram

$$
\begin{tikzcd}
\wedge V_A \arrow[r, "\iota_f"] \arrow[d, "\sim"'] & \wedge V_A \otimes \wedge W \arrow[d, "\sim"] \\
A_{\mathrm{PL}}(A) \arrow[r, "A_{\mathrm{PL}}(f)"'] & A_{\mathrm{PL}}(X)
\end{tikzcd}
\tag{3.2}
$$

whose vertical arrows are quasi-isomorphisms, where A_{PL} denotes the polynomial de Rham functor; see [15, Theorem 3.1] for the existence of a minimal relative Sullivan model for a map.

We say that a map $f : X \to A$ is *relatively finite* if the vector space W is of finite dimension for the minimal relative Sullivan model $\wedge V_A \otimes \wedge W$ for the map f. We observe that $\Theta(f)$ is tame in the sense in Definition 2.4 if the map f is relatively finite.

In order to relate the sequence $\theta(\iota_f)$ to a persistence object, we recall the floor function $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$. Then, we have an extended tame persistence CDGA $\Theta'(f)$ defined by $\Theta'(f) := (\lfloor\ \rfloor)^*\theta(\iota_f)$. It is immediate that $\Theta'(f)$ is tame if f is relatively finite.

Remark 3.2. It follows from [19, 4.6 Theorem] that a minimal relative Sullivan algebra ι_f associated with a map $f : X \to A$ is uniquely determined up to isomorphism and then so is $\Theta'(f)$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$.

Let $\mathrm{Func}(I, \mathsf{Top}_0)$ be the functor category from I the category consisting of two objects and nontrivial one arrow to Top_0 the category of path-connected topological spaces. The main result in this section is described as follows.

Theorem 3.3. *The function Θ' gives rise to a contravariant functor*

$$\Theta : \mathrm{Func}(I, \mathsf{Top}_0) \to \mathrm{Ho}(\mathsf{et}([0,\infty), \mathsf{CDGA})) = \mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+}))$$

where the right-hand side denotes the homotopy category of $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$.

The proof of the theorem is postponed to Section 3.2.

We consider the colimit functor $\mathrm{colim} : \mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+}) \to \mathsf{CDGA}$ whose right adjoint is the constant diagram functor c. Since c preserves fibrations as well as trivial fibrations, it follows that the pair (colim, c) is a Quillen adjunction. Thus, we have a left derived functor $\mathbf{L}(\mathrm{colim}) : \mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})) \to \mathrm{Ho}(\mathsf{CDGA})$; see, for example, [11, Proposition 9.3].

Proposition 3.4. *The diagram of functors*

$$
\begin{tikzcd}
\mathrm{Func}(I, \mathsf{Top}_0)^{\mathrm{op}} \arrow[r, "\Theta"] & \mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})) \arrow[d, "\mathbf{L}(\mathrm{colim})"] \\
\mathsf{Top}_0^{\mathrm{op}} \arrow[u, "i"] \arrow[r, "q\circ A_{\mathrm{PL}}(\ )"'] & \mathrm{Ho}(\mathsf{CDGA})
\end{tikzcd}
\tag{3.3}
$$

is commutative, where $q : \mathsf{CDGA} \to \mathrm{Ho}(\mathsf{CDGA})$ denotes the localization functor and i is the functor defined by $i(X) = X \to *$ the trivial map.

Proof. The commutativity follows from the fact that the image of the functor Θ consists of cofibrant objects; see Proposition 2.6. $\square$

3.2. **Cylinder and path objects in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$.** In order to develop homotopy theory for extended tame persistence CDGAs, we clarify a cylinder object and a path object for $\theta(\iota)$ associated with a relative Sullivan algebra ι in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$.

We recall the cylinder object which is used when defining the left homotopy in CDGA; see [19, Chapter 5]. Let $\wedge Z$ be a Sullivan algebra and $\nabla : \wedge Z \otimes \wedge Z \to \wedge Z$ the multiplication. Define graded vector spaces $\overline{Z}$ and $\widehat{Z}$ by $\overline{Z}^n = Z^{n+1}$ and $\widehat{Z}^n = Z^n$, respectively. We write $\overline{v}$ and $\widehat{v}$ for the elements in $\overline{Z}$ and $\widehat{Z}$ corresponding to v. Then, the map ∇ is decomposed as

$$\wedge Z \otimes \wedge Z \xrightarrow{i} (\wedge Z)^I := (\wedge Z \otimes \wedge Z \otimes \wedge \overline{Z}, \widetilde{d}) \xrightarrow[\cong]{\zeta} (\wedge Z \otimes \wedge \overline{Z} \otimes \wedge \widehat{Z}, D) \xrightarrow[\sim]{\rho} \wedge Z,$$

where the map i is the natural inclusion, ρ is defined by $\rho(\widehat{v}) = 0 = \rho(\overline{v})$, $\rho(v) = v$ and $D(\overline{v}) = \widehat{v}$. Moreover, the map ζ is an isomorphism of algebras defined by $\zeta(\overline{v}) = \overline{v}$, $\zeta(v \otimes 1 \otimes 1) = v$ and $\zeta(1 \otimes v \otimes 1) = \sum_{n=0}^{\infty} \frac{\theta^n}{n!}(v)$, where $\theta = sD + Ds$ with the derivation s of degree -1 defined by $s(v) = \overline{v}$, $s(\overline{v}) = 0 = s(\widehat{v})$. The differential $\widetilde{d}$ is defined by $\widetilde{d} = \zeta^{-1} D \zeta$. We observe that $\zeta(1 \otimes v \otimes 1) = v + \widehat{v} + \sum_{n=1}^{\infty} \frac{(sD)^n}{n!}(v)$.

We see that for objects F and F' in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$, the pointwise tensor product $F \otimes F'$ is the coproduct of F and F'. The pointwise multiplication gives rise to a morphism $\nabla : \theta(\iota) \otimes \theta(\iota) \to \theta(\iota)$, where $\theta(\iota)$ is the persistence CDGA associated with a relative Sullivan algebra $\iota : \wedge V \to \wedge V \otimes \wedge W$. We define a persistence CDGA $\theta(\iota)^I$ by the sequence

$$(\wedge V)^I \to (\wedge V \otimes \wedge W^{\leq 1})^I \to (\wedge V \otimes \wedge W^{\leq 2})^I \to \cdots .$$

Then, we have a commutative diagram

$$\begin{tikzcd} \theta(\iota) \otimes \theta(\iota) \arrow[r, "g"'] \arrow[rr, bend left, "\nabla"] & \theta(\iota)^I \arrow[r, "\pi"'] & \theta(\iota) \end{tikzcd} \tag{3.4}$$

in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$, where g and π are induced by the natural inclusion and the projection π mentioned above, respectively.

Lemma 3.5. *In the commutative diagram (3.4), the map g is a cofibration and π is a trivial fibration in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$. It turns out that the persistence CDGA $\theta(\iota)^I$ is a cylinder object for $\theta(\iota)$.*

Proof. The definitions of the fibration and the weak equivalence described in Theorem 2.5 enable us to deduce that π is a trivial fibration. In order to show that i is a cofibration, we recall the construction with the diagram (2.3). Then, we have a commutative diagram

$$\begin{tikzcd} (\wedge V \otimes \wedge W^{\leq k})^{\otimes 2} \arrow[r, "\theta(\iota)^{\otimes 2}(k<k+1)"] \arrow[d, "g(k)"'] & (\wedge V \otimes \wedge W^{\leq k+1})^{\otimes 2} \arrow[d, "\bar{g}(k+1)"] \arrow[dd, bend left=60, "g(k+1)"] \\ (\wedge V \otimes \wedge W^{\leq k})^{\otimes 2} \otimes \wedge(\overline{V \oplus W^{\leq k}}) \arrow[r] \arrow[dr, "(\theta(\iota)^I)(k<k+1)"'] & (\wedge V \otimes \wedge W^{\leq k+1})^{\otimes 2} \otimes \wedge(\overline{V \oplus W^{\leq k}}) \arrow[d, "\hat{g}(k+1)"] \\ & (\wedge V \otimes \wedge W^{\leq k+1})^{\otimes 2} \otimes \wedge(\overline{V \oplus W^{\leq k+1}}) \end{tikzcd}$$

in which the inside square is a pushout. It follows that $\hat{g}$ is a relative Sullivan algebra and hence g is a cofibration. $\square$

For a minimal relative Sullivan algebra ι, we consider a commutative diagram

$$
\begin{tikzcd}
\theta(\iota) \arrow[r, "h"'] \arrow[rr, bend left, "\Delta"] & \theta(\iota)\otimes\wedge(t,dt) \arrow[r, "p"'] & \theta(\iota)\times\theta(\iota),
\end{tikzcd}
\tag{3.5}
$$

where Δ denotes the diagonal map, h is the canonical inclusion and p is the epimorphism defined by $p = ev_0 \times ev_1$. We regard $\wedge(t, dt)$ as a constant persistence CDGA.

Lemma 3.6. *In the commutative diagram (3.5), the map h is a trivial cofibration and p is a fibration in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$. It turns out that the persistence CDGA $\theta(\iota)\otimes\wedge(t, dt)$ is a path object for $\theta(\iota)$.*

Proof. It is readily seen that h is a weak equivalence. Then, it suffices to show that h is a cofibration. The diagram (2.3) for h is given by

$$
\begin{tikzcd}
\wedge V\otimes\wedge W^{\le k} \arrow[r, "\theta(\iota)(k<k+1)"] \arrow[d, "h(k)"'] & \wedge V\otimes\wedge W^{\le k+1} \arrow[d, "\bar{h}(k+1)"] \arrow[dd, bend left=60, "h(k+1)"] \\
(\wedge V\otimes\wedge W^{\le k})\otimes\wedge(t,dt) \arrow[r] \arrow[dr, "(\theta(\iota)\otimes\wedge(t,dt))(k<k+1)"'] & (\wedge V\otimes\wedge W^{\le k+1})\otimes\wedge(t,dt) \arrow[d, "\hat{h}(k+1)"] \\
 & (\wedge V\otimes\wedge W^{\le k+1})\otimes\wedge(t,dt).
\end{tikzcd}
$$

We see that $\hat{h}(k+1)$ is the identity map and then h is a cofibration in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$. $\square$

In order to prove Theorem 3.3, we use a relative homotopy in the sense in [19, 9.13 Definition].

Lemma 3.7. *Let H be a relative homotopy between relative Sullivan algebras ι and ι' in CDGA; that is, H fits in the commutative diagram*

$$
\begin{tikzcd}
(\wedge V)^I \arrow[r, "\iota"] \arrow[d, "H"'] & (\wedge V\otimes\wedge W)^I \arrow[d, "H"] \\
\wedge V' \arrow[r, "\iota'"'] & \wedge V'\otimes\wedge W'.
\end{tikzcd}
$$

Then, one has $H(\theta(\iota)^I(k)) \subset \theta(\iota')(k)$.

Proof. This result follows from the fact that H preserves the degree of CDGAs. $\square$

Proof of Theorem 3.3. The proof heavily relies on the properties of left homotopies on relative Sullivan algebras investigated in [19]. Let $f : X \to A$ and $g : Y \to B$ be continuous maps and $(h, k) : f \to g$ a morphism in $\mathrm{Func}(I, \mathsf{Top}_0)$. Then, the construction in [19, (10,10)] allows us to obtain a diagram

$$
\begin{tikzcd}
\wedge V' \arrow[rrr, "\iota_g"] \arrow[dr, "\gamma'"', "\sim"] \arrow[ddd, "\varphi"'] & & & \wedge V'\otimes\wedge W' \arrow[dl, "\eta'", "\sim"'] \arrow[ddd, "\psi"] \\
 & A_{\mathrm{PL}}(B) \arrow[r, "g^*"] \arrow[d, "k^*"'] & A_{\mathrm{PL}}(Y) \arrow[d, "h^*"] & \\
 & A_{\mathrm{PL}}(A) \arrow[r, "f^*"] & A_{\mathrm{PL}}(X) & \\
\wedge V \arrow[rrr, "\iota_f"'] \arrow[ur, "\gamma", "\sim"'] & & & \wedge V\otimes\wedge W \arrow[ul, "\eta"', "\sim"]
\end{tikzcd}
\tag{3.6}
$$

in which ι_f and ι_g are relative Sullivan models for f and g, respectively, for the right and left squares, $(k^*\gamma', h^*\eta') \sim_{\mathrm{rel}} (\gamma\varphi, \eta\psi)$ and all the remaining squares commute; see [19, 9.13 Definition] for the definition of the relative homotopy relation $\sim_{\mathrm{rel}}$. Moreover, if a pair (φ_1, ψ_1) of morphisms from ι_g to ι_f satisfies the condition above, then we have $(\varphi_1, \psi_1) \sim_{\mathrm{rel}} (\varphi, \psi)$. This follows from [19, 9.19 Theorem]. In order to consider left homotopy in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$, we may use the cylinder object, which is introduced in Lemma 3.5, for an extended tame CDGA $\theta(\iota_f)$. Thus, it follows from Lemma 3.7 that a relative homotopy in CDGA gives rise to a homotopy in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$. Therefore, the assignment Θ on the objects is well defined.

The functoriality of Θ follows from the uniqueness of the pair (φ, ψ) up to homotopy and the fact that the composite of morphisms of CDGAs preserves the equivalence relation on relative homotopies; see [19, 9.17 and 9.18 Propositions]. $\square$

Remark 3.8. Let $\iota_f : \wedge V_A \to \wedge V_A \otimes \wedge W$ be a minimal relative Sullivan model for a map $f : X \to A$. Then, by definition, the persistence CDGA $\Theta(f)$ has the form $\wedge V_A \otimes \wedge W^{\leq n}$ on the interval $[n, n+1)$.

Remark 3.9. The property (3.1) enables us to deduce that the structure map $\theta(\iota)^{s<t}$ in $\theta(\iota)$ for a relative Sullivan algebra ι is a minimal relative Sullivan algebra. By Proposition 2.6 (ii), we see that for each map $f : X \to Y$, the persistence CDGA $\Theta'(f)$ is cofibrant in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$. Moreover, every object in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$ is fibrant. Thus, we may consider $\Theta'(f)$ itself without a fibrant-cofibrant replacement in $\mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+}))$.

Remark 3.10. Given a Sullivan representative $\varphi_f : \mathfrak{M}(A) = \wedge V_A \to \wedge V_X = \mathfrak{M}(X)$ for a map $f : X \to A$, we have a minimal relative Sullivan algebra $i_f : \wedge V_A \to \wedge V_A \otimes \wedge W$ for φ_f; that is, i_f satisfies the condition that $\eta_f \circ i_f = \varphi_f$ for some quasi-isomorphism $\eta_f : \wedge V_A \otimes \wedge W \to \wedge V_X$. The result [16, Proposition 2.22] enables us to deduce that i_f also fits in the commutative diagram (3.2) instead of ι_f. This implies that the persistence CDGA $\theta(i_f)$ obtained by i_f coincides with $\theta(\iota_f)$ up to isomorphism.

As seen above, a homotopy h between morphisms $\varphi, \psi \colon F \to G$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$ is a morphism of the form $F^I \to G$ or $F \to G \otimes \wedge(t, dt)$, where F^I is a cylinder object for F and $G \otimes \wedge(t, dt)$ is a path object for G. We observe that the homotopy h gives rise to a pointwise homotopy $h(i)$ between $\varphi(i), \psi(i) \colon F(i) \to G(i)$ for each i but the inverse does not necessarily hold. The following remark explains how to construct a right homotopy between objects in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$ associated with minimal relative Sullivan algebras.

Remark 3.11. Let $h : \wedge V \otimes \wedge W \to \wedge V' \otimes \wedge W' \otimes \wedge(t, dt)$ be a morphism of CDGAs between minimal relative Sullivan algebras $\iota : \wedge V \to \wedge V \otimes \wedge W$ and $\iota' : \wedge V' \to \wedge V' \otimes \wedge W'$. We do not necessarily assume that $h(V) \subset \wedge V'$.

Let ε be a positive real number. Suppose that $h(v)$ is in $\wedge V' \otimes \wedge (W'^{\leq \lfloor \varepsilon \rfloor}) \otimes \wedge(t, dt)$ for $v \in V$. Then, since $h(w)$ is in $\wedge V' \otimes \wedge (W'^{\leq i}) \otimes \wedge(t, dt)$ for $w \in W^{\leq i}$, it follows that the restriction of h gives rise to a map $\widetilde{h} : \theta(\iota) \to \theta(\iota')^{\varepsilon} \otimes \wedge(t, dt)$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$.

4. A Postnikov tower meets an extend tame persistence CDGA

It is well known that the minimal model for a simply-connected space X is related to a Postnikov tower for X via Hirsch extensions. As we did not find any literature explaining in detail the same relationships between a relative Sullivan model for a map f and a Postnikov tower for f, we clarify such a relationship from the viewpoint of persistence theory in this section.

4.1. **A partial Quillen equivalence and the functor Θ.** We begin with the notion of a *partial Quillen equivalence* introduced in [28]. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ denote a Quillen adjunction between model categories with the unit η and the counit ϵ; see [11] for derived functors. Let $q_X : QX \to X$ be a cofibrant replacement of $X \in \mathcal{C}$ and $r_Y : Y \to RY$ a fibrant replacement of $Y \in \mathcal{D}$. Then, we define the *derived unit* and the *derived counit* by the composites

$$\widetilde{\eta_X} : X \xrightarrow{\eta_X} UFX \xrightarrow{U_{r_{FX}}} URFX \quad \text{and} \quad \widetilde{\epsilon_Y} : FQUY \xrightarrow{F_{q_{UY}}} FUY \xrightarrow{\epsilon_Y} Y,$$

respectively.

Lemma 4.1. *The following conditions are equivalent.*

(i) *For every cofibrant object X of $\mathcal{C}$ and every fibrant object Y of $\mathcal{D}$, the derived unit $\widetilde{\eta_X}$ and the derived counit $\widetilde{\epsilon_Y}$ are weak equivalences.*

(ii) *For every cofibrant object X of $\mathcal{C}$ and every fibrant object Y of $\mathcal{D}$, a map $f : X \to U(Y)$ is a weak equivalence in $\mathcal{C}$ if and only if its adjoint $f^\flat : F(X) \to Y$ is a weak equivalence in $\mathcal{D}$.*

As a consequence, if each $\widetilde{\eta_X}$ and each $\widetilde{\epsilon_Y}$ are weak equivalences for every cofibrant object X and every fibrant object Y, then the derived functors $\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathbf{R}U$ are inverse equivalences of categories.

Proof. See, for example, [29, Lemma C.3.6]. The latter half follows from [11, Theorem 9.7 (ii)]. In fact, the condition (ii) is the sufficient one in the theorem. □

Definition 4.2. (cf. [28, Definition 2.4]) Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a Quillen adjunction. Suppose $\mathcal{C}_0 \subset \mathcal{C}$ and $\mathcal{D}_0 \subset \mathcal{D}$ are full subcategories such that the following hold:

(1) The two subcategories are closed under weak equivalences; that is, if C is weakly equivalent to C_0 with $C_0 \in \mathcal{C}_0$, then $C \in \mathcal{C}_0$ too, and analogously for $\mathcal{D}_0$.

(2) $FX \in \mathcal{D}_0$ for a cofibrant object $X \in \mathcal{C}_0$ and $UY \in \mathcal{C}_0$ for a fibrant object $Y \in \mathcal{D}_0$

(3) Every objectwise derived unit $X \to URFX$ is a weak equivalence for a cofibrant $X \in \mathcal{C}_0$. Dually, every objectwise derived counit $FQUY \to Y$ is a weak equivalence for a fibrant $Y \in \mathcal{D}_0$.

Then, we call the adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ a *partial Quillen equivalence for the subcategories $\mathcal{C}_0$ and $\mathcal{D}_0$.*

Remark 4.3. Originally, the notion of a partial Quillen equivalence in [28] is defined for two model categories which admits functorial factorizations. Moreover, the second condition requires that $F\mathcal{C}_0 \subset \mathcal{D}_0$ and $U\mathcal{D}_0 \subset \mathcal{C}_0$. As mentioned in [28, The first paragraph in Page 5], (**) : the restriction of the derived adjunction $\mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D})$ to full subcategories consisting of objects $\mathcal{C}_0$ and $\mathcal{D}_0$, respectively, gives an equivalence of categories.

While the condition (2) in Definition 4.2 is different from the original one and the model category that we deal with does not require the functorial factorizations, the assertion (**) remains valid. This follows from the fact that the unit and counit of the derived functors in Lemma 4.1 give rise to isomorphisms on the objects in the full subcategories of the homotopy categories; see the proof of [11, Theorem 9.7 (ii)]. Observe that the conditions (1) and (2) yield the well-definedness of the restrictions of the derived functors.

To describe the Sullivan–de Rham equivalence theorem [5, 10.1 Theorem] in terms of the partial Quillen equivalence, we recall finiteness properties of a CDGA and a simplicial set.

Definition 4.4. (i) A CDGA A is of finite $\mathbb{Q}$-type if each M^i is finite dimensional for the minimal model M for A.
(ii) A simplicial set X is of finite $\mathbb{Q}$-type if the ith rational cohomology group of X for $i \geq 0$ is finite dimensional.

Theorem 4.5. (cf. [5, 10.1 Theorem] and [28, Theorem 3.4].) *Let $\mathcal{C}_0$ be the full subcategory of CDGA consisting of connected CDGAs of finite $\mathbb{Q}$-type. Let $\mathcal{D}_0$ denote the full subcategory of sSet consisting of nilpotent path-connected rational simplicial sets of finite $\mathbb{Q}$-type. Then, the adjoint pair $\langle\ \rangle : \mathsf{CDGA} \rightleftarrows \mathsf{sSet}^{\mathrm{op}} : A_{\mathrm{PL}}(\ )$ of the realization functor $\langle\ \rangle$ and the polynomial de Rham functor $A_{\mathrm{PL}}(\ )$ gives rise to a partial Quillen equivalence for the subcategories $\mathcal{C}_0$ and $\mathcal{D}_0^{\mathrm{op}}$.*

Proof. It is readily seen that the condition (1) holds. The well-definedness of the functors in [5, 10.1 Theorem] allows us to deduce that the condition (2) holds. We consider the condition (3). Since the adjunction map φ_Y dealt with in the proof of [5, 10.1 Theorem (i)] is nothing but the objectwise derived counit $\widetilde{\epsilon_Y}$, it follows form the proof that $\widetilde{\epsilon_Y}$ is a weak equivalence. Moreover, the proof of [5, 10.1 Theorem (ii)] yields that the unit η_X is a weak equivalence for a cofibrant $X \in \mathcal{C}_0$ and then so is the derived unit $\widetilde{\eta_X}$. We observe that the homotopy category of fibrant and cofibrant objects of a model category $\mathcal{M}$, which is equivalent to $\mathrm{Ho}(\mathcal{M})$, is considered in [5, Sections 8, 9 and 10]. □

The following lemma is a variant of [28, Lemma 5.2].

Proposition 4.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a partial Quillen equivalence for the subcategories $\mathcal{C}_0$ and $\mathcal{D}_0$. Then, the induced Quillen adjunction $F_* : \mathsf{et}(\mathcal{C}^{\mathbb{R}_+}) \rightleftarrows \mathsf{et}(\mathcal{D}^{\mathbb{R}_+}) : U_*$ is a partial Quillen equivalence for the subcategories $\mathsf{et}(\mathcal{C}_0^{\mathbb{R}_+})$ and $\mathsf{et}(\mathcal{D}_0^{\mathbb{R}_+})$.*

Proof. The proof is verbatim the same as that of the original result. Instead of the derived counit and the derived unit obtained by the cofibrant and fibrant replacement functors in the proof of [28, Lemma 5.2], we consider the derived counit and the derived unit defined at the beginning of this section. Moreover, in our proof, it is necessary to pay attention to the difference in the definitions of the partial Quillen equivalence as mentioned in Remark 4.3.

It is readily seen that the condition (1) holds. By Proposition 2.6, we see that a cofibrant object of $\mathsf{et}(\mathcal{C}^{\mathbb{R}_+})$ is pointwise cofibrant. By definition, a fibrant object of $\mathsf{et}(\mathcal{C}^{\mathbb{R}_+})$ is pointwise fibrant. Then, the condition (2) holds. We verify that the condition (3) holds. Let $X \in \mathsf{et}(\mathcal{C}_0^{\mathbb{R}_+})$ be a cofibrant object. Proposition 2.6 (ii) yields that X_0 is a cofibrant object and the transition morphism $X(s<t) : X(s) \to X(t)$ for any $s < t$ in $\mathbb{R}_+$ is cofibration in $\mathcal{C}_0$. Thus, we see that $X(t)$ is cofibrant in $\mathcal{C}_0$ for $t \in \mathbb{R}_+$. Let $\underline{r}_Y : Y \to \underline{R}Y$ be a fibrant replacement of $Y \in \mathsf{et}(\mathcal{D}_0^{\mathbb{R}_+})$. Observe that $(\underline{R}Y)(t)$ is in $\mathcal{D}_0$ and then $\underline{R}Y$ is an object in $\mathsf{et}(\mathcal{D}_0^{\mathbb{R}_+})$. We prove that the composite

$$\widetilde{\eta_X} : X \xrightarrow{\eta_{*X}} U_*F_*X \xrightarrow{U_*\underline{r}_{F_*X}} U_*\underline{R}F_*X$$

is a weak equivalence in $\mathsf{et}(\mathcal{C}_0^{\mathbb{R}_+})$, where η_* denotes the unit of $F_* \dashv U_*$, which is indeed componentwise the unit η of $F \dashv U$. Therefore, we need to check that the composite

$$(\widetilde{\eta_X})(t) : X(t) \xrightarrow{\eta(X(t))} UFX(t) \xrightarrow{U\underline{r}_{F_*X}(t)} U(\underline{R}F_*X)(t)$$

is a weak equivalence in $\mathcal{C}_0$ for all $t \in \mathbb{R}_+$. The model category structure on $\mathsf{et}(\mathcal{D}^{\mathbb{R}_+})$ described in Theorem 2.5 implies that $(\underline{R}F_*X)(t)$ is fibrant. Moreover, by Proposition 2.6 (i), we see that $(\underline{r}_{F_*X})(t)$ is an acyclic cofibration. Let $r_Y : Y \to RY$ be a fibrant replacement of $Y \in \mathcal{D}_0$. Then, there exists a lift h which makes the diagram

$$\begin{tikzcd}
FX(t) \arrow[r, tail, "r_{FX(t)}", "\sim"'] \arrow[d, tail, "(\underline{r}_{F_*X})(t)"', "\sim"] & R(FX(t)) \\
(\underline{R}F_*X)(t) \arrow[ru, dotted, "h"'] &
\end{tikzcd}$$

commutative. Since h is a weak equivalence between fibrant objects, it follows from the dual to [11, 9.9 Lemma (K. Brown)] that the right Quillen functor U takes h to a weak equivalence. Then, we have the commutative diagram

$$\begin{tikzcd}
X(t) \arrow[r, "\eta_{X(t)}"] \arrow[rr, dotted, bend left] & UFX(t) \arrow[r, "U_{(\underline{r}_{F_*X})(t)}"] \arrow[rr, bend right, "U_{r_{FX(t)}}"'] & U\underline{R}(F_*X)(t) \arrow[r, "U_h", "\sim"'] & UR(FX(t)).
\end{tikzcd}$$

As mentioned above, the object $X(t)$ is cofibrant in $\mathcal{C}_0$. Moreover, by assumption, the adjoint $F \dashv U$ is a partial Quillen equivalence. Therefore, the total composite, which is an objectwise derived unit of $F \dashv U$ at $X(t)$, is a weak equivalence. Since U_h is a weak equivalence, it follows from the 2-out-of-3 property that the dotted composite, which is nothing but the morphism $(\widetilde{\eta_X})(t)$, is also a weak equivalence.

The dual argument works to show that the objectwise derived counit is a weak equivalence. □

Remark 4.7. Let $\mathcal{C}$ be a model category and $\mathsf{tame}(\mathcal{C}^{\mathbb{R}_+})$ denote the full subcategory of $\mathsf{et}(\mathcal{C}^{\mathbb{R}_+})$ comprised of tame objects; see Definition 2.4. The category $\mathsf{tame}(\mathcal{C}^{\mathbb{R}_+})$ is endowed with the model category structure described in Theorem 2.5, which is defined originally in [9]. Proposition 4.6 remains valid even if 'et' is changed with 'tame'. This follows from the proof of the proposition.

In order to describe the main result in this section, we recall the definition of a Postnikov tower for a map.

Definition 4.8. [18, Chapter VI Definition 3.9] Let $f : X \to Y$ be a morphism of simplicial sets. Then a Postnikov tower for f is a tower of spaces $\{X_n, q_{n+1} : X_{n+1} \to X_n\}_{n\geq 0}$ equipped with maps of towers

$$\{i_n\} : \{X\} \to \{X_n\}, \quad \{p_n\} : \{X_n\} \to \{Y\}$$

such that

(1) the composite $p_n \circ i_n$ is f for any n;
(2) for any choice of vertex of X, the map $(i_n)_* : \pi_k(X) \to \pi_k(X_n)$ is an isomorphism for $k \leq n$;
(3) for any choice of vertex of X, the map $(p_n)_* : \pi_k(X_n) \to \pi_k(Y)$ is an isomorphism for $k > n+1$;
(4) for any choice of vertex v of X, there is an exact sequence

$$0 \longrightarrow \pi_{n+1}(X_n) \longrightarrow \pi_{n+1}(Y) \xrightarrow{\partial} \pi_n(F_v),$$

where F_v is the homotopy fiber of f at v and ∂ is the connecting homomorphism in the long exact sequence of the homotopy fibration $F_v \to X \to Y$.

Here, $\{X\}$ and $\{Y\}$ denote the constant towers of X and Y, respectively.

We have a special Postnikov tower $\{X(n)\}$ for a Kan fibration $f : X \to B$ with fibrant B which is called the *Moore–Postnikov tower* for f; see [18, VI, 3].

For q simplices $\sigma, \tau : \Delta^q \to X$, we define a relation $\sigma \sim_n \tau$ if

i) $f \circ \sigma = f \circ \tau$ and
ii) $\sigma_| = \tau_| : \mathrm{sk}_n\Delta^q \to X$.

The relation gives an equivalence relation on X and then the projections $i_n : X \to X(n) := X/\sim_n$ and well-defined maps $p_n : X(n) \to B$ induced by f are defined. The consideration in [18, Chapter VI] yields that $\{\{X(n)\}, \{i_n\}, \{p_n\}\}$ is a Postnikov tower for $f : X \to B$. Observe that there exist canonical projections $q_n : X(n) \to X(n-1)$ and the map $p_n : X(n) \to B$ is a fibration with $p_n = p_{n-1} \circ q_n$. Moreover, it follows that the construction is functorial. Thus, we have a functor MP from the subcategory consisting of Kan fibrations with fibrant bases of $\mathrm{Func}(\mathrm{I}, \mathsf{sSet})$ to $\mathsf{et}(\mathsf{sSet}^{(\mathbb{Z}_+,\leq)^{\mathrm{op}}})$ defined by sending a Kan fibration f to the Moore–Postnikov tower for f.

Let $S_*(\ )$ be the singular simplex functor and $\mathcal{H}$ the homotopy fibration functor. Let $\lfloor\ \rfloor : (\mathbb{R}_+, \leq) \to (\mathbb{Z}_+, \leq)$ denote the floor function. Observe that f is a Serre fibration if and only if $S_*(f)$ is a Kan fibrations; see [18, Page 11]. We denote by $\mathcal{F}_{\mathrm{sc}}$ the full subcategory of $\mathrm{Func}(\mathrm{I}, \mathsf{Top}_0)^{\mathrm{op}}$ consisting of maps between simply-connected spaces of finite $\mathbb{Q}$-type. Observe that for a simply-connected space X, the space X is of finite $\mathbb{Q}$-type if and only if the vector spaces $H^*(X;\mathbb{Q})$ for $i \geq 1$ are finite dimensional; see [14, Proposition 12.2].

Theorem 4.9. *With the same notation as in Theorem 4.5, one has a diagram of categories and functors*

$$\begin{tikzcd}
\mathrm{Func}(\mathrm{I}, \mathsf{Top})^{\mathrm{op}} \arrow[rr, bend left, "mp:=(\lfloor\ \rfloor)^*\circ\mathrm{MP}\circ S_*(\ )\circ\mathcal{H}"] & \mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+}) \arrow[r, shift left, "\langle\ \rangle_*"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathsf{et}((\mathsf{sSet}^{\mathrm{op}})^{\mathbb{R}_+}) \arrow[l, shift left, "A_{\mathrm{PL}}(\ )_*"] \arrow[d] \\
\mathrm{Func}(\mathrm{I}, \mathsf{Top}_0)^{\mathrm{op}} \arrow[u] \arrow[r, "\Theta"] & \mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})) \arrow[r, shift left, "\mathbf{L}\langle\ \rangle_*"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(\mathsf{et}((\mathsf{sSet}^{\mathrm{op}})^{\mathbb{R}_+})) \arrow[l, shift left, "\mathbf{R}A_{\mathrm{PL}}(\ )_*"] \\
\mathcal{F}_{\mathrm{sc}} \arrow[u] \arrow[ur, "\Theta_{\mathrm{res}}"'] \arrow[r, dotted, "\Theta_{\mathrm{res}}"'] \arrow[rr, bend right, "(Rationalization)\circ(mp)=:(mp)_{\mathbb{Q}}"'] & \mathrm{Full}(\mathsf{et}(\mathcal{C}_0^{\mathbb{R}_+})) \arrow[u] \arrow[r, shift left, "\mathbf{L}\langle\ \rangle_*"] \arrow[r, phantom, "\simeq"] & \mathrm{Full}(\mathsf{et}((\mathcal{D}_0^{\mathrm{op}})^{\mathbb{R}_+})) \arrow[l, shift left, "\mathbf{R}A_{\mathrm{PL}}(\ )_*"] \arrow[u]
\end{tikzcd} \tag{4.1}$$

for which the following assertions hold.

(i) *The adjunction* $(\langle\ \rangle_*, A_{\mathrm{PL}}(\ )_*)$ *induced by* $(\langle\ \rangle, A_{\mathrm{PL}}(\ ))$ *pointwise gives rise to a partial Quillen equivalence for the subcategories* $\mathsf{et}(\mathcal{C}_0^{\mathbb{R}_+})$ *and* $\mathsf{et}((\mathcal{D}_0^{\mathrm{op}})^{\mathbb{R}_+})$. *As a consequence, the adjunction in the bottom of the lower-right square gives an equivalence of the categories.*

(ii) *The diagram starting at* $\mathcal{F}_{\mathrm{sc}}$ *and ending at* $\mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+}))$ *via* $\mathsf{et}((\mathsf{sSet}^{\mathrm{op}})^{\mathbb{R}_+})$ *is commutative.*

(iii) *The lower diagram comprised of* Θ_{res}, *the adjoint and the rationalization* $(mp)_{\mathbb{Q}}$ *is commutative.*

Here, Θ *is the functor described in Theorem 3.3,* Θ_{res} *is its restriction and* $\mathrm{Full}(\mathcal{E})$ *denotes the full subcategory of the homotopy category whose objects are in* $\mathcal{E}$. *Moreover, The (Rationalization) is the composite* $\mathbf{L}\langle\ \rangle_* \circ \mathbf{R}A_{\mathrm{PL}}(\ )_*$.

Remark 4.10. (i) We have an isomorphism $\mathrm{Func}(J, \mathcal{C}^{\mathrm{op}}) \cong \mathrm{Func}(J^{\mathrm{op}}, \mathcal{C})^{\mathrm{op}}$ between functor categories. While the codomain of the functor mp is the full subcategory $(\mathsf{et}((\mathsf{sSet})^{\mathbb{R}_+^{\mathrm{op}}}))^{\mathrm{op}}$ of the functor category $\mathrm{Func}(\mathbb{R}_+^{\mathrm{op}}, \mathsf{sSet})^{\mathrm{op}}$ consisting of extended tame copersistence simplicial sets, we use the category $\mathsf{et}((\mathsf{sSet}^{\mathrm{op}})^{\mathbb{R}_+})$ as the codomain up to the isomorphism of the categories.

(ii) The (Rationalization) gives indeed rise to the *levelwise* $\mathbb{Q}$*-localization.* In fact, The counit $\varepsilon_X : \mathbf{L}\langle\ \rangle_* \circ \mathbf{R}A_{\mathrm{PL}}(X)_* \to X$ is indeed the levelwise $\mathbb{Q}$-localization in $\mathsf{sSet}^{\mathrm{op}}$; see [5, 11.2] and [14, Theorem 17.12]. Since each object in $\mathsf{et}((\mathsf{sSet}^{\mathrm{op}})^{\mathbb{R}_+})$ is fibrant, it follows that $\mathbf{R}A_{\mathrm{PL}}(\ )_* = A_{\mathrm{PL}}(\ )_*$.

(iii) The domain of the composite $\mathrm{MP} \circ S_*(\ ) \circ \mathcal{H} : \mathrm{Func}(\mathrm{I}, \mathsf{Top}) \to \mathsf{sSet}^{(\mathbb{Z}_+, \leq)^{\mathrm{op}}}$ is a simplicial model category of towers of simplicial sets; see [18, Chapter VI, 1]. Moreover, we see that the adjoint

$$\lfloor\ \rfloor^* : (\mathsf{sSet}^{(\mathbb{Z}_+, \leq)^{\mathrm{op}}})^{\mathrm{op}} = (\mathsf{sSet}^{\mathrm{op}})^{(\mathbb{Z}_+, \leq)} \rightleftarrows \mathsf{et}((\mathsf{sSet}^{\mathrm{op}})^{\mathbb{R}_+}) : \mathrm{res}$$

of the the functor induced by the floor function and the restriction is a Quillen adjunction.

Corollary 4.11. *For objects* f *and* g *in* $\mathcal{F}_{\mathrm{sc}}$, *it holds that*

$$d_{\mathrm{IHC}}(\Theta(f), \Theta(g)) = d_{\mathrm{IHC}}(mp(f)_{\mathbb{Q}}, mp(g)_{\mathbb{Q}}),$$

where the right-hand side denotes the interleaving distance between copersistence simplicial spaces in the homotopy category.

Proof. This follows from the commutativity of the bottom diagram in (4.1) and the equivalence of categories. $\square$

Proof of Theorem 4.9. (i) The assertion follows from Proposition 4.6. Remark 4.3 and Theorem 4.5 yield the latter half of the assertion.

(ii) We give an appropriate cofibrant replacement of the object $(A_{\mathrm{PL}}(\ )_* \circ mp)(f)$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$ for a map $f : X \to B$. Consider the commutative diagram

$$\begin{tikzcd}
\wedge V_B \arrow[r, "j_1"'] \arrow[d, "="'] & \wedge V_B \otimes \wedge W_1 \arrow[r, dashed, "j_2"'] \arrow[d, "\sim"'] & \cdots \arrow[r, dashed] & \wedge V_B \otimes \wedge W_{n-1} \arrow[r, dashed, "j_n"'] \arrow[d, "\sim"'] & \wedge V_B \otimes \wedge W_n \arrow[r, dashed] \arrow[d, "\sim"'] & \cdots \\
\wedge V_B \arrow[r] \arrow[d, "\sim"'] & \wedge V_B \otimes \wedge Z_1 \arrow[r] \arrow[d, "\sim"'] & \cdots \arrow[r] & \wedge V_B \otimes \wedge Z_{n-1} \arrow[r] \arrow[d, "\sim"'] & \wedge V_B \otimes \wedge Z_n \arrow[r] \arrow[d, "\sim"'] & \cdots \\
A_{\mathrm{PL}}(B) \arrow[r, "A_{\mathrm{PL}}(q_1)"] & A_{\mathrm{PL}}(X(1)) \arrow[r, "A_{\mathrm{PL}}(q_2)"] & \cdots \arrow[r] & A_{\mathrm{PL}}(X(n-1)) \arrow[r, "A_{\mathrm{PL}}(q_n)"] & A_{\mathrm{PL}}(X(n)) \arrow[r] & \cdots
\end{tikzcd} \tag{4.2}$$

with solid arrows in which the second sequence consists of cofibrations and the third one is induced by the MP tower of f. Moreover, the retraction to minimal relative Sullivan algebras gives rise to the first line; that is, each vertical map is the inclusion and a quasi-isomorphism. This follows from [14, Theorem 14.9]. We show that there exist dots arrows such that each square is commutative and the first sequence is isomorphic to $\Theta(f)$ in $\mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+}))$.

The construction in [14, pages 204 and 205] enables us to obtain a morphism j_n which fits the commutative diagram

$$\begin{tikzcd}
 & \wedge V_B \otimes \wedge W_{n-1} \arrow[r] \arrow[d, "j_n"] & \wedge W_{n-1} \arrow[d, "\widetilde{j_n}"] \\
\wedge V_B \arrow[ur] \arrow[dr] & & \\
 & \wedge V_B \otimes \wedge W_n \arrow[r] & \wedge W_n
\end{tikzcd}$$

and the square in (4.2) containing j_n and $A_{\mathrm{PL}}(q_n)$ as the top and bottom arrows, respectively, is homotopy commutative. Moreover, the map $\widetilde{j_n}$ induced by j_n is a Sullivan representative for the natural map $(q_n)_| : F(n-1) \to F(n)$, where $F(n)$ is the fiber of the fibration $p_n : X(n) \to B$. We observe that the space $F(n)$ is the nth stage of the Moore–Postnikov tower of the fiber F of f. Moreover, since B is simply connected by assumption, it follows that $\wedge W_n$ is a minimal model for $F(n)$; see the proof of [18, Theorem 3.11]. Then, we see that $W_n = W_n^{\leq n}$ and $j_n : W_{n-1} \to W_n^{\leq n-1}$ is an isomorphism. Therefore, we may assume that j_n is an inclusion. Thus, the colimit $\mathrm{colim}_n(\wedge V_B \otimes \wedge W_n)$ is regarded as a relative Sullivan algebra of the form $\iota' : \wedge V_B \to V_B \otimes \wedge\widetilde{W}$.

By virtue of [16, Proposition 2.22], we may replace inductively vertical arrows connecting the first and third sequences in (4.2) with quasi-isomorphisms so that the each square is strictly commutative. It turns out that $\theta(\iota')$ is a cofibrant replacement for $A_{\mathrm{PL}}(mp(f))$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$.

In what follows, we show that $\theta(\iota')$ is isomorphic to $\theta(\iota_f)$. The universality of the colimit gives rise to a morphism u which makes the square consisting of solid arrows and u in the diagram

$$\begin{tikzcd}
\wedge V_B \arrow[r] \arrow[d, "="'] & \wedge V_B \otimes \wedge\widetilde{W} \arrow[r] \arrow[d, dotted, "\Phi"] \arrow[dd, dotted, bend left, "u"] & \wedge\widetilde{W} \arrow[d, dotted, "\overline{\Phi}"] \arrow[dd, dotted, bend left, "\overline{u}"] \\
\wedge V_B \arrow[r] \arrow[d, "\sim"'] & \wedge V_B \otimes \wedge W \arrow[r] \arrow[d, "\sim"', "v"] & \wedge W \arrow[d, "\sim"', "\overline{v}"] \\
A_{\mathrm{PL}}(B) \arrow[r, "A_{\mathrm{PL}}(f)"'] & A_{\mathrm{PL}}(X) \arrow[r] & A_{\mathrm{PL}}(F)
\end{tikzcd}$$

commutative. The lifting property ([14, Proposition14.6]) implies that there exists a morphism Φ such that the upper-left square is commutative and $v \circ \Phi \simeq u$ rel $\wedge V_B$. Moreover, the relative homotopy gives a homotopy between $\overline{v} \circ \overline{\Phi}$ and $\overline{u}$, where $\overline{v}$, $\overline{\Phi}$ and $\overline{u}$ are morphisms induced by v, Φ and u, respectively.

By [14, Proposition 15.5], we see that $\overline{v}$ is a quasi-isomorphism. Therefore, the restriction of $\overline{\Phi}$ to $\wedge W_n$ gives rise to a Sullivan representative for i_n in a Moore-Postnikov tower $\{i_n\} : \{F\} \to \{F(n)\}$ of the fiber F. This implies that $Q(\overline{\Phi}) : W_n \to W^{\leq n}$ is an isomorphism. Observe that $\wedge\widetilde{W}$ and $\wedge W$ are minimal. Then, the morphism $\overline{\Phi}$ is an isomorphism. It follows from [15, Proposition 3.10] that Φ is an isomorphism.

(iii) The result (i) and the commutativity in (ii) yield the assertion (iii). $\square$

Remark 4.12. With the same notation as in the proof of Theorem 4.9, in the category of simplicial sets, we have a commutative diagram

$$\begin{tikzcd}
 & & \lim\langle \wedge V_B \otimes \wedge(W^{\leq n})\rangle \arrow[d, "\mathrm{pr}_n"] \\
S_*(X)_{\mathbb{Q}} = \langle \wedge V_B \otimes \wedge W\rangle \arrow[r, "\langle\eta\rangle", "\cong"'] \arrow[rr, bend right, "\langle(\mathrm{inc.})_n\rangle"'] & \langle \mathrm{colim} \wedge V_B \otimes \wedge(W^{\leq n})\rangle \arrow[ur, "\cong", "u"'] \arrow[r, "\langle i_n\rangle"'] & \langle \wedge V_B \otimes \wedge(W^{\leq n})\rangle,
\end{tikzcd}$$

where η : colimit $\wedge V_B \otimes \wedge W^{\leq n} \to \wedge V_B \otimes \wedge W$ is a canonical isomorphism. The realization functor $\langle\ \rangle$ is the left adjoint to the functor A_{PL} and then $\langle\ \rangle$ preserves the colimit. Thus, the evident map u is an isomorphism. As a consequence, the universality of the limit enables us to conclude that the composite $u \circ \langle\eta\rangle$ is the evident map $S_*(X)_{\mathbb{Q}} = \langle \wedge V_B \otimes \wedge W\rangle \to \lim\langle \wedge V_B \otimes \wedge(W^{\leq n})\rangle$ which is an isomorphism.

4.2. **The geometric realization of an extended tame CDGA.** In this subsection, we work in the category of topological spaces and we discuss the geometric realization of the extended tame CDGA $\theta(\iota)$ obtained from a minimal relative Sullivan algebra $\iota : \wedge V \to \wedge V \otimes \wedge W$. Here, we assume that $W = W^{\geq 1}$. However, the condition that $V^1 = 0$ is not necessarily assumed.

While the original definition (Definition 4.8) of a Postnikov tower of a map is given for simplicial sets, we here consider its analogue for path-connected topological spaces. In the end of the section, we give geometric realizations of specific extended tame CDGAs described in Sections 5 and 8.

Let $|\cdot|$ be the spatial realization functor; see [14, §17]. By applying the functor $|\cdot|$ to the sequence $\theta(\iota)$, we obtain a tower of topological spaces

$$\{|\theta(\iota)(n)|\} : |\theta(\iota)(0)| \longleftarrow |\theta(\iota)(1)| \longleftarrow |\theta(\iota)(2)| \longleftarrow \cdots \longleftarrow |\theta(\iota)(n)| \longleftarrow \cdots$$

Each structure map $\theta(\iota)(n) \to \theta(\iota)(n{+}1)$ of $\theta(\iota)$ is a relative Sullivan algebra. Then, by [14, Proposition 17.9], we see that the sequence above is a tower of fibrations.

Let i_n and p_n be the spatial realizations of the inclusions $\theta(\iota)(n) \to \wedge V \otimes \wedge W$ and $\wedge V \to \theta(\iota)(n)$, respectively.

Proposition 4.13. *The tower $\{|\theta(\iota)(n)|\}_{n\geq 0}$ equipped with $\{i_n\}$ and $\{p_n\}$ is a Postnikov tower for the map $|\iota| : |\wedge V \otimes \wedge W| \to |\wedge V|$ in the sense in Definition 4.8.*

Proof. To prove that $\{|\theta(n)|\}$ is a Postnikov tower, we verify the conditions (1)–(4) in Definition4.8.

(1) Clearly, $p_n \circ i_n = |\iota|$ for each n by the naturality of the spatial realization.

(2) In the case $n = 0$, $|\wedge V \otimes \wedge W|$ and $|\theta(\iota)(0)| = |\wedge V|$ are connected. Thus, it is immediate that $(i_0)_* : \pi_0(|\wedge V \otimes \wedge W|) \to \pi_0(|\theta(\iota)(0)|)$ is an isomorphism. Suppose that $n \geq 1$. By [14, Proposition 17.9], we see that i_n is a fibration with fiber $\left|\wedge(W^{>n}), \bar{d}\right|$. Here $(\wedge(W^{>n}), \bar{d})$ is the quotient of $\wedge V \otimes \wedge W$ by the ideal generated by $\left(\wedge V \otimes \wedge(W^{\leq n})\right)^+$. Since $(\wedge W^{>n}, \bar{d})$ is a Sullivan algebra with generators only in degrees $k > n$, it follows from [15, §1.7 Theorem 1.1 (iii)] that and the formality of S^n that

$$\pi_k\left(\left|\wedge(W^{>n}), \bar{d}\right|\right) \cong \left[\left(\wedge(W^{>n}), \bar{d}\right), A_{\mathrm{PL}}(S^k)\right] \cong \left[\left(\wedge(W^{>n}), \bar{d}\right), H^*(S^k;\mathbb{Q})\right].$$

Since the right-hand side homotopy set consists only of the trivial morphism for $k \leq n$, we have $\pi_k\left(\left|\wedge(W^{>n}), \bar{d}\right|\right) = 0$ for $k \leq n$. Thus, the homotopy long exact sequence of homotopy groups associated with i_n implies the condition (2).

(3) The proof of the condition (3) is essentially the same as that of (2). In particular, the result [14, Proposition 17.9] allows us to deduce that $p_n : |\theta(\iota)(n)| \to |\wedge V|$ is a fibration with fiber $\left|\wedge(W^{\leq n}), \bar{d}\right|$. We also see that $\pi_k\left(\left|\wedge(W^{\leq n}), \bar{d}\right|\right) = 0$ for $k \geq n+1$ from [15, §1.7 Theorem 1.1 (iii)]. By the long exact sequence of homotopy groups associated with p_n, we conclude that condition (3) is satisfied.

(4) Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_{n+1}(|\theta(\iota)(n)|) \arrow[r, "(p_n)_*"] & \pi_{n+1}(|\wedge V|) \arrow[r, "\partial"] & \pi_{n+1}(F) \\
& \pi_{n+1}(|\wedge V \otimes \wedge W|), \arrow[u, "(i_n)_*"] \arrow[ur, "|\iota|_*"'] & &
\end{tikzcd}$$

where F is the fiber of $|\iota|$. The commutativity of this diagram follows from (1). We shall show that the top sequence in the diagram is exact. By the same argument as in (3), we see that $(p_n)_*$ is a monomorphism. Similarly, we can verify that $(i_n)_*$ is an epimorphism by the same argument as in (2). It follows from the commutativity of the diagram that $\operatorname{Ker}\partial = \operatorname{Im}|\iota|_* = \operatorname{Im}((p_n)_* \circ (i_n)_*) = \operatorname{Im}(p_n)_*$. This completes the proof. □

Remark 4.14. In Proposition 4.13, we do not assume that the each component $\theta(i)$ of the persistence CDGA is relatively finite; that is, $\dim \theta(i)^n < \infty$ for $n \geq 0$. Then, it does not seem that Theorem 4.9 (iii) yields Proposition 4.13.

We provide examples each of which is a decomposition of a map f whose pointwise rationalization is a Postnikov tower for the map $f_{\mathbb{Q}}$. These serve as specific geometric realizations of extended tame CDGAs described in Examples 5.7 and 8.4.

Example 4.15. Let $i : S^2 \hookrightarrow \mathbb{C}P^2$ be the inclusion. Recall that $\mathbb{C}P^2$ is obtained by attaching a 4-dimensional disk D^4 to S^2 via the Hopf map $h : S^3 \to S^2$. We show that a Postnikov tower for the rationalization $i_{\mathbb{Q}}$ is given by

$$\mathbb{C}P^2_{\mathbb{Q}} \xleftarrow{=} \mathbb{C}P^2_{\mathbb{Q}} \xleftarrow{=} \mathbb{C}P^2_{\mathbb{Q}} \xleftarrow{(\mathrm{pr}_1)_{\mathbb{Q}}} (X_3)_{\mathbb{Q}} \xleftarrow{\bar{f}_{\mathbb{Q}}} S^2_{\mathbb{Q}} \xleftarrow{=} S^2_{\mathbb{Q}} \xleftarrow{=} \cdots, \tag{4.3}$$

where X_3 is the pullback $\mathbb{C}P^2 \times_{K(\mathbb{Z},4)} PK(\mathbb{Z},4)$ of the path fibration $p : PK(\mathbb{Z},4) \to K(\mathbb{Z},4)$, which is defined by $p(\gamma) = \gamma(1)$, along a map $f : \mathbb{C}P^2 \to K(\mathbb{Z},4)$. The map $f : \mathbb{C}P^2 \to K(\mathbb{Z},4)$ is defined as the composite $g \circ q$, where $q : \mathbb{C}P^2 \to S^4$ is the quotient map that collapses S^2 to the base point. The map $g : S^4 \to K(\mathbb{Z},4)$ represents the fundamental class of S^4 under the identification $[S^4, K(\mathbb{Z},4)] \cong H^4(S^4;\mathbb{Z})$. Since $f \circ i$ is the trivial map by the definition of f, the universal property of the pullback induces a unique map $\bar{f} : S^2 \to \mathbb{C}P^2 \times_{K(\mathbb{Z},4)} PK(\mathbb{Z},4)$ that makes the following diagram commutative:

$$\begin{tikzcd}
S^2 \arrow[dr, "\bar{f}"] \arrow[drr, bend left, "\text{trivial map}"] \arrow[ddr, bend right, "i"'] & & \\
& \mathbb{C}P^2 \times_{K(\mathbb{Z},4)} PK(\mathbb{Z},4) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & PK(\mathbb{Z},4) \arrow[d, "p"] \\
& \mathbb{C}P^2 \arrow[r, "f = g \circ q"] & K(\mathbb{Z},4).
\end{tikzcd} \tag{4.4}$$

The associated maps $\{i_n\}$ and $\{p_n\}$ in the Postnikov tower are defined by

$$i_n = \begin{cases} i_{\mathbb{Q}} & (n = 0, 1, 2) \\ \bar{f}_{\mathbb{Q}} & (n = 3) \\ id & (n \geq 4) \end{cases} \quad \text{and} \quad p_n = \begin{cases} id & (n = 0, 1, 2) \\ (\mathrm{pr}_1)_{\mathbb{Q}} & (n = 3) \\ i_{\mathbb{Q}} & (n \geq 4). \end{cases}$$

We shall verify that this tower satisfies conditions (1)–(4) of the definition of a Postnikov tower. Condition (1) is clear from the definitions of i_n and p_n. Condition (2) is straightforward for $n \neq 3$. To verify condition (2) for $n = 3$ and $k \leq 3$ in the non-rational case, we consider the fiber sequence

$$K(\mathbb{Z}, 3) \cong \Omega K(\mathbb{Z}, 4) \longrightarrow X_3 \xrightarrow{\mathrm{pr}_1} \mathbb{C}P^2. \tag{4.5}$$

Since $(\mathrm{pr}_1)_* : \pi_k(X_3) \to \pi_k(\mathbb{C}P^2)$ and $i_* : \pi_k(S^2) \to \pi_k(\mathbb{C}P^2)$ are isomorphisms for $k < 3$, it follows that $\bar{f}_* : \pi_k(S^2) \to \pi_k(X_3)$ is an isomorphism.

We consider the case $k = 3$. Observe that $i \circ h : S^3 \to \mathbb{C}P^2$ is null-homotopic. We can choose a null-homotopy $H : S^3 \times I \to \mathbb{C}P^2$ defined by the composite $H = \Phi \circ \alpha$, where $\Phi : D^4 \to \mathbb{C}P^2$ is the characteristic map of the 4-cell and $\alpha : S^3 \times I \to D^4$ is given by $\alpha(u, t) = (1 - t)u + tb$ for a basepoint $b \in S^3 = \partial D^4$. By the construction of H, the composition $q \circ H$ induces a relative homeomorphism

$$q \circ H : (S^3, \{b\}) \times (I, \partial I) \xrightarrow{\cong} (S^4, *).$$

By the homotopy lifting property of the fibration pr_1, there exists a homotopy $G : S^3 \times I \to X_3$ such that the following diagram commute:

$$\begin{tikzcd} S^3 \times \{0\} \arrow[r, "\bar{f} \circ h"] \arrow[d, hook] & X_3 \arrow[d, "\mathrm{pr}_1"] \\ S^3 \times I \arrow[r, "H"] \arrow[ur, "G"] & \mathbb{C}P^2. \end{tikzcd}$$

Explicitly, the map G is given by $G(u, t) = (H(u, t), \gamma_{u,t})$, where $\gamma_{u,t} \in PK(\mathbb{Z}, 4)$ is the path defined by $\gamma_{u,t}(s) = g \circ q \circ H(u, ts)$ for $s \in [0, 1]$. Since $H(u, 1)$ is the base point, the path $\gamma_{u,1} = g \circ q \circ H(u, \cdot\,)$ is contained in $\Omega K(\mathbb{Z}, 4)$. This defines a map $\phi : S^3 \to \Omega K(\mathbb{Z}, 4)$ by $\phi(u) = \gamma_{u,1}$ which makes the diagram

$$\begin{tikzcd} S^3 \arrow[r, "\mathrm{ad}(q \circ H)"'] \arrow[rr, bend left, "\phi"] \arrow[d, "h"'] & \Omega S^4 \arrow[r, "\Omega g"'] & \Omega K(\mathbb{Z}, 4) \arrow[d, "\iota"] \\ S^2 \arrow[rr, "\bar{f}"] & & X_3. \end{tikzcd}$$

commutative up to homotopy with the homotopy induced by G. Here, $\mathrm{ad}(q \circ H)$ is the adjoint of $q \circ H$, and ι denotes the inclusion of the fiber. Since $\pi_3(\mathbb{C}P^2) = 0$, which follows from the long exact sequence associated with the S^1-bundle $S^1 \to S^5 \to \mathbb{C}P^2$, it is readily seen that $\pi_3(\iota)$ is an isomorphism. The composite $q \circ H$ gives rise to a homeomorphism $\Sigma S^3 \to S^4$. Then, we see that $\mathrm{ad}(q \circ H)$ is a generator of $\pi_3(\Omega S^4) \cong \mathbb{Z}$. Furthermore, since g represents the fundamental class, Ωg induces an isomorphism on $\pi_3(\Omega S^4)$. Thus, ϕ represents a generator of $\pi_3(\Omega K(\mathbb{Z}, 4))$, and the commutativity of the diagram implies that $\bar{f}_* : \pi_3(S^2) \to \pi_3(X_3)$ is an isomorphism. Therefore, condition (2) is satisfied for $n = 3$.

In the rational setting, it is well known that $\pi_k(S^2_{\mathbb{Q}}) = 0$ for $k > 3$ and $\pi_k(\mathbb{C}P^2_{\mathbb{Q}}) = 0$ for $k > 5$. The long exact sequence of the fibration (4.5) implies that $\pi_k((X_3)_{\mathbb{Q}}) \cong \pi_k(\mathbb{C}P^2_{\mathbb{Q}})$ for $k > 4$. Furthermore, the rational homotopy group of the homotopy fiber F of $i : S^2 \to \mathbb{C}P^2$ are given by

$$\pi_k(F) \otimes \mathbb{Q} \cong \pi_k(F_{\mathbb{Q}}) \cong \begin{cases} \mathbb{Q} & (k = 3, 4) \\ 0 & (\text{otherwise}). \end{cases}$$

Combining these observations, conditions (3) and (4) are readily seen to be satisfied.

We next describe the Sullivan model that corresponds to the Postnikov tower of $i_{\mathbb{Q}}$. Let $\mathfrak{M}(S^2) = (\wedge(x, y), d)$ and $\mathfrak{M}(\mathbb{C}P^2) = (\wedge(x, z), d)$ be minimal Sullivan models of S^2 and $\mathbb{C}P^2$, respectively. Their differentials satisfy $d(x) = 0$, $d(y) = x^2$, and $d(z) = x^3$, where $|x| = 2$, $|y| = 3$, and $|z| = 5$. A Sullivan representative for i is given by $i^* : \wedge(x, z) \to \wedge(x, y)$, $i^*(x) = x$ and $i^*(z) = xy$. Then, we have a minimal relative Sullivan model $\iota : \wedge(x, z) \to \wedge(x, z, v, w)$ of i^* via the quasi-isomorphism defined by

$$\eta : \wedge(x, z, v, w) \xrightarrow{\sim} \wedge(x, y),$$

where $\eta(v) = y$, $\eta(w) = 0$, $\eta(z) = xy$, $dv = x^2$ and $dw = xy - z$; see Remark 3.10.

The relative Sullivan model for $p : PK(\mathbb{Z}, 4) \to K(\mathbb{Z}, 4)$ is of the form

$$(\wedge(w), 0) \hookrightarrow (\wedge(w, v), d),$$

where $d(v) = w$ and $d(w) = 0$; see [14, §15(b)Example 2] for the model of $K(\mathbb{Z}, 4)$. By the choice of f, the morphism $f^* : \wedge(w) \to \wedge(x, z)$ given by $f^*(w) = x^2$ provides a Sullivan representative for f. It follows from [14, §15 Proposition 15.8] that

$$(\wedge(x, z), d) \otimes_{(\wedge(w),0)} (\wedge(w, v), d) \cong (\wedge(x, z, v), d)$$

is a Sullivan model for $X_3 = \mathbb{C}P^2 \times_{K(\mathbb{Z},4)} PK(\mathbb{Z}, 4)$ and the inclusion $\iota_2 : (\wedge(x, z), d) \to (\wedge(x, z, v), d)$ is a Sullivan representative for pr_1. Moreover, we see that a morphism $\bar{f}^* : (\wedge(x, z, v), d) \to (\wedge(x, y), d)$ defined by $\bar{f}^*(x) = x$, $\bar{f}^*(z) = 0$, $\bar{f}^*(v) = y$ is a Sullivan representative for $\bar{f}$. Indeed, the commutativity of the lower-left triangle in the diagram (4.4) implies that any Sullivan representative $\varphi : (\wedge(x, z, v), d) \to (\wedge(x, y), d)$ for $\bar{f}$ must satisfy $\varphi(x) = x$. Since φ commutes with the differentials, it follows that φ is uniquely determined as $\bar{f}^*$. We also see that the inclusion $\iota_3 : (\wedge(x, z, v), d) \to (\wedge(x, z, v, w), d)$ provides a relative Sullivan model of $\bar{f}^*$ via the quasi-isomorphism η. Therefore, the sequence

$$\begin{array}{ccccccccc} & & 2 & & 3 & & 4 & & \\ \cdots & \xrightarrow{=} & (\wedge(x, z), d) & \xrightarrow{\iota_2} & (\wedge(x, z, v), d) & \xrightarrow{\iota_3} & (\wedge(x, z, v, w), d) & \xrightarrow{=} & \cdots \end{array}$$

is a cofibrant replacement of the image of the sequence (4.3) by $(\lfloor\ \rfloor)^* \circ A_{\mathrm{PL}}(\ )_*$, and this coincides with $\theta(\iota)$.

Example 4.16. Let $p : PS^2 \to S^2$ be the path fibration of S^2. A Postnikov tower for the rationalization $p_{\mathbb{Q}}$ is given by

$$S^2_{\mathbb{Q}} \xleftarrow{h_{\mathbb{Q}}} S^3_{\mathbb{Q}} \xleftarrow{\tilde{\mathrm{ev}}(\,\cdot\,,1)_{\mathbb{Q}}} PS^2_{\mathbb{Q}} \xleftarrow{=} PS^2_{\mathbb{Q}} \xleftarrow{=} \cdots \tag{4.6}$$

Here, h is the Hopf map. In fact, since h is a fiber bundle, it is a Hurewicz fibration from [25, Chapter 7, §4]. We define the map $\tilde{\mathrm{ev}} : PS^2 \times I \to S^3$ as a lift in the following diagram

$$\begin{tikzcd} PS^2 \times \{0\} \arrow[r, "*"] \arrow[d, "\text{incl.}"'] & S^3 \arrow[d, "h"] \\ PS^2 \times I \arrow[r, "\mathrm{ev}"] \arrow[ur, "\tilde{\mathrm{ev}}"] & S^2, \end{tikzcd}$$

where ev in the bottom row is the evaluation map. The associated maps $\{i_n\}$ and $\{p_n\}$ in the Postnikov tower are defined by

$$i_n = \begin{cases} p_{\mathbb{Q}} & (n=0) \\ \tilde{\mathrm{ev}}(\,\cdot\,,1)_{\mathbb{Q}} & (n=1) \\ id & (n \geq 2) \end{cases} \quad \text{and} \quad p_n = \begin{cases} id & (n=0) \\ h_{\mathbb{Q}} & (n=1) \\ p_{\mathbb{Q}} & (n \geq 2). \end{cases}$$

It is straightforward to verify that this construction satisfies conditions (1)–(4) in Definition 4.8. Conditions (1) and (2) follow directly from the definition of the maps. In particular, condition (2) for $n = 1$ is satisfied since both $S^3_{\mathbb{Q}}$ and $PS^2_{\mathbb{Q}}$ are simply connected. We also see that, in the rational setting, conditions (3) and (4) are immediately satisfied since $PS^2_{\mathbb{Q}}$ is contractible and the homotopy groups $\pi_k(S^2_{\mathbb{Q}})$ and $\pi_k(S^3_{\mathbb{Q}})$ vanish for $k > 3$.

It follows that a Sullivan representative for the Hopf map $h : S^3 \to S^2$ is given by $h^* : (\wedge(x, y), d) \to (\wedge(e), 0)$, $h^*(x) = 0$, $h^*(y) = e$, where $|e| = 3$. For the relative Sullivan algebra $\iota_1 : (\wedge(x, y), d) \to (\wedge(x, y, z), d)$, a quasi-isomorphism

$$\eta : (\wedge(x,y,z), d) \xrightarrow{\simeq} (\wedge(e), 0)$$

defined by $\eta(x) = 0$, $\eta(y) = e$, $\eta(z) = 0$ provides a relative Sullivan model for h^*. Finally, it suffices to consider a model for $\tilde{\mathrm{ev}}(\,\cdot\,, 1)$. It is readily seen that the inclusion $\iota_2 : (\wedge(x, y, z), d) \to (\wedge(x, y, z, w), d)$ is a relative Sullivan model for $\tilde{\mathrm{ev}}(\,\cdot\,, 1)$. Indeed, since $(\wedge(x, y, z, w), d)$ is acyclic, any CDGA morphism to $(\wedge(x, y, z, w), d)$ is null-homotopic. Therefore, the sequence

$$\begin{array}{ccccccc} 0 & & 1 & & 2 & & \\ (\wedge(x,y),d) & \xrightarrow{\iota_1} & (\wedge(x,y,z),d) & \xrightarrow{\iota_2} & (\wedge(x,y,z,w),d) & \xrightarrow{=} & \cdots \end{array}$$

is a Sullivan model for the Postnikov tower (4.6), and this coincides with $\theta(\iota)$.

Remark 4.17. Let P be the sequence whose pointwise rationalization is the tower (4.3). Then, it follows from the latter half of Example 4.15 that $A_{\mathrm{PL}}(\ )_*(P) \cong \Theta(i : S^2 \hookrightarrow \mathbb{C}P^2)$ in $\mathrm{Full}(\mathsf{et}(\mathcal{D}_0^{\mathbb{R}_+}))$. Then, we see that the sequence $\mathbf{L}\langle\ \rangle_*(\Theta(i))$ is isomorphic to the tower (4.3) and then (4.3) is a Postnikov tower for $i_{\mathbb{Q}}$. In Example 4.15, we prove the fact with explicit constructions of structure maps of the tower. In particular, we pay attention to show that $\bar{f}_* : \pi_k(S^2) \to \pi_k(X_3)$ is an isomorphism for $k \leq 3$ in the non-rational case.

In the same way, it is proved that the sequence (4.6) is a Postnikov tower for $p_{\mathbb{Q}}$ in Example 4.16.

5. The interleaving distance between maps in the homotopy category

We introduce a bivariate homotopy invariant for maps with the functor Θ in Theorem 3.3 and the interleaving distance in the homotopy category $\mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+}))$.

5.1. **A pseudodistance on the homotopy set of maps.** For maps $f : X \to A$ and $g : Y \to B$ between path-connected spaces, we consider the interleaving distance $d_{\mathrm{IHC}}(\Theta(f), \Theta(g))$ in the homotopy category $\mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+}))$. We remark that $\Theta(f)$ is a cofibrant object in the category of extended tame persistence CDGAs; see Proposition 2.6. Thanks to this fact, we can compute the interleaving distance between $\Theta(f)$ and $\Theta(g)$ with homotopy.

While the proofs of the following results are standard, the assertions are fascinating.

Proposition 5.1. *Let $f : X \to A$ and $g : Y \to B$ be maps between simply-connected spaces.*

(i) $d_{\mathrm{IHC}}(\Theta(f), \Theta(g)) = 0$ *if and only if the rationalizations of maps f and g are isomorphic in* $\mathrm{Func}(I, \mathrm{Ho}(\mathsf{Top}))$*; that is, one has a homotopy commutative diagram*

$$\begin{tikzcd} X_{\mathbb{Q}} \arrow[r, "f_{\mathbb{Q}}"] \arrow[d, "h"', "\simeq"] & A_{\mathbb{Q}} \arrow[d, "\simeq"', "k"] \\ Y_{\mathbb{Q}} \arrow[r, "g_{\mathbb{Q}}"'] & B_{\mathbb{Q}} \end{tikzcd}$$

in Top *in which vertical arrows are homotopy equivalences.*

(ii) *Suppose that $d_{\mathrm{IHC}}(\Theta(f), \Theta(g)) < \infty$. Then, the rational homotopy type of X coincides with that of Y.*

Proof. (i) We use the same notation as in Remark 3.10. Let φ_f and φ_g be Sullivan representatives for f and g, respectively. We choose minimal relative Sullivan algebras i_f and i_g in Remark 3.10.

The "only if" part is proved. For any $\delta > 0$, the persistence CDGAs $(\lfloor\ \rfloor)^*\theta(i_f)$ and $(\lfloor\ \rfloor)^*\theta(i_g)$ are δ-interleaved in the homotopy category and then $\theta(i_f)$ and $\theta(i_g)$ are 0-interleaved in the homotopy category. Therefore, we have a morphism $\varphi : \theta(i_f) \to \theta(i_g)$ of persistence CDGAs, which is a weak equivalence. The morphism gives rise to a commutative diagram

$$\begin{tikzcd} \theta(i_f)(0) \arrow[r] \arrow[d, "\varphi(0)"'] \arrow[rr, bend left, "\varphi_f"] & \mathrm{colim}_n \theta(i_f)(n) \arrow[r, "\eta_f", "\simeq"'] \arrow[d, "\Phi := \mathrm{colim}\,\varphi(n)"] & \wedge V_X = \mathfrak{M}(X) \arrow[d, dotted, "\psi"] \\ \theta(i_g)(0) \arrow[r] \arrow[rr, bend right, "\varphi_g"'] & \mathrm{colim}_n \theta(i_g)(n) \arrow[r, "\simeq", "\eta_g"'] & \wedge V_Y = \mathfrak{M}(Y) \end{tikzcd} \tag{5.1}$$

of solid arrows in CDGA in which $\varphi(0)$ and Φ are isomorphisms on the rational cohomology. By using homotopy inverses ξ_f and ξ_g of η_f and η_g, respectively, we have a dots arrow ψ with $\Phi \circ \xi_f \simeq \xi_g \circ \psi$. Then, we see that the right square is homotopy commutative. By realizing the maps, we have the result.

The "if" part follows from the fact that $\theta(i_f)$ is isomorphic to $\theta(i_g)$. In fact, in the diagram (3.6), the small square with γ' is homotopy commutative. Then, we see that the big square with η is homotopy commutative. It follows from [14, Proposition 12.9] that the big diagram itself is homotopy commutative. Observe that η is a quasi-isomorphism. The result [16, Proposition 2.22] makes the big diagram commutative replacing ψ with an extension $\widehat{i_f\varphi}$ of $i_f \circ \varphi$ which satisfies the condition that $\widehat{i_f\varphi} \simeq \psi$. The map ψ is a quasi-isomorphism and hence so is $\widehat{i_f\varphi}$. Moreover, the map φ is an isomorphism and two relative Sullivan algebras

$\wedge V \otimes \wedge W$ and $\wedge V' \otimes \wedge W'$ are minimal. Therefore, it follows from [14, Theorem 14.11] that $\widehat{i_f \varphi}$ is an isomorphism. We have $\Theta'(f) \cong \Theta'(g)$ as a persistence CDGA.

We observe that the equivalence between left and right homotopies in the proof above is used.

(ii) Let $\iota_f : \wedge V \to \wedge V \otimes \wedge W$ and $\iota_g : \wedge V \to \wedge V \otimes \wedge Z$ be relative Sullivan models for f and g, respectively. Let (φ, ψ) be an ε-interleaving between $\theta(\iota_f)$ and $\theta(\iota_g)$. Then, the morphisms $\varphi(i) : \wedge V \otimes W^{\leq i} \to \wedge V \otimes Z^{\leq i+\varepsilon}$ of CDGAs induce a morphism $\Phi : \wedge V \otimes W \to \wedge V \otimes \wedge Z$ of CDGAs. Observe that $\Phi \circ i_k = j_{k+\varepsilon} \circ \varphi(k) : \wedge V \otimes \wedge W^{\leq k} \to \wedge V \otimes \wedge Z$, where i_k and $j_{k+\varepsilon}$ are the canonical inclusions.

We show the surjectivity of Φ_*. For an element x in $H^n(\wedge V \otimes \wedge Z)$, there exists an element u in $H^n(\wedge V \otimes \wedge Z^{\leq n+1})$ such that $(j_{n+1})_*(u) = x$. Let α be the element $(i_{n+1+\varepsilon})_* \circ \psi(n+1)_*(u)$ in $H^n(\wedge V \otimes \wedge W)$. Then, we see that

$$\begin{aligned}\Phi_*(\alpha) &= (\Phi_* \circ (i_{n+1+\varepsilon})_* \circ \psi(n+1)_*)(u)\\ &= ((j_{n+1+2\varepsilon})_* \circ \varphi(n+1+\varepsilon)_* \circ \psi(n+1)_*)(u)\\ &= ((j_{n+1+2\varepsilon})_* \circ \iota_g(n+1 < n+1+2\varepsilon)_*)(u) = (j_{n+1})_*(u) = x.\end{aligned}$$

The injectivity of Φ_* is proved as follows. Suppose that $\Phi_*(y) = 0$ for an element $y \in H^n(\wedge V \otimes \wedge W)$. Then, there exists an element v in $H^n(\wedge V \wedge W^{\leq n+1})$ such that $(i_{n+1})_*(v) = y$. The definition of Φ enables us to deduce that $\Phi_* \circ (i_{n+1})_* = (j_{n+1+\varepsilon})_* \circ \varphi(n+1)_*$. The map $(j_{n+1+\varepsilon})_*$ is an isomorphism on the cohomology $H^n(\wedge V \otimes \wedge Z^{\leq n+1+\varepsilon})$. Thus, we have $\varphi(n+1)_*(v) = 0$. This yields that

$$\begin{aligned}0 &= ((i_{n+1+2\varepsilon})_* \circ \psi(n+1+\varepsilon)_* \circ \varphi(n+1)_*)(v)\\ &= ((i_{n+1+2\varepsilon})_* \circ \iota_f(n+1 < n+1+2\varepsilon)_*)(v) = (i_{n+1})_*(v) = y.\end{aligned}$$

As a consequence, we see that the morphism Φ is a quasi-isomorphism. The same argument as in (i) with the right-hand square in (5.1) allows us to obtain the result. $\square$

When considering the distance in $\mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+}))$ via the functor Θ, our interest goes to maps whose domains coincide.

Theorem 5.2. *The function $d : [X,Y] \times [X,Y] \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined by $d([f],[g]) = d_{\mathrm{IHC}}(\Theta(f), \Theta(g))$ is a well-defined pseudodistance on $[X,Y]$. Assume further that X and Y are simply connected. Then, $d([f],[g]) = 0$ if and only if there exist self homotopy equivalences α and β on $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$, respectively, such that $\beta \circ f_{\mathbb{Q}} \simeq g_{\mathbb{Q}} \circ \alpha$.*

The following lemma is used in proving the theorem above.

Lemma 5.3. *Suppose that $f : X \to Y$ is homotopic to $g : X \to Y$. Then, persistence object $\theta(\iota_f)$ is isomorphic to $\theta(\iota_g)$, where ι_f and ι_g are minimal relative Sullivan models for f and g, respectively.*

Proof. We consider a diagram of solid arrows

$$\begin{tikzcd}
\wedge V_Y \arrow[r, "\iota_f"] \arrow[rr, bend left, "\iota_g"] \arrow[d, "u"', "\sim"] & \wedge V_Y \otimes \wedge W \arrow[r, dotted, "\varphi"] \arrow[d, "\sim"', "w"] & \wedge V_Y \otimes \wedge W' \arrow[dl, bend left, "\sim", "w'"'] \\
A_{\mathrm{PL}}(Y) \arrow[r, "{A_{\mathrm{PL}}(f),\ A_{\mathrm{PL}}(g)}"'] & A_{\mathrm{PL}}(X) &
\end{tikzcd}$$

in which $A_{\mathrm{PL}}(f) \circ u \simeq A_{\mathrm{PL}}(g) \circ u$; see [14, Proposition 12.6]. The Lifting lemma [14, Lemma 12.4] allows us to obtain the map φ with $w' \circ \varphi \simeq w$. Observe that φ is a quasi-isomorphism. We see that $w' \circ (\varphi \circ \iota_f) \simeq w' \circ \iota_g$ and then $\varphi \circ \iota_f \simeq \iota_g$. The result [16, Proposition 2.22] implies that $\widehat{\iota_g} \circ \iota_f = \iota_g$ for some $\widehat{\iota_g}$ with $\widehat{\iota_g} \simeq \varphi$. Since $\widehat{\iota_g}$ is a quasi-isomorphism, it follows from [14, Theorem 14.11] and the minimality of ι_f and ι_g that $\widehat{\iota_g}$ is an isomorphism and hence $\theta(\iota_f)$ is isomorphic to $\theta(\iota_g)$. □

Proof of Theorem 5.2. The well-definedness follows from Lemma 5.3. Proposition 5.1 (i) implies the latter half of the assertion. □

5.2. **Interleaving distance to $\Theta(id)$.** Let $f\colon X \to Y$ be a map between simply connected spaces. In this subsection, we investigate the distance $d_{\mathrm{IHC}}(\Theta(f\colon X \to Y), \Theta(id\colon X \to X))$ provided f is relatively finite in the sense in Section 3.1. As a consequence, the distance is determined under some assumption.

Let $\wedge V_X$ and $\wedge V_Y$ be minimal Sullivan models for X and Y, respectively. Denote by $f^*\colon \wedge V_Y \to \wedge V_X$ the model for the map $f\colon X \to Y$. Take the minimal relative Sullivan model $\eta\colon \wedge V_Y \otimes \wedge W \xrightarrow{\simeq} \wedge V_X$ for f^*. We observe that $\Theta(f)(i) = \wedge V_Y \otimes \wedge W^{\leq i}$ and $\Theta(id)(i) = \wedge V_X$ for each $i \in \mathbb{Z}_+$.

Let N be the maximum integer such that $W^N \neq 0$.

Proposition 5.4. *The extended tame persistence CDGAs $\Theta(f)$ and $\Theta(id)$ are N-interleaved in the homotopy category. In other words, $d_{\mathrm{IHC}}(\Theta(f), \Theta(id)) \leq N$.*

Proof. Note that the quasi-isomorphismη is surjective since the linear part of η is surjective by [14, Proposition 14.13] and the minimality of $\wedge V_X$. Hence [14, Lemma 12.4] implies that there is a morphism $\rho\colon \wedge V_X \to \wedge V_Y \otimes \wedge W$ of CDGAs which makes the diagram

$$\begin{tikzcd} & \wedge V_Y \otimes \wedge W \arrow[d, "\eta"', "\simeq"] \\ \wedge V_X \arrow[ur, dotted, "\rho"] \arrow[r, "id"'] & \wedge V_X \end{tikzcd}$$

commutes strictly, i.e. $\eta\rho = id$. Here η is a homotopy equivalence of CDGAs and ρ is the homotopy inverse of η. Thus $\rho\eta \simeq id$.

Now we define $\varphi\colon \Theta(f) \Rightarrow \Theta(id) \circ T_N$ and $\psi\colon \Theta(id) \Rightarrow \Theta(f) \circ T_N$ by the restrictions of η and ρ, respectively. Note that ψ is well defined since $\wedge V_Y \otimes \wedge W^{\leq N} = \wedge V_Y \otimes \wedge W$. It is easy to show that the pair (φ, ψ) is an N-interleaving in the homotopy category; see Remark 3.11. □

Moreover, we can determine $d_{\mathrm{IHC}}(\Theta(f), \Theta(id))$ explicitly under an assumption on the model for f^*.

Proposition 5.5. *Assume that any morphism $\alpha\colon \wedge V_X \to \wedge V_Y \otimes \wedge W^{<N}$ has no homotopy retract. Then, it holds that $d_{\mathrm{IHC}}(\Theta(f), \Theta(id)) = N$.*

Proof. By Proposition 5.4, it is enough to show that $\Theta(f)$ and $\Theta(id)$ are *not* ε-interleaved in the homotopy category for any $\varepsilon < N$. Assume that there is such an ε-interleaving (φ, ψ), where $\varphi\colon \Theta(f) \Rightarrow \Theta(id) \circ T_N$ and $\psi\colon \Theta(id) \Rightarrow \Theta(f) \circ T_N$. We may assume $N - 1 \leq \varepsilon < N$. Then, we have $\psi(0)\colon \wedge V_X \to \wedge V_Y \otimes \wedge W^{\leq \varepsilon} = \wedge V_Y \otimes \wedge W^{<N}$ and $\varphi(\varepsilon)\psi \simeq id$, which is a contradiction. □

Example 5.6. By using Proposition 5.5, we determine the interleaving distance between $\Theta(* : S^m \to S^m)$ and $\Theta(id : S^m \to S^m)$ in the homotopy category.

(i) We prove that the distance $d_{\mathrm{IHC}}(\Theta(* : S^{2n} \to S^{2n}), \Theta(id : S^{2n} \to S^{2n})) = 4n-1$. Let $\mathfrak{M}(S^{2n}) = (\wedge(x,y), d)$ be the minimal Sullivan model for S^{2n}, where $d(x) = 0$, $d(y) = x^2$, $|x| = 2$. The relative Sullivan model for $*$ is a relative Sullivan algebra of the form

$$\iota_* : \wedge(x,y) \to \wedge(x,y) \otimes \wedge(z,w,x',y') \cong \wedge(x,y,z,w) \otimes \wedge(x',y').$$

Here, $d(z) = x$, $d(w) = xz - y$ and $\wedge(x',y')$ is a copy of $\wedge(x,y)$; see also Subsection 8.4 for the Sullivan models.

Let $W = \mathrm{span}_{\mathbb{Q}}\{z,w,x',y'\}$ and $N = 4n-1$. Then, any morphism $\alpha : \wedge(x,y) \to \wedge(x,y) \otimes \wedge W^{<N} \cong \wedge(x,y) \otimes \wedge(z,w,x')$ has no homotopy retract. Indeed, if there exists a homotopy retract β of α, then the induced composite on homology

$$H(\wedge(x,y)) \xrightarrow{H(\alpha)} H(\wedge(x,y) \otimes \wedge(z,w,x')) \xrightarrow{H(\beta)} H(\wedge(x,y))$$

is equal to the identity map of $H(\wedge(x,y))$. However, $H(\alpha)$ is trivial since $H(\wedge(x,y)) \cong \mathbb{Q}[x]/(x^2)$ and $H(\wedge(x,y) \otimes \wedge(z,w,x')) \cong \mathbb{Q}[x']$ as algebras, which contradicts $H(\beta) \circ H(\alpha) = id$. Therefore, $d_{\mathrm{IHC}}(\Theta(*), \Theta(id)) = 4n-1$ from Proposition 5.5.

(ii) We show that $d_{\mathrm{IHC}}(\Theta(* : S^{2n+1} \to S^{2n+1}), \Theta(id : S^{2n+1} \to S^{2n+1})) = 2n+1$. The persistence CDGAs $\Theta(*)$ and $\Theta(id)$ have the forms

$$\begin{array}{lccccc} & 0 & 2n-1 & 2n & 2n+1 & \\ \Theta(*) : & \wedge(x) = \cdots = & \wedge(x) \rightarrowtail & \wedge(x,y) \rightarrowtail & \wedge(x,y,z) = \cdots, \\ \Theta(id) : & \wedge(x) = \cdots = & \wedge(x) \longrightarrow & \wedge(x) \longrightarrow & \wedge(x) = \cdots, \end{array}$$

where $d(y) = x$, $d(z) = 0$, $\deg(x) = \deg(z) = 2n+1$ and $\deg(y) = 2n$. Since $H(\wedge(x,y), d) = \mathbb{Q}$, it follows that every morphism $\alpha\colon \wedge(x) \to \wedge(x,y)$ has no homotopy retract. By virtue of Proposition 5.5, we have the result.

Example 5.7. Let $i : S^2 \hookrightarrow \mathbb{C}P^2$ be the inclusion. In this example, we determine the distance $d_{\mathrm{IHC}}(\Theta(i : S^2 \to \mathbb{C}P^2), \Theta(id : S^2 \to S^2))$. Recall from Example 4.15 the Sullivan representative i^* for i and its minimal relative Sullivan model $\wedge(x,z) \hookrightarrow \wedge(x,z,v,w)$ defined via the quasi-isomorphism $\eta : \wedge(x,z,v,w) \xrightarrow{\sim} \wedge(x,y)$.

We see that $\Theta(i)$ and $\Theta(id)$ are 3-interleaved in the homotopy category. In fact, the interleaving (φ, ψ) between the two persistence CDGAs is given as follows.

- $\varphi : \Theta(i) \Rightarrow \Theta(id) \circ T_3$ is a natural transformation defined by a restriction of η.
- $\psi : \Theta(id) \Rightarrow \Theta(i) \circ T_3$ is a natural transformation induced by

$$\rho : \wedge(x,y) \to \wedge(x,z,v,w),\ \rho(x) = x,\ \rho(y) = v.$$

It is readily seen that $\varphi^3\psi = \Theta(id)(* < *+6)$ by definition. We define a homotopy $\Phi : \wedge(x,z,v,w) \to \wedge(x,z,v,w) \otimes \wedge(t,dt)$ by

$$\Phi(x) = x, \quad \Phi(z) = z \otimes t + xv \otimes (1-t) - w \otimes dt, \quad \Phi(v) = v, \quad \Phi(w) = w \otimes t.$$

Then, the restriction of Φ gives rise to a homotopy $\widetilde{\Phi} : \Theta(i) \to \Theta(i)^6 \otimes \wedge(t,dt)$ which yields $\Theta(i)(* < *+6) \simeq \psi^3\varphi$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$; see Remark 3.11. It follows that (φ, ψ) is 3-interleaving between $\Theta(i)$ and $\Theta(id)$.

Next, we show that $\Theta(i)$ and $\Theta(id)$ are not ε-interleaved in the homotopy category for any $2 \le \varepsilon < 3$. Assume that $\Theta(i)$ and $\Theta(id)$ are ε-interleaved in the homotopy category. Let $\varphi' : \Theta(i) \Rightarrow \Theta(id) \circ T_\varepsilon$ and $\psi' : \Theta(id) \Rightarrow \Theta(i) \circ T_\varepsilon$ be the natural transformation which gives the ε-interleaving. Then, we have the following commutative diagram in $\mathrm{Ho}(\mathsf{CDGA})$:

$$\begin{tikzcd} \Theta(id)(0) \arrow[rr, "\Theta_1(0\leq 2\varepsilon)"] \arrow[dr, "\psi'(0)"'] & & \Theta(id)(2\varepsilon) \\ & \Theta(i)(\varepsilon). \arrow[ur, "\varphi'(\varepsilon)"'] & \end{tikzcd}$$

By applying the homology functor to the diagram, we have the commutative diagram

$$\begin{tikzcd} \mathbb{Q}[x]/(x^2) \arrow[rr, "id"] \arrow[dr, "H(\psi'(0))"'] & & \mathbb{Q}[x]/(x^2) \\ & \mathbb{Q}[x]/(x^3) \arrow[ur, "H(\varphi'(\varepsilon))"'] & \end{tikzcd}$$

in the category of commutative graded algebras. However, any algebra map from $\mathbb{Q}[x]/(x^2)$ to $\mathbb{Q}[x]/(x^3)$ are trivial, which contradicts the commutativity of the diagram. Therefore, we conclude that $d_{\mathrm{IHC}}(\Theta(i), \Theta(id)) = 3$.

It is worthwhile mentioning that Proposition 5.5 is not applicable when determining the interleaving distance $d_{\mathrm{IHC}}(\Theta(i), \Theta(id))$. Indeed, the CDGA map $\alpha : \wedge(x, y) \to \wedge(x, z) \otimes \wedge(v)$, $\alpha(x) = x$, $\alpha(y) = v$ has a homotopy retract $\beta : \wedge(x, z) \otimes \wedge(v) \to \wedge(x, y)$ defined by $\beta(x) = x$, $\beta(z) = 0$ and $\beta(v) = y$.

When $X = *$, the assumption in Proposition 5.5 cannot be satisfied. In this case, we have the following proposition, which implies that the distance is strictly smaller than N.

Proposition 5.8. *Let Y be a space with the base point $* \in Y$, $f\colon * \to Y$ the inclusion of the base point, and $\wedge V_Y$ the minimal Sullivan model for Y. Assume that $f\colon * \to Y$ is relatively finite in the sense in Section 3.1, i.e. V_Y is finite dimensional. Then, it holds that $d_{\mathrm{IHC}}(\Theta(f\colon * \to Y), \Theta(id\colon * \to *)) = N/2$.*

Proof. In this case, the minimal relative Sullivan model $\eta\colon \wedge V_Y \otimes \wedge W \xrightarrow{\simeq} \mathbb{Q}$ for $f\colon * \to Y$ satisfies $W^n = (V_Y)^{n+1}$ for $n \in \mathbb{N}$. Recall that $N = \max\{n \mid W^n \neq 0\}$.

First, we show that $\Theta(f)$ and $\Theta(id)$ are $N/2$-interleaved in the homotopy category. Note that $\Theta(f)(i) = \wedge V_Y \otimes \wedge W \simeq \mathbb{Q}$ and $\Theta(id)(i) = \mathbb{Q}$ for any $i \geq N$. Hence, the trivial maps $\varphi\colon \Theta(f) \Rightarrow \Theta(id) \circ T_{N/2}$ and $\psi\colon \Theta(id) \Rightarrow \Theta(f) \circ T_{N/2}$ give an $\frac{N}{2}$-interleaving (φ, ψ) in the homotopy category.

Next ,we show that $\Theta(f)$ and $\Theta(id)$ are not ε-interleaved in the homotopy category for any $\varepsilon < \frac{N}{2}$. Assume that there is such an ε-interleaving (φ, ψ), where $\varphi\colon \Theta(f) \Rightarrow \Theta(id) \circ T_{N/2}$ and $\psi\colon \Theta(id) \Rightarrow \Theta(f) \circ T_{N/2}$. Since $\Theta(id)(\varepsilon) = \mathbb{Q}$, the inclusion map $\Theta(f)(0 < 2\varepsilon)\colon \wedge V_Y \to \wedge V_Y \otimes \wedge W^{<2\varepsilon}$ is null-homotopic and so is $\wedge V_Y \to \wedge V_Y \otimes \wedge W^{<N}$. Hence, its linear part $(V_Y, 0) \to (V_Y \oplus W^{<N}, d)$ is also null-homotopic. But this is a contradiction since $\dim V_Y > \dim W^{<N}$. This completes the proof. $\square$

6. Examples where d_{IHC} is greater than d_{Coh}

The homology functor H gives rise to a functor $H : \mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})) \to \mathsf{et}(\mathsf{GA}^{\mathbb{R}_+})$ to the category of extended tame persistence graded algebras. We here recall the *cohomology interleaving distance* $d_{\mathrm{CohI}}(F, G)$ of extended tame persistence CDGAs F and G defined by $d_{\mathrm{CohI}}(F, G) := d_{\mathrm{I}}(H(F), H(G))$, where d_{I} denotes the interleaving distance defined in $\mathsf{et}(\mathsf{GA}^{\mathbb{R}_+})$.

The cohomology interleaving distance also detects persistence of graded algebras in the sense, for example, that $d_{\mathrm{CohI}}(\Theta(X \to *), \Theta(Y \to *)) > 0$ even if $H^*(X;\mathbb{Q}) \cong H^*(Y;\mathbb{Q})$ as $\mathbb{Q}$-graded algebras.

Example 6.1. Let X be the formal space $(S^3 \times S^8)\#(S^3 \times S^8)$ and Y a non-formal 11-dimensional manifold such that $\mathfrak{M}(Y) = (\wedge(x,y,z), d)$ with $|x| = |y| = 3, |z| = 5$, $dx = dy = 0$ and $dz = xy$. Then $H^*(X;\mathbb{Q}) = \wedge(x,y) \otimes \mathbb{Q}[u,w]/I \cong H^*(Y;\mathbb{Q})$ with $|u| = |w| = 8$ and $I = (xy, xu, yw, u^2, uw, w^2, xw - yu)$. Note that the minimal Sullivan model for X is given by

$$\mathfrak{M}(X) = (\wedge(x, y, z, v_7, v_7', u, w, v_9, v_9', v_9'', v_{10}, v_{10}', v_{10}'', \ldots), d) \tag{6.1}$$

with $dx = dy = du = dw = 0$, $dz = xy$, $dv_7 = xz$, $dv_7' = yz$, $dv_9 = xv_7$, $dv_9' = yv_7'$, $dv_9'' = xv_7' + yv_7$, $dv_{10} = xu$, $dv_{10}' = yw$ and $dv_{10}'' = xw - yu, \ldots$; see [16, Example 3.6]. Here, $|x| = |y| = 3$, $|z| = 5$ and $|v_i| = i$. Then, we see that $d_{\mathrm{CohI}}(\Theta(f), \Theta(g)) > 0$ for $f : X \to *$ and $g : Y \to *$. Indeed, if the distance is zero, then the map $\varphi(8) : H(\Theta(g))(8) \to H(\Theta(f))(8)$ induces a map from $\mathbb{Q}[u]/(u^2)$ to $\mathbb{Q}[u]$ or $\mathbb{Q}[w]$ by degree reasons in the model (6.1), which is a contradiction.

For the rest of this section, we give examples of maps $f, g : X \to A$ with

$$d_{\mathrm{IHC}}(\Theta(f), \Theta(g)) \gneqq d_{\mathrm{CohI}}(\Theta(f), \Theta(g)).$$

Proposition 6.2. *For the trivial map $f = * : S^3 \to S^2$ and the Hopf map $h : S^3 \to S^2$, one has*

(1) $d_{\mathrm{IHC}}(\Theta(f), \Theta(h)) = 3$
(2) $d_{\mathrm{CohI}}(\Theta(f), \Theta(h)) = 2$.

Proof. In the proof, we will only present non-trivial assignments for generators when defining a morphism of CDGA's.

(1) Let $\mathfrak{M}(S^2) = (\wedge(x,y), d) \to (\wedge(x, y, \bar{x}, \bar{y}, \tilde{y}), d)$ be the minimal relative Sullivan algebra for f and $\mathfrak{M}(S^2) = (\wedge(x,y), d) \to (\wedge(x, y, \bar{x}), d)$ the minimal relative Sullivan algebra for g, where the differentials are defined by $d(\bar{x}) = x$, $d(\bar{y}) = y - x\bar{x}$ and $d(\tilde{y}) = 0$. Then, the persistence CDGA $\Theta(f)$ has the form $\wedge(x,y)$ on $[0,1)$, $\wedge(x,y,\bar{x})$ on $[1,2)$, $\wedge(x,y,\bar{x},\bar{y})$ on $[2,3)$ and $\wedge(x,y,\bar{x},\bar{y},\tilde{y})$ on $[3,\infty)$; see Remark 3.8. The persistence CDGA $\Theta(h)$ is defined by $\wedge(x,y)$ on $[0,1)$ and $\wedge(x,y,\bar{x})$ on $[1,\infty)$.

For $\varepsilon \geq 3$, a part of an interleaving $\varphi(i) : \Theta(f)(i) \to \Theta(h)(i+\varepsilon)$ is defined by $\varphi(i)(\tilde{y}) = y - x\bar{x}$ and sending the other generators to zero. In addition, we define $\psi(i) : \Theta(h)(i) \to \Theta(f)(i+\varepsilon)$ by $\psi(i)(y) = \tilde{y}$ and sending the others to zero. It follows that $\Theta(f)$ and $\Theta(h)$ are ε-interleaved in the homotopy category. In fact, for $i \geq 0$, we have $\varphi^\varepsilon \circ \psi \simeq \Theta(h)(* < * + 2\varepsilon)$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$ by restricting the homotopy $H : \wedge(x,y,\bar{x}) \to \wedge(x,y,\bar{x}) \otimes \wedge(t, dt)$ given by $H(\bar{x}) = \bar{x}t$, $H(x) = xt - \bar{x}dt$ and $H(y) = y - x\bar{x}(1-t^2)$. Moreover, we have $\psi^\varepsilon \circ \varphi \simeq \Theta(f)(* < * + 2\varepsilon)$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$ by the restriction of the homotopy $H : \wedge(x,y,\bar{x},\bar{y},\tilde{y}) \to \wedge(x,y,\bar{x},\bar{y},\tilde{y}) \otimes \wedge(t,dt)$ given by $H(\bar{x}) = \bar{x}t$, $H(x) = xt - \bar{x}dt$, $H(\bar{y}) = \bar{y}t^2$, $H(y) = yt^2 + 2\bar{y}tdt$ and $H(\tilde{y}) = \tilde{y}$; see Remark 3.11.

Suppose that $2 \leq \varepsilon < 3$ and (φ, ψ) is an ε-interleaving in the homotopy category. Then, we see that $\varphi(\varepsilon) \circ \psi(0)$ is homotopic to the structure map $\Theta(h)(0 < 2\varepsilon)$, which is the inclusion $\wedge(x,y) \to \wedge(x,y,\bar{x})$. By applying the functor $H(Q(\ ))$ to the right triangle in (2.2), we have $H(Q(\Theta(h)(0 < 2\varepsilon)))(y) = y$. On the other hand, since $H(Q(\Theta(h)(0 < 2\varepsilon)))$ factors through $H(Q(\Theta(f)(\varepsilon))) = \mathbb{Q}$, it follows that $H(Q(\Theta(h)(0 < 2\varepsilon)))(y) = 0$, which is a contradiction. Hence, we obtain the result (1).

(2) Under the same notation as in the proof of (1), the persistence CDGA $H(\Theta(f))$ is given by $\mathbb{Q}[x]/(x^2)$ on $[0,1)$, $\wedge(y)$ on $[1,2)$, $\mathbb{Q}$ on $[2,3)$ and $\wedge(\tilde{y})$ on $[3,\infty)$. Moreover, the persistence CDGA $H(\Theta(h))$ has the form given by $\mathbb{Q}[x]/(x^2)$ on $[0,1)$ and $\wedge(y)$ on $[1,\infty)$. For $\varepsilon \geq 2$, we define $\varphi(i) : H(\Theta(f))(i) \to H(\Theta(h))(i+\varepsilon)$ by $\varphi(i)(\tilde{y}) = y$. Another part of the interleaving $\psi(i) : H(\Theta(h))(i) \to H(\Theta(f))(i+\varepsilon)$ is defined by $\psi(i)(y) = \tilde{y}$. We see that $H(\Theta(f))$ and $H(\Theta(h))$ are ε-interleaved. If $1 \leq \varepsilon < 2$, then $\varphi(1+\varepsilon) \circ \psi(1)$ is the constant map since $H(\Theta(f))(\varepsilon) = \mathbb{Q}$. On the other hand, $\Theta(1 < 1+2\varepsilon)$ is the identity map. This yields that $H(\Theta(f))$ and $H(\Theta(h))$ are not ε-interleaved. We have the result (2). $\square$

We consider another example which gives the inequality $d_{\mathrm{CohI}} < d_{\mathrm{IHC}}$. Let $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ denote the connected sum, whose minimal Sullivan model is of the form

$$\wedge V = (\wedge(x_1, x_2, y_1, y_2), d) \quad \text{with} \quad dx_i = 0, dy_1 = x_1^2 + x_2^2 \quad \text{and} \quad dy_2 = x_1 x_2,$$

where $|x_i| = 2$ and $|y_i| = 3$; see [16, Example 3.8].

We consider a map $f_k : S^3 \to \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ which corresponds to the dual of y_k via the identification $\pi_3\left(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}\right) \otimes \mathbb{Q} \cong \mathrm{Hom}(V^3, \mathbb{Q})$ for $k = 1, 2$; see [14, Theorem 15.11]. A Sullivan representative for f_k is given by $f_k^* : \wedge V \to (\wedge(e), 0)$, $f_k^*(x_i) = 0$, $f_k^*(y_i) = e$ $(k = i)$ and $f_k^*(y_i) = 0$ $(k \neq i)$ with $|e| = 3$. Here, $(\wedge(e), 0)$ is a Sullivan model of S^3. Then, we obtain the minimal relative Sullivan model η for f_k^* defined by

$$\eta : (\wedge V \otimes \wedge(\bar{x}_1, \bar{x}_2, \bar{y}_k), d) \longrightarrow (\wedge(e), 0), \;\; \eta(\bar{x}_i) = 0, \;\; \eta(\bar{y}_k) = 0,$$

where the differential of $\wedge V \otimes \wedge(\bar{x}_1, \bar{x}_2, \bar{y}_j)$ is defined by $d(\bar{x}_i) = x_i$, $d(\bar{y}_1) = y_2 - x_1\bar{x}_2$ and $d(\bar{y}_2) = y_1 - x_1\bar{x}_1 - x_2\bar{x}_2$; see Remark 3.10.

Proposition 6.3. *One has*

(1) $d_{\mathrm{CohI}}(\Theta(f_1), \Theta(f_2)) = 0$,
(2) $d_{\mathrm{IHC}}(\Theta(f_1), \Theta(f_2)) > 0$.

To prove the proposition, we consider automorphisms on $\wedge V$ mentioned above. The following lemma can be proved by a straightforward calculation.

Lemma 6.4. *Any isomorphism $\varphi \colon \wedge V \to \wedge V$ of Sullivan algebras can be written as*

$$\varphi(x_1) = \lambda x_1 + \bar{\lambda} x_2, \qquad \varphi(x_2) = \sigma\bar{\lambda} x_1 + \sigma\lambda x_2,$$
$$\varphi(y_1) = (\lambda^2 + \bar{\lambda}^2) y_1 + 4\lambda\bar{\lambda} y_2, \qquad \varphi(y_2) = \sigma\lambda\bar{\lambda} y_1 + \sigma(\lambda^2 + \bar{\lambda}^2) y_2,$$

where $\sigma \in \{\pm 1\}$ and $\lambda, \bar{\lambda} \in \mathbb{Q}$ with $\lambda^2 \neq \bar{\lambda}^2$.

By using the above lemma, we prove the following lemma, which will be used in the proof of Proposition 6.3.

Lemma 6.5. *For any isomorphism $\varphi : \wedge V \to \wedge V$, the coefficient of y_2 in $Q\varphi(y_2)$ is nonzero. Here, $Q\varphi : V \to V$ is the linear part of φ.*

Proof. By Lemma 6.4, the coefficient of y_2 in $Q\varphi(y_2)$ is $\sigma(\lambda^2 + \bar{\lambda}^2)$. Since $\lambda, \bar{\lambda} \in \mathbb{Q}$ and $\lambda^2 \neq \bar{\lambda}^2$, we have $\lambda^2 + \bar{\lambda}^2 > 0$. Hence $\sigma(\lambda^2 + \bar{\lambda}^2) \neq 0$. $\square$

We are ready to prove Proposition 6.3.

Proof of Proposition 6.3. The assertion (1) is trivial since $H\Theta(f_1) \cong H\Theta(f_2)$ as persistence CGAs. In order to prove the assertion (2), we show that $\Theta(f_1)$ and $\Theta(f_2)$ are not ε-interleaved in the homotopy category for $0 \leq \varepsilon < 1/2$. Assume that $\Theta(f_1)$ and $\Theta(f_2)$ are ε-interleaved in the homotopy category. Then, we have an ε-interleaving (φ, ψ) with $\varphi : \Theta(f_1) \Rightarrow \Theta(f_2) \circ T_\varepsilon$ and $\psi : \Theta(f_2) \Rightarrow \Theta(f_1) \circ T_\varepsilon$. Since φ is a natural transformation and the structure map $\Theta(f_1)(0 \to \varepsilon)$ is the identity map, it follows that $\varphi(0) = \varphi(\varepsilon)$. Furthermore, the homotopy commutative diagrams

$$\begin{tikzcd} \Theta(f_1)(0) \arrow[rr, "id"] \arrow[dr, "\varphi(0)"'] & & \Theta(f_1)(2\varepsilon) \\ & \Theta(f_2)(\varepsilon), \arrow[ur, "\psi(\varepsilon)"'] & \end{tikzcd} \qquad \begin{tikzcd} & \Theta(f_1)(\varepsilon) \arrow[dr, "\varphi(\varepsilon)"] & \\ \Theta(f_2)(0) \arrow[ur, "\psi(0)"] \arrow[rr, "id"] & & \Theta(f_2)(2\varepsilon) \end{tikzcd}$$

show that $\varphi(0)$ is a quasi-isomorphism on $\wedge V = \Theta(f_1)(0) = \Theta(f_2)(\varepsilon)$. It follows from [14, Proposition 12.10] that $\varphi(0)$ is an isomorphism. Now, the natural transformation φ induces the commutative diagram

$$\begin{tikzcd} \Theta(f_1)(0) \arrow[r, "\Theta(f_1)(0\leq 2)"] \arrow[d, "\varphi(0)"'] & \Theta(f_1)(2) \arrow[d, "\varphi(2)"] \\ \Theta(f_2)(\varepsilon) \arrow[r, "\Theta(f_2)(\varepsilon\leq\varepsilon+2)"] & \Theta(f_2)(\varepsilon+2). \end{tikzcd}$$

Applying the homology functor to the linear part of the diagram described above, we obtain the commutative diagram

$$\begin{tikzcd} V \arrow[r, "q_1"] \arrow[d, "Q\varphi(0)"'] & H\left(Q\left(\wedge V \otimes \wedge(\bar{x}_1, \bar{x}_2, \bar{y}_2)\right), d_1\right) \arrow[d, "H(Q\varphi(2))"] \\ V \arrow[r, "q_2"] & H\left(Q\left(\wedge V \otimes \wedge(\bar{x}_1, \bar{x}_2, \bar{y}_1)\right), d_1\right), \end{tikzcd}$$

where q_1 and q_2 are quotient maps. It is readily seen that $q_1(y_2) = 0$, that is, $H(Q\varphi(2)) \circ q_1(y_2) = 0$. On the other hand, $q_2 \circ Q\varphi(0)(y_2) \neq 0$ from Lemma 6.5, which contradicts the commutativity of the diagram above. □

Remark 6.6. The proof of Lemma 6.5 depends on the fact that the equation $\lambda^2 + \bar{\lambda}^2 = 0$ has no non-trivial solution in $\mathbb{Q}$.

If we take coefficients in $\mathbb{C}$, the isomorphism φ defined by

$$\varphi(x_1) = x_1 + \sqrt{-1}x_2, \qquad \varphi(x_2) = \sqrt{-1}x_1 + x_2,$$
$$\varphi(y_1) = 4\sqrt{-1}y_2, \qquad \varphi(y_2) = \sqrt{-1}y_1$$

is a counterexample for Lemma 6.5. Moreover, the extension

$$\tilde{\varphi}\colon (\wedge V \otimes \wedge(\bar{x}_1, \bar{x}_2, \bar{y}_1), d) \to (\wedge V \otimes \wedge(\bar{x}_1, \bar{x}_2, \bar{y}_2), d)$$

of φ given by

$$\tilde{\varphi}(\bar{x}_1) = \bar{x}_1 + \sqrt{-1}\bar{x}_2, \qquad \tilde{\varphi}(\bar{x}_2) = \sqrt{-1}\bar{x}_1 + \bar{x}_2,$$
$$\tilde{\varphi}(\bar{y}_1) = \sqrt{-1}\bar{y}_2 - \bar{x}_1\bar{x}_2$$

is an isomorphism of relative Sullivan algebras over $\wedge V$. Hence $d_{\mathrm{IHC}}(\Theta(f_1), \Theta(f_2)) = 0$ and Proposition 6.3 (2) does not hold with coefficients in $\mathbb{C}$.

7. Formalities of a persistence CDGA

The purposes of this section are to introduce formalities of an object in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$, and to compare them.

Definition 7.1. A persistence CDGA F in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$ is *H-formal* if there exists a zig-zag sequence of weak equivalences between F and $(H(F), 0)$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$.

In order to define other formalities of extended tame persistence CDGAs, we consider the commutative diagram

$$\begin{tikzcd} \mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+}) \arrow[r, "j"] \arrow[d, "\pi"'] & \mathsf{CDGA}^{\mathbb{R}_+} \arrow[d, "q_*"] \\ \mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})) \arrow[r, dotted, "j_*"'] & \mathrm{Ho}(\mathsf{CDGA})^{\mathbb{R}_+}, \end{tikzcd}$$

where π and q are the localization maps, j is the embedding and j_* are unique map induced by j. Observe that $q_* \circ j$ sends a weak equivalence to an isomorphism.

We recall interleaving distances defined in Section 2. By virtue of [7, Proposition 3.6], we have

Proposition 7.2. *For objects F and G in $\mathsf{et}(\mathcal{M}^{\mathbb{R}_+})$,*

$$d_{\mathrm{HC}}(j_*(F), j_*(G)) \leq d_{\mathrm{IHC}}(F, G) \leq d_{\mathrm{HI}}(F, G) \leq d_{\mathrm{I}}(F, G).$$

By using these distances, we define formalities of a persistence CDGA.

Definition 7.3. Let F be an extended tame persistence CDGA.
(1) F is *d_{HI}-formal* if $d_{\mathrm{HI}}(F, H(F)) = 0$; see [23, The revised version].
(2) F is *d_{IHC}-formal* if $d_{\mathrm{IHC}}(F, H(F)) = 0$.
(3) F is *d_{HC}-formal* if $d_{\mathrm{HC}}(j_*(F), j_*(H(F))) = 0$.
Here $H(F)$ is regarded as a CDGA with the trivial differential.

By Remark 2.10, we see that d_{HI} is a pseudodistance on the class of objects in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$. Therefore, the same arguments as in the proofs of [23, Proposition 2.11] and Lemma [23, Lemma 2.14] enable us to deduce the following result.

Lemma 7.4. *Suppose that extended tame persistence CDGAs F and G are d_{HI}-formal. Then $d_{\mathrm{HI}}(F, G) = d_{\mathrm{CohI}}(F, G)$.*

As mentioned in the Introduction, every persistence cochain complexes is d_{HI}-formal; see [23, Proposition 3.5]. The result yields the equality $d_{\mathrm{CohI}} = d_{\mathrm{IHC}}$ on the category of persistence cochain complexes; see [23, Theorem 3.3]. On the other hand, Lemma 7.4, Propositions 6.2 and 6.3 imply that there exist non d_{HI}-formal objects in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$; see Remark 7.15. Thus, we are interested in relationships among the formalities in Definition 7.3 and the H-formality.

Proposition 7.5. *For an extended tame persistence CDGA F, consider the following implications*

$$\begin{tikzcd} F \textit{ is } H\textit{-formal} \arrow[r, Rightarrow, "(1)"] & F \textit{ is } d_{\mathrm{HI}}\textit{-formal} \arrow[dl, Rightarrow, "(2)"'] \\ F \textit{ is } d_{\mathrm{IHC}}\textit{-formal} \arrow[u, Rightarrow, "(5)"] \arrow[r, Rightarrow, shift left, "(3)"] & F \textit{ is } d_{\mathrm{HC}}\textit{-formal.} \arrow[l, Rightarrow, shift left, "(4)"] \end{tikzcd}$$

(i) *The implications (1), (2) and (3) hold in general.*
(ii) *Let* $\{\tau_n\}_{n\geq 0}$ *be a sequence which discretises* F*. Suppose that there exists a positive number* α *such that* $\tau_{n+1} - \tau_n \geq \alpha$ *for each* n*. Then, the implications (4) and (5) hold.*

Remark 7.6. As a consequence of the proposition above, we see that formalities in Definition 7.3 and the H-formality are equivalent one another under an appropriate assumption, for example, in the case where the given persistence CDGA is tame; see Definition 2.4.

The proof of Proposition 7.5 is given after proving Proposition 7.9 below, which relates the d_{IHC}-formality to the *formalizability* of a map in the sense of Thomas; see [33, V].

Definition 7.7. A morphism $\alpha : (B, d_B) \to (C, d_C)$ of CDGAs is *formalizable* if there exists a homotopy commutative diagram

$$\begin{tikzcd} (B, d_B) \arrow[r, "\alpha"] & (C, d_C) \\ \wedge V_B \arrow[u, "m_B", "\sim"'] \arrow[r, "m_\alpha"] \arrow[d, "\theta_B"', "\sim"] & \wedge V_C \arrow[u, "\sim", "m_C"'] \arrow[d, "\sim"', "\theta_C"] \\ H^*(B) \arrow[r, "H(\alpha)"] & H^*(C) \end{tikzcd} \tag{7.1}$$

in which vertical arrows are quasi-isomorphisms and $\wedge V_B$ and $\wedge V_C$ are minimal models for B and C, respectively.

A continuous map $f : X \to Y$ between path-connected spaces is *formalizable* if the morphism $A_{\mathrm{PL}}(f) : A_{\mathrm{PL}}(Y) \to A_{\mathrm{PL}}(X)$ of CDGAs is formalizable. A path-connected space X is formal if the trivial map $X \to *$ is formalizable.

In the original definition [33, V.3 (iii)], it is required that $H(m_B) = H(\theta_B)$. This requirement is satisfied by connecting the diagram (7.1) with the diagram obtained by applying the homology functor to (7.1). Moreover, even if m_α in (7.1) is replaced with a relative Sullivan model $\iota\colon \wedge V_B \to \wedge V_B \otimes W$ for α, we have a homotopy commutative diagram (7.1) with a lift m_α which satisfies the condition that $\eta \circ m_\alpha \simeq \iota$ for a minimal model $\eta : \wedge V_C \xrightarrow{\sim} \wedge V_B \otimes W$.

Remark 7.8. Let $p : PX \to X$ be the path fibration on a formal space X. For $\alpha = A_{\mathrm{PL}}(p)$, we choose a Sullivan representative as the map m_α in the diagram (7.1) above. Since $H(A_{\mathrm{PL}}(PX)) \cong \mathbb{Q}$, it follows that the lower square in (7.1) is commutative. Thus, we see that $p : PX \to X$ is formalizable.

Proposition 7.9. *Let* F *be the extended tame persistence CDGA* $\Theta(f)$ *for a map* $f : X \to Y$*. If* $\Theta(f)$ *is* d_{IHC}*-formal, then the map* f *is formalizable.*

Proof. We consider a relative Sullivan model $C(i) \to C(i+1)$ for $H(\Theta(f))(i) \to H(\Theta(f))(1)$ for each $i \geq 0$. We regard C as a persistence object in $\mathsf{CDGA}^{(\mathbb{Z}_+,\leq)}$. Then, we have a cofibrant replacement $\gamma : C' := (\lfloor\ \rfloor)_* C \to H(\Theta(f))$ of $H(\Theta(f))$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$. Since $H(\Theta(f))$ is weakly equivalent to C', it follows form the assumption that $0 = d_{\mathrm{IHC}}(\Theta(f), H(\Theta(f))) = d_{\mathrm{IHC}}(\Theta(f), C')$. This implies that $\Theta(f)$ and C' are δ-interleaved in the homotopy category for $\delta < \frac{1}{2}$.

Let (φ, ψ) be the δ-interleaving of $\Theta(f)$ and C'. Since $2\delta < 1$, it follows that $\psi(n) = \psi^\delta(n)$ and $\psi^\delta(n) \circ \varphi(n) \simeq id_{\Theta(f)(n)}$ for each non-negative integer n. Moreover, we see that $\varphi(n) = \varphi^\delta(n)$ and $\varphi^\delta(n) \circ \psi(n) \simeq id_{C'(n)}$. These facts imply that

φ gives rise to a morphism $\widetilde{\varphi} : \Theta(f) \to C'$ of persistence CDGAs for which the component $\widetilde{\varphi}(n)$ is a quasi-isomorphism for $n \geq 0$. Thus, we have a commutative diagram

$$\begin{tikzcd} \Theta(f)(n) \arrow[r, "\Theta(f)(n<n+1)"] \arrow[d, "\widetilde{\varphi}(n)"', "\sim"] & \Theta(f)(n+1) \arrow[d, "\sim"', "\widetilde{\varphi}(n+1)"] \\ C'(n) \arrow[r, "C'(n<n+1)"'] & C'(n+1) \end{tikzcd} \tag{7.2}$$

for each non-negative integer n. The commutativity of the diagrams yields the commutative diagram

$$\begin{tikzcd} \wedge V \arrow[r] \arrow[d, "\widetilde{\varphi}"', "\sim"] & \wedge V \otimes \wedge(W^{\leq m}) \arrow[r, "\iota_m"] \arrow[d, "\widetilde{\varphi}(m)"', "\sim"] & \mathrm{colim}_n(\wedge V \otimes \wedge(W^{\leq n})) \arrow[d, "k_1"] \\ C'(0) \arrow[r] \arrow[d, "\gamma(0)"', "\sim"] & C'(m) \arrow[r, "i_m"] \arrow[d, "\gamma(m)"', "\sim"] & \mathrm{colim}_n C'(n) \arrow[d, "k_2"] \\ H^*(\wedge V) \arrow[r] & H^*(\wedge V \otimes \wedge(W^{\leq m})) \arrow[r, "H^*(i)"] & H^*(\wedge V \otimes \wedge W), \end{tikzcd} \tag{7.3}$$

where ι_m and i_m are the canonical maps and k_1 is the evident map given by maps $\widetilde{\varphi}(m)$. Moreover, the morphism k_2 of CDGAs is induced by the composites

$$C'(n) \to H^*(\Theta(f)(n)) \to H^*(\mathrm{colim}_n\Theta(f)(n)) = H^*(\wedge V \otimes \wedge W)$$

of natural maps. Observe that $\Theta(f)(n) = \wedge V \otimes \wedge(W^{\leq i})$ and $\mathrm{colim}_n(\wedge V \otimes \wedge(W^{\leq n})) = \wedge V \otimes \wedge W$. To complete the proof, it suffices to show that the composite $k_2 \circ k_1$ is a quasi-isomorphism. We consider $H^\ell(k_2 \circ k_1)$ on the ℓth cohomology. Then, for a sufficient large integer $m(>> \ell)$, it follows that $H^\ell(\iota_m)$, $H^\ell(i_m)$ and $H^\ell(i)$ are isomorphisms. We have the result. □

Proof of Proposition 7.5. (i) By definition, the H-formality implies the d_{HI}-formality. Proposition 7.2 enables us to conclude that the implications (2) and (3) hold.

(ii) As for the implication (5), by applying the same argument as in obtaining the commutative diagram (7.2) in the proof of Proposition 7.9, we have a commutative diagram

$$\begin{tikzcd} F(\tau_n) \arrow[r, "F(\tau_n<\tau_{n+1})"] \arrow[d, "\widetilde{\varphi}(\tau_n)"', "\sim"] & F(\tau_{n+1}) \arrow[d, "\sim"', "\widetilde{\varphi}(\tau_{n+1})"] \\ C'(\tau_n) \arrow[r, "C'(\tau_n<\tau_{n+1})"'] & C'(\tau_{n+1}). \end{tikzcd} \tag{7.4}$$

Here C' is a cofibrant replacement of $H(F)$; see the paragraph after Proposition 2.6. Observe that we use a positive number δ with $2\delta < \alpha$ for obtaining the diagram (7.4). This yields that F is H-formal.

We prove that the implication (4) holds. The same argument as above enables us to obtain the diagram (7.4) which is homotopy commutative. Here, the equalities in the argument are replaced with those up to homotopy. We observe that the δ-interleaving (φ, ψ) of F and C' is considered in the category $\mathrm{Ho}(\mathsf{CDGA})^{([0,\infty),\leq))}$.

By applying [16, Proposition 2.22 (Extension of homotopies)] to the composite $C'(0 < \tau_0) \circ \widetilde{\varphi}(0)$, we have a strictly commutative diagram (7.2) for $n = 0$ replacing $\widetilde{\varphi}(\tau_0)$ with $\widehat{\varphi}(\tau_0)$ which is homotopic to $\widetilde{\varphi}(\tau_0)$. We observe that $\widehat{\varphi}(\tau_0)$ is a quasi-isomorphism. An inductive argument with the replacement enables us to obtain a strictly commutative diagram

$$\begin{tikzcd} F(\tau_n) \arrow[r, "F(\tau_n<\tau_{n+1})"] \arrow[d, "\widehat{\varphi}(\tau_n)"', "\sim"] & F(\tau_{n+1}) \arrow[d, "\sim"', "\widehat{\varphi}(\tau_{n+1})"] \\ C'(\tau_n) \arrow[r, "C'(\tau_n<\tau_{n+1})"'] & C'(\tau_{n+1}) \end{tikzcd}$$

in which the vertical arrows are quasi-isomorphisms. The family $\{\widetilde{\varphi}(0), \widehat{\varphi}(\tau_n)\}_{n\geq 0}$ gives rise to a weak equivalence between F and $H(F)$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$. This yields that F is d_{IHC}-formal. $\square$

Lemma 7.10. *The persistence CDGA $\theta(f)$ is H-formal if and only if $\theta(f)(n < n+1) : \theta(f)(n) \to \theta(f)(n+1)$ is formalizable for each $n \geq 0$.*

Proof. To prove the 'only if' part, we apply [22, Lemma A.1] to a zig-zag of weak equivalences connecting $\theta(f)$ and $H(\theta(f))$. Then, we have a commutative diagram

$$\begin{tikzcd} \theta(f)(n) \arrow[r, "\theta(f)(n<n+1)"] \arrow[d, "\sim"'] & \theta(f)(n+1) \arrow[d, "\sim"] \\ H(\theta(f)(n)) \arrow[r, "H(\theta(f)(n<n+1))"'] & H(\theta(f)(n+1)) \end{tikzcd}$$

for each $n \geq 0$. By using a Sullivan representative of $\theta(f)(n < n+1)$, we have a homotopy commutative diagram as (7.1), which allows us to deduce that $\theta(f)(n < n+1) : \theta(f)(n) \to \theta(f)(n+1)$ is formalizable.

In order to prove the 'if' part, we lift the Sullivan representative $\alpha_n : \mathfrak{M}_n \to \mathfrak{M}_{n+1}$ of $\theta(f)(n < n+1)$ in the homotopy commutative diagram, which gives the formalizability, to a relative Sullivan algebra $\iota_n : \mathfrak{M}_n \to \mathfrak{M}_n \otimes \wedge W_n$ with a quasi-isomorphism $\eta_{n+1} : \mathfrak{M}_n \otimes \wedge W_n \xrightarrow{\sim} \mathfrak{M}_{n+1}$. Thus, we have $\eta_{n+1} \circ \iota_n = \alpha_n$. The same inductive argument as in the proof of Proposition 7.5(5) with [16, Proposition 2.22] enables us to obtain the result. $\square$

Example 7.11. Since the Hopf map $h : S^3 \to S^2$ is not formalizable, it follows from Propositions 7.5 and 7.9 that $\Theta(h)$ in Proposition 6.2 is not d_{HI}-formal. Here, it is proved that $d_{\mathrm{IHC}}(\Theta(h), H(\Theta(h))) = 1$. The same argument as in Proposition 6.2 yields that $d_{\mathrm{IHC}}(\Theta(h), H(\Theta(h))) \leq 1$. We observe that a cofibrant replacement C of $H(\Theta(h))$ is given by $\wedge(x, y)$ on $[0, 1)$ and $\wedge(x, y, \bar{x}, \bar{y}, \tilde{y})$ on $[1, \infty)$.

In order to prove $d_{\mathrm{IHC}}(\Theta(h), H(\Theta(h))) \geq 1$, suppose that $\Theta(h)$ and $H(\Theta(h))$ are δ-interleaved in the homotopy category for some $\delta < 1$ with a δ-interleaving (φ, ψ) for which $\psi : \Theta(h) \to C^\delta$ is a morphism in $\mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+}))$. Then, since $\Theta(h)(0) \to \Theta(h)(2\delta)$ is the identity, by applying the functor $H(Q(\ ))$, it follows that $\psi(0)(y) = \lambda y$ in $C(\delta)$ for some nonzero rational number λ. Therefore, the naturality of ψ implies that $\psi(1+\delta)(y) = \lambda y$ in $C(1+2\delta)$. Since $\varphi(1+2\delta)\circ\psi(1+\delta) \simeq id_{\wedge(x,y,\bar{y})}$, we see that $(H(Q(\varphi(1+2\delta))) \circ H(Q(\psi(1+\delta))))(y) = y$. However, we have $y = 0$ in $H(Q(C(1+2\delta)))$, which is a contradiction.

By combining Lemma 7.10 with the following result due to Thomas [33], we have a necessary and sufficient condition for $\theta(f)$ to be H-formal with *Eilenberg–Moore models* (E.M. models) for $\theta(f)$ and $H(\theta(f))$. An E.M. model is regarded as a relative Sullivan algebra in which *lower degrees* are considered explicitly; see [33, II. 2] for more details.

Proposition 7.12. (cf. [33, V.6. Proposition]) *Let $\alpha : (B, d_B) \to (C, d_C)$ be the morphism of CDGAs in the upper homotopy commutative diagram in (7.1). Then, the following conditions are equivalent.*
(i) α is formalizable.
(ii) There exist an isomorphism $(id, \Psi, \overline{\Psi})$ between an E.M. model $(\mathcal{F}, \eta)$ of m_α and an E.M. model $(\mathcal{F}', \eta')$ of $H(\alpha)\circ\theta_B$ for some quasi-isomorphism $\theta_B : \mathfrak{M}_B \to H^*(B)$ with $H(\theta_B) = H(m_B)$ such that $(\Psi - id)$ strictly lowers the filtration degree and $(\eta')^* \circ \Psi^* = H(m_C) \circ \eta^*$.

Example 7.13. The persistence CDGA $\Theta(p : PS^{2n} \to S^{2n})$ in Example 8.4 below is not H-formal. To see this, suppose that $\theta(p)$ is H-formal. Then, by Lemma 7.10, we see that the morphism $\alpha := \theta(p)(0 < 2n-1) : \wedge(x,y) \to \wedge(x,y,z)$ is formalizable. Observe that α is a E.M. model for α itself. We have a quasi-isomorphism $\theta_B : \wedge(x,y) \to H(\wedge(x,y),d) = \mathbb{Q}[x]/(x^2)$ with $H(\theta_B) = id$. For the trivial map $t : H(\wedge(x,y),d) = \mathbb{Q}[x]/(x^2) \to H(\wedge(x,y,z)) = \wedge(xz-y)$, the composite $t \circ \theta_B$ admits an E.M. model of the form $\wedge(x,y) \to \wedge(x,y,z,w) \otimes \wedge(z')$ with the quasi-isomorphism $\eta' : \wedge(x,y,z,w)\otimes\wedge(z') \to \wedge(xz-y)$ defined by $\eta(u) = 0$ for $u = x, y, z, w$ and $\eta'(z') = xz - y$. It follows from Proposition 7.12 that these E.M. models are isomorphic to each other, which is a contradiction.

We see that $p : PS^{2n} \to S^{2n}$ is formalizable; see Remark 7.8. Thus, it follows that the converse of Proposition 7.9 does not hold in general.

In order to describe an example of an extended tame persistence CDGA $\Theta(f)$ which is H-formal, we recall terminology from [32].

A fibration $f : X \to Y$, whose fiber F and base are simply connected, is said to be *weakly homotopically trivial* (W.H.T.) if the connecting map of the rational fibration $F_{\mathbb{Q}} \to X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is the zero map; see [32, page 75]. For example, a fibration with a section is so. We observe that a fibration is W.H.T if and only if $\pi_*(X)\otimes\mathbb{Q} \cong \pi_*(F)\otimes\mathbb{Q} \oplus \pi_*(Y)\otimes\mathbb{Q}$.

Proposition 7.14. *For a W.H.T. fibration $f : X \to Y$, if X is formal space with $\dim \pi_*(X)\otimes\mathbb{Q} < \infty$, then $\Theta(f)$ is H-formal; see the paragraph before Proposition 8.2 for a W.H.T. fibration.*

Proof. From [12, Theorem II], the minimal model $\mathfrak{M}(X)$ for X is a two stage model, i.e., $\mathfrak{M}(X) = (\wedge V, d)$ for which $V = V_0 \oplus V_1$, $dV_0 = 0$ and $\{d(v_i)\}$ gives a regular sequence in $\wedge V_0$ for some generators v_i of V_1. Then, any two sub-DGAs $\mathfrak{M}_a$ and $\mathfrak{M}_b$ of $\mathfrak{M}(X)$ represents formal spaces and there is a commutative diagram:

$$\begin{tikzcd}
\mathfrak{M}_a \arrow[r, "i"] \arrow[d, "\eta_1"', "\simeq"] & \mathfrak{M}_b \arrow[d, "\eta_2", "\simeq"'] \\
(H^*(\mathfrak{M}_a), 0) \arrow[r, "i^*"'] & (H^*(\mathfrak{M}_b), 0)
\end{tikzcd}$$

in which i is the inclusion $\mathfrak{M}_a = \wedge V_a \subset \mathfrak{M}_b = \wedge V_b$. Here, the quasi-isomorphism η_i is given by $H^*(\mathfrak{M}_i) = \wedge V_{i,0}/(dV_{i,1})$ for $V_{i,0} \subset V_0$ and $V_{i,1} \subset V_1$ for $i = a$ and b. Then, the inclusion i gives rise to $\Theta(f)$ which is H-formal. $\square$

Remark 7.15. In view of Proposition 6.3 and Lemma 7.4, one of the maps f_1 and f_2 in the proposition is not d_{HI}-formal. Indeed, the both of maps $f_k : S^3 \to \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ are *not* formalizable. To see this, suppose that the f_k is formalizable for $k = 1, 2$. Then, the homotopy commutativity of the lower square in Definition 7.1 enables us to deduce that a Sullivan representative for f_k is homotopic to the trivial map, which is a contradiction to the choice of the map.

8. More calculations of the distances d_{IHC} of maps

This section is devoted to gathering computational examples and upper bounds of the distance $d_{\mathrm{IHC}}(\Theta(f), \Theta(g))$ for maps f and g. In particular, the example of the distance for path fibrations is the first one that we have investigated at the beginning of our work due to the result in Proposition 5.1 (ii).

The following example deals with maps which are not relatively finite.

Example 8.1. Let $h : X = S^3 \vee S^3 \to S^3$ be the map that sends the second S^3 to the base point and $g : X = S^3 \vee S^3 \to *$ the constant map. The minimal model of X is given by $\mathfrak{M}(S^3 \vee S^3) = (\wedge(x, y, z, u, w, ...), d)$ with $|x| = |y| = 3$, $|z| = 5$, $|u| = |w| = 7$, $d(x) = d(y) = 0$, $d(z) = xy$, $d(u) = xz$ and $d(w) = yz$. The minimal relative Sullivan model for h is given by the inclusion $(\wedge(x), 0) \to (\wedge(x, y, z, u, w, ...), d)$. Moreover, the inclusion $\iota_g : (\mathbb{Q}, 0) \to (\wedge(x, y, z, u, w, ...), d)$ is the minimal relative Sullivan model for g.

(i)(cf. Proposition 5.4) Then, it follows that $d_{\mathrm{IHC}}(\Theta(id_X), \Theta(g)) = \infty$. In fact, suppose that there exists a δ-interleaving (φ, ψ) between $\Theta(id_X)$ and $\Theta(g)$. By applying the functor $H \circ Q$ to the homotopy relation $\psi^\delta \circ \varphi \simeq \Theta(f)(* < * + \delta)$, we have $((H \circ Q)(\psi^\delta) \circ (H \circ Q)(\varphi))(i) = id_{(H\circ Q)\mathfrak{M}(X)}$ for each $i \geq 0$. The composite in the left-hand side factors through a finite dimensional vector space. The right-hand side is the identity map on the infinite dimensional vector space $(H \circ Q)(\mathfrak{M}(X)) \cong Q\mathfrak{M}(X)$, which is a contradiction.

(ii) We have $d_{\mathrm{IHC}}(\Theta(h), \Theta(g)) = 3$. Indeed, for $\varepsilon \geq 3$, they are ϵ-interleaved since the natural transformations φ and ψ are defined by identities. Also, for $\varepsilon < 3$, they are not homotopy ε-interleaved since $H(\psi(\varepsilon) \circ \varphi(0))(x) = 0$.

We observe that the W.H.T. condition for a fibration $f : X \to Y$ is equivalent to that the minimal relative Sullivan model of f gives a minimal model for X. Then, the minimal model for Y is regarded as a sub CDGA of that for X.

Proposition 8.2. *Let $f : X \to Y$ and $g : X \to Z$ be W.H.T. fibrations. Let $\mathfrak{M}(Y) = (\wedge V_1, d)$ and $\mathfrak{M}(Z) = (\wedge V_2, d)$ be minimal models for Y and Z, respectively. Suppose that $\dim V_i < \infty$ for $i = 1$ and 2. Then, it holds that*

$$\begin{aligned} d_{\mathrm{HC}}(j_*(\Theta(f)), j_*(\Theta(g))) &\leq d_{\mathrm{IHC}}(\Theta(f), \Theta(g)) \leq d_{\mathrm{HI}}(\Theta(f), \Theta(g)) \\ &\leq d_{\mathrm{I}}(\Theta(f), \Theta(g)) \leq \max_{i\in\{1,2\}}\{\max\{n \mid V_i^n \neq 0\}\}. \end{aligned}$$

Observe that the upper bound in the proposition above depends only on the minimal models for codomains of f and g.

Proof of Proposition 8.2. We prove the last inequality. Other ones follow from Proposition 7.2.

Without lost of generality, we assume that inclusions $\wedge V_1 \to \wedge(V_1 \oplus W_1)$ and $\wedge V_2 \to \wedge(V_2 \oplus W_2)$ for some W_1 and W_2 give rise to the persistence CDGAs $\Theta(f)$ and $\Theta(g)$, respectively. Let $\mathfrak{M}(X) = (\wedge V, d)$ be the minimal model for the space X. Then, the W.H.T. condition and the uniqueness of the model enable us to obtain isomorphisms $\widetilde{f} : \wedge(V_1 \oplus W_1) \xrightarrow{\cong} \wedge V$ and $\widetilde{g} : \wedge(V_2 \oplus W_2) \xrightarrow{\cong} \wedge V$ of CDGAs.

Let N be the right-hand side integer in the inequality that we show. We define N-interleavings $\varphi(i) : \Theta(f)(i) \to \Theta(g)(i + N)$ and $\psi(i) : \Theta(g)(i) \to \Theta(f)(i + N)$ by the restriction of $\widetilde{g}^{-1} \circ \widetilde{f}$ and $\widetilde{f}^{-1} \circ \widetilde{g}$ to the domains, respectively. Since $V_i = V_i^N$ for $i = 1$ and 2, it follows that (φ, ψ) is a well-defined N-interleaving. $\square$

Principal G-bundles over a space X are classified by the homotopy set $[X, BG]$ up to isomorphism, where BG denotes the classifying space of G. Thus, Theorem 5.2 enables us to consider the pseudodistance between G-bundles. In the example below, we determine the distance d between principal S^1-bundles over $\mathbb{C}P^n$.

Example 8.3. We identify the isomorphism class of an S^1-principal bundle over $\mathbb{C}P^n$ with the homotopy class of its classifying map $\xi_k : \mathbb{C}P^n \to BS^1$, which corresponds to an integer $k \in \mathbb{Z} \cong H^2(\mathbb{C}P^n; \mathbb{Z}) \cong [\mathbb{C}P^n, BS^1]$. The pseudodistance $d(\xi_k, \xi_l)$ in Theorem 5.2 is given by

$$d(\xi_k, \xi_l) = \begin{cases} 0 & (k, l \neq 0 \quad \text{or} \quad k = l = 0), \\ 2 & (\text{otherwise}). \end{cases}$$

This result can be verified by a direct calculation using the Sullivan models as follows. Let $\mathfrak{M}(BS^1) = (\wedge(u), 0)$ and $\mathfrak{M}(\mathbb{C}P^n) = (\wedge(x, y), d)$ be minimal Sullivan models for BS^1 and $\mathbb{C}P^n$, respectively. Here, $|u| = |x| = 2$, $|y| = 2n + 1$, $dx = 0$ and $dy = x^{n+1}$. A Sullivan representative for ξ_k is given by the morphism $\xi_k^* : \mathfrak{M}(BS^1) \to \mathfrak{M}(\mathbb{C}P^n)$ defined by $\xi_k^*(u) = kx$. Then, the relative Sullivan model for ξ_k^* is constructed as follows.

- For $k \neq 0$, we have a relative Sullivan model $(\wedge(u), 0) \to (\wedge(u, y_k), d)$ of ξ_k^* via the quasi-isomorphism $\eta : (\wedge(u, y_k), d) \to \mathfrak{M}(\mathbb{C}P^n)$ defined by $\eta(u) = kx$ and $\eta(y_k) = y$, where $d(y_k) = u^{n+1}/k^{n+1}$.
- For $k = 0$, a relative Sullivan model $(\wedge(u), 0) \to (\wedge(u, z), d) \otimes (\wedge(x, y), d)$ of ξ_0^* via the quasi-isomorphism $\eta : (\wedge(u, z), d) \otimes (\wedge(x, y), d) \to \mathfrak{M}(\mathbb{C}P^n)$ given by the projection to $\mathfrak{M}(\mathbb{C}P^n)$, where $dz = u$.

It is readily seen that the relative Sullivan models for ξ_k^* and ξ_l^* for any $k, l \neq 0$ are isomorphic. This implies that $d(\xi_k, \xi_l) = 0$ immediately for any $k, l \neq 0$. The case $k = l = 0$ is trivial.

Moreover, $\Theta(\xi_k)$ and $\Theta(\xi_0)$ are 2-interleaved in the homotopy category for any $k \neq 0$. An interleaving (φ, ψ) is given as follows.

- $\varphi : \Theta(\xi_k) \Rightarrow \Theta(\xi_0) \circ T_2$ is a natural transformation defined by a restriction of the morphism $(\wedge(u, y_k), d) \to (\wedge(u, z), d) \otimes (\wedge(x, y), d)$, $u \mapsto x$, $y_k \mapsto y/k^{n+1}$.
- $\psi : \Theta(\xi_0) \Rightarrow \Theta(\xi_k) \circ T_2$ is a natural transformation defined by a restriction of the morphism $(\wedge(u, z), d) \otimes (\wedge(x, y), d) \to (\wedge(u, y_k), d)$, $u \mapsto 0$, $z \mapsto 0$, $x \mapsto u$, $y \mapsto k^{n+1} y_k$.

Similarly to Example 5.7, it can be shown that $\Theta(\xi_k)$ and $\Theta(\xi_0)$ are not ε-interleaved in the homotopy category for $0 \leq \varepsilon < 2$. This leads to the conclusion that $d(\xi_k, \xi_0) = 2$.

In the rest of this section, we consider extended tame persistence CDGAs $\Theta(p)$ for path fibrations p.

Example 8.4. (i) Let $p : PS^{2n} \to S^{2n}$ be the path fibration, where $n \geq 1$. The relative Sullivan model for the map p is a relative Sullivan algebra of the form

$$(\wedge(x, y), d) \rightarrowtail (\wedge(x, y, z, w), d) \simeq \mathbb{Q}$$

in which $d(y) = x^2$, $d(z) = x$ and $d(w) = xz - y$, where $\deg(x) = 2n$. Thus, we see that $\theta(\iota_p)$, but not $\Theta(p)$, has the form

$$\begin{array}{ccccc} 0 & & 2n-1 & & 4n-2 \\ \wedge(x,y) = \cdots = \wedge(x,y) & \rightarrowtail & \wedge(x,y,z) = \cdots = \wedge(x,y,z) & \rightarrowtail & \wedge(x,y,z,w) = \cdots . \end{array}$$

Moreover, the cohomology $H(\theta(\iota_p))$ of the persistence CDGA $\theta(\iota_p)$ has the form

$$\begin{array}{ccccc} 0 & & 2n-1 & & 4n-2 \\ \mathbb{Q}[x]/(x^2) = \cdots = \mathbb{Q}[x]/(x^2) & \xrightarrow{t} & \wedge(xz-y) = \cdots = \wedge(xz-y) & \xrightarrow{t} & \mathbb{Q} = \cdots , \end{array}$$

where t denotes the trivial map. Thus, we have

$$H(\Theta(p)) \cong [0,\infty)_0 \oplus [0, 2n-1)_{2n} \oplus [2n-1, 4n-2)_{4n-1}$$

as a persistence graded *module*.

(ii) Let $q : PS^{2n-1} \to S^{2n-1}$ be the path fibration, where $n \geq 1$. It follows that $\theta(\iota_q)$ and $H(\theta(\iota_q))$ have the form

$$\begin{array}{ccc} 0 & & 2n-2 \\ \wedge(u) = \cdots = \wedge(u) & \rightarrowtail & \wedge(u,v) = \cdots \end{array} \quad \text{and} \quad \begin{array}{ccc} 0 & & 2n-2 \\ \wedge(u) = \cdots = \wedge(u) & \xrightarrow{t} & \mathbb{Q} = \cdots , \end{array}$$

respectively. We see that $H(\Theta(q)) \cong [0,\infty)_0 \oplus [0, 2n-2)_{2n-1}$ as a persistence graded module. Observe that the numbers of bars which appear in $H(\Theta(p))$ and $H(\Theta(q))$ are different from each other.

We conclude this section by giving an upper bound of d_{IHC} for path fibrations.

Proposition 8.5. *Let $\wedge V_X$ and $\wedge V_Y$ be the minimal models for simply-connected spaces X and Y, respectively. Suppose that $\dim V_X < \infty$ and $\dim V_Y < \infty$. For the path fibrations $f : PX \to X$ and $g : PY \to Y$, one has $d_{\mathrm{IHC}}(\Theta(f), \Theta(g)) \leq (\max\{\max\{i \mid V_X^i \neq 0\} - 1, \max\{i \mid V_Y^i \neq 0\} - 1\})/2 =: N$.*

Proof. It follows from [14, Theorem 15.3], which gives a model for a fibration, that $\mathfrak{M}(PX) = \wedge V_X \otimes \wedge \overline{V}_X$ and $\mathfrak{M}(PY) = \wedge V_Y \otimes \wedge \overline{V}_Y$. Define morphisms $\varphi(i) : \Theta(f)(i) \to \Theta(g)(i+N)$ and $\psi(i) : \Theta(g)(i) \to \Theta(f)(i+N)$ by the trivial maps, respectively. Then, we see that $\Theta(f)$ and $\Theta(g)$ are N-homotopy interleaved in the homotopy category with the interleaving (φ, ψ).

In fact, since the path space PY is contractible, it follows that the unit $\eta : \mathbb{Q} \to \mathfrak{M}(PY)$ is a quasi-isomorphism. Then, Whitehead theorem ([18, Chapter II, Theorem 1.10]) enables us to conclude that η is a homotopy equivalence with the homotopy inverse γ. We observe that the domain and codomain of η are cofibrant-fibrant objects in CDGA. Let H be a homotopy from $\eta \circ \gamma$ to $id_{\mathfrak{M}(PY)}$. It follows from the choice of the integer N that $\overline{V}_Y^{\leq 2N} = \overline{V}_Y$. Then, Remark 3.11 allows us to obtain a homotopy $\varphi^\varepsilon \circ \psi \simeq \Theta(g)(* < * + 2N)$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$ with the restriction of the homotopy H. By the same argument as above, we see that $\psi^\varepsilon \circ \varphi \simeq \Theta(f)(* < * + 2N)$ in $\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})$. $\square$

9. Perspective

As seen in the Introduction, the result [23, Theorem 3.3] asserts that the interleaving distances and the cohomology interleaving distance coincide for persistence cochain complexes. The starting point of the manuscript has been to find algebraic or geometric examples which show the difference between the cohomology interleaving distance and the interleaving distance in the homotopy category. To this end, we have prepared the categorical framework in Sections 3 and 4. This section describes future perspective based on the framework.

9.1. **Extended tame persistence differential graded Lie algebras.** As seen in [28, Theorem 6.9], we may replace the category CDGA in Theorem 4.9 with the category cdgl of complete differential graded Lie algebras (cdgls), which is deeply considered in [3]. In fact, the category cdgl is endowed with a cofibrantly generated model structure; see [3, Chapter 8]. Thus, by combining Proposition 4.6, [3, Corollary 8.2] and [13, Theorem 2.10], we have a partial Quillen equivalence between $\mathsf{et}(\mathsf{sSet}^{\mathbb{R}_+})$ and $\mathsf{et}(\mathsf{cdgl}^{\mathbb{R}_+})$ on reduced nilpotent rational simplicial sets and homologically nilpotent connected cdgls. We may investigate extended tame persistence simplicial sets as well as towers, which mentioned in Definition 4.8 and Remark 4.10, with their Lie models from the homotopical point of view.

It is worthwhile mentioning that in the consideration of Lie models, the finiteness condition for persistence objects assumed in Theorem 4.9 is not needed.

9.2. **(Co)fibration categories.** Let $\mathcal{C}$ be a cofibration category in the sense of Baues [2]. Then, following the proof of [9, Theorem 2.2], we may give a (non trivial) cofibration model category structure to $\mathsf{et}(\mathcal{C}^{\mathbb{R}_+})$. In fact, for example, by considering the terminal object virtually, the existence of a fibrant model follows from the same argument as that in the proofs of the factorization axiom and the left lifting property in the model category. Such a cofibration category structure on $\mathsf{et}(\mathcal{C}^{\mathbb{R}_+})$ will be considered in [31]. Then, in view of the investigation in [27], we may develop the homotopy theory on persistence spaces in *coarse geometry*.

There are some versions of a cofibration (fibration) category. Therefore, it will be important work to consider the expansion of the (co)fibration category structure to the category of extended tame functors for each version. The (co)fibration category structures in [8] and [1] are applicable to investigating *persistence digraphs* and *persistence C^*-algebras*.

9.3. **Multiparameter persistence objects.** One might expect developments of persistence CDGAs and related objects in multiparameter persistence theory. As a consequence of Proposition 3.4, one has a commutative diagram

$$\begin{tikzcd}
(\mathrm{Func}(I, \mathsf{Top}_0))^{\mathrm{op}})^{(\mathbb{R}^n,\leq)} \arrow[r, "\Theta_*"] & (\mathrm{Ho}(\mathsf{et}(\mathsf{CDGA}^{\mathbb{R}_+})))^{(\mathbb{R}^n,\leq)} \arrow[d, "(\mathbf{L}(\mathrm{colim}))_*"] \\
(\mathsf{Top}_0^{\mathrm{op}})^{(\mathbb{R}^n,\leq)} \arrow[u, "i_*"] \arrow[r, "(q\circ A_{PL}(\ ))_*"'] & (\mathrm{Ho}(\mathsf{CDGA}))^{(\mathbb{R}^n,\leq)}.
\end{tikzcd} \tag{9.1}$$

Thus, we obtain naturally $(n+1)$-parameter persistence objects from n-parameter persistence objects via Θ_*. In [34], Zhou considers the interleaving distance and related distances between persistence spaces with the functor $(q \circ A_{PL}(\ ))_*$ in the case where $n = 1$.

The E.M. model mentioned in Section 7 has the lower degrees adding to the dimensions of the generators. It seems that we have two parameter persistence object modifying $\Theta(f)$ with the second degree.

Moreover, we might be able to deal with the objects described in Sections 9.1 and 9.2 as multiparameter persistence ones improving the diagram (4.1).

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Department of Mathematics, Faculty of Science, Shinshu University, Matsumoto, Nagano 390-8621, Japan
Email address: `kuri@math.shinshu-u.ac.jp`

Nippon Institute of Technology, Gakuendai, Miyashiro-machi, Minamisaitama-gun, Saitama 345-8501, Japan
Email address: `naito.takahito@nit.ac.jp`

Department of Science and Technology, Graduate School of Medicine, Science and Technology, Shinshu University, Matsumoto, Nagano 390-8621, Japan
Email address: `23hs601d@shinshu-u.ac.jp`

Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, Aichi 464-8601, Japan
Email address: `shun.wakatsuki@math.nagoya-u.ac.jp`

Faculty of Education, Kochi University, Akebono-cho, Kochi 780-8520, Japan
Email address: `tyamag@kochi-u.ac.jp`