A singular de Rham algebra and spectral sequences in diffeology

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6 June 2021
Global Diffeology Seminar
Online on Zoom

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§1. A stratifold as a diffeological space

The embedding $C^\infty(\ ) : \mathbf{Mfd} \to \mathbb{R}\text{-}\mathbf{Alg}$

Definition 1.1 (Sikorski (1971).)

A *differential space* is a pair $(S, \mathcal{C})$ consisting of a topological space S and an $\mathbb{R}$-subalgebra $\mathcal{C}$ of the $\mathbb{R}$-algebra $C^0(S)$ of continuous real-valued functions on S, which is assumed to be *locally detectable* and C^∞*-closed*.

Local detectability : $f \in \mathcal{C}$ if and only if for any $x \in S$, there exist an open neighborhood U of x and an element $g \in \mathcal{C}$ such that $f|_U = g|_U$.

C^∞-closedness : For each $n \geq 1$, each n-tuple $(f_1, ..., f_n)$ of maps in $\mathcal{C}$ and each smooth map $g : \mathbb{R}^n \to \mathbb{R}$, the composite $h : S \to \mathbb{R}$ defined by $h(x) = g(f_1(x), ..., f_n(x))$ belongs to $\mathcal{C}$.

For $x \in S$, T_xS : the vector space consisting of derivations on the $\mathbb{R}$-algebra $\mathcal{C}_x$ of the germs at x (*tangent space*).

Definition 1.2 (Kreck (2010))

A *stratifold* is a differential space $(S, \mathcal{C})$ such that the following four conditions hold:

1. S is a locally compact Hausdorff space with countable basis;
2. the *skeleta* $sk_k(S) := \{x \in S \mid \dim T_xS \leq k\}$ are closed in S;
3. for each $x \in S$ and open neighborhood U of x in S, there exists a *bump function* at x subordinate to U
4. the *strata* $S^k := sk_k(S) - sk_{k-1}(S)$ are k-dimensional smooth manifolds such that restriction along $i : S^k \hookrightarrow S$ induces an isomorphism of stalks $i^* : \mathcal{C}_x \overset{\cong}{\to} C^\infty(S^k)_x$ for each $x \in S^k$.

- The 'cone' is a stratifold.
- A *parametrized stratifold* (p-stratifold) is constructed from a stratifold attaching other finite manifolds with boundaries.

A continuous map $f : (S, \mathcal{C}) \to (S', \mathcal{C}')$ is a *morphism of stratifolds* if $\phi \circ f \in \mathcal{C}$ for any $\phi \in \mathcal{C}'$. We denote by **Stfd** the category of stratifolds.

Proposition 1.3 (Aoki–K ('17))

There is a functor $k : \mathbf{Stfd} \to \mathbf{Diff}$ *defined by* $k(S, \mathcal{C}) = (S, \mathcal{D}_{\mathcal{C}})$ *and* $k(f) = f$ *for a morphism* $f : S \to S'$ *of stratifolds, where*

$$\mathcal{D}_{\mathcal{C}} := \left\{ u : U \to S \;\middle|\; \begin{array}{l} U : \text{open in } \mathbb{R}^q, q \geq 0, \\ \phi \circ u \in C^\infty(U) \text{ for any } \phi \in \mathcal{C} \end{array} \right\},$$

Let M *be a manifold and* $(S, \mathcal{C})$ *a stratifold. Then the functor* $k : \mathbf{Stfd} \to \mathbf{Diff}$ *induces a bijection*

$$k_* : \mathrm{Hom}_{\mathbf{Stfd}}((M, C^\infty(M)), (S, \mathcal{C})) \overset{\cong}{\to} \mathrm{Hom}_{\mathbf{Diff}}((M, \mathcal{D}_{C^\infty(M)}), (S, \mathcal{D}_{\mathcal{C}})).$$

$$\begin{tikzcd}
\mathbf{Mfd} \arrow[r, "j", "\text{fully faithful}"'] \arrow[rr, bend left, "\ell\text{: fully faithful}"] & \mathbf{Stfd} \arrow[r, "k"] & \mathbf{Diff} \arrow[r, shift left, "D"] \arrow[r, phantom, "\perp"] & \mathbf{TopSp} \arrow[l, shift left, "C"]
\end{tikzcd}$$

Observe that the functor k above is nothing but the functor Π in the sense of Batubenge, I-Zemmour, Karshon and Watts ('17).

§2. The de Rham complex due to Souriau

For an open set U of $\mathbb{R}^n$, let $\mathcal{D}^X(U)$ be the set of plots with U as the domain and $\Omega^*_{\text{de Rham}}(U)$ the usual de Rham complex of U. Let **Open** be the category consisting of open sets of Euclidian spaces and smooth maps between them.

$$\Omega^p(X) := \left\{ \begin{tikzcd} \textbf{Open}^{\text{op}} \arrow[r, bend left, "\mathcal{D}^X", ""{name=U, below}] \arrow[r, bend right, "\Omega^p_{\text{de Rham}}"', ""{name=D, above}] & \textbf{Sets} \arrow[Rightarrow, from=U, to=D, "\omega"] \end{tikzcd} \;\middle|\; \omega \text{ is a natural transformation} \right\}$$

with the cochain algebra structure induced by that of $\Omega^*_{\text{de Rham}}(U)$.

Remark 2.1

Let M be a manifold and $\Omega^*_{\text{deRham}}(M)$ the usual de Rham complex of M. Recall the *tautological map* $\theta : \Omega^*_{\text{deRham}}(M) \to \Omega^*(M)$ defined by

$$\theta(\omega) = \{p^*\omega\}_{p\in\mathcal{D}^M}.$$

Then it follows that θ is an isomorphism of cochain algebras.

Iglesias-Zemmour (Canad. J. Math. 65 (2013)) has introduced an integration map of the form

$$\int^{IZ} : \Omega^*(X) \longrightarrow C^*_{\text{cube}}(X)$$

to the cubic cochain complex.

For the irrational torus $T^2_\gamma = \mathbb{R}/(\mathbb{Z} + \gamma\mathbb{Z})$, ($\gamma$: irrational) with the quotient diffeology. We see that

$$\Omega^*(T^2_\gamma) \cong (\wedge^*(\mathbb{R}^1), d = 0)$$

and then $H^1(\Omega(T^2_\gamma)) \cong \mathbb{R}^1$.
On the other hand, by the Hurewicz theorem in **Diff** enables us to conclude that

$$H^1(C^*_{\text{cube}}(T^2_\gamma)) \cong \mathbb{R}^2.$$

- One might expect a new de Rham complex for which the 'de Rham theorem' holds. (For connecting de Rham calculus and homotopy theory.)

§3. The singular de Rham complex

- The cubic de Rham complex (Iwase – Izumida '19)
- The singular de Rham complex (K '20)

$\mathbb{A}^n := \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i = 1\}$: a diff-space with subdiffeology of the manifold $\mathbb{R}^{n+1}$

Define a simplicial DGA $(A^*_{DR})_\bullet$ as follows.
For each $n \geq 0$, $(A^*_{DR})_n := \Omega^*(\mathbb{A}^n)$ and define a simplicial set

$$S^D_\bullet(X) := \{\{\sigma : \mathbb{A}^n \to X \mid \sigma : C^\infty\text{–map}\}\}_{n\geq 0}$$

Moreover, we have a simplicial map

$$S^D_\bullet(X) \to S^D_\bullet(X)_{\text{sub}} := \{\{\sigma : \Delta^n_{\text{sub}} \to X \mid \sigma \text{ is a } C^\infty\text{-map}\}\}_{n\geq 0}$$

induced by the inclusion $j : \Delta^n_{\text{sub}} \to \mathbb{A}^n$.

Let $\boldsymbol{\Delta}$ be the category which has posets $[n] := \{0, 1, ..., n\}$ for $n \geq 0$ as objects and non-decreasing maps $[n] \to [m]$ for $n, m \geq 0$ as morphisms. By definition, a simplicial set is a contravariant functor from $\boldsymbol{\Delta}$ to **Sets** the category of sets.

$$A^*_{DR}(S^D_\bullet(X)) := \left\{ \begin{tikzcd} \boldsymbol{\Delta}^{\mathrm{op}} \arrow[r, bend left, "S^D_\bullet(X)"{name=U}] \arrow[r, bend right, "(A^*_{DR})_\bullet"'{name=D}] & \mathbf{Sets} \arrow[Rightarrow, from=U, to=D, "\omega"] \end{tikzcd} \;\middle|\; \omega : \text{ a natural transformation} \right\}$$

Definition 3.1 (For connecting new de Rham to the original one.)

The *factor map* $\alpha : \Omega^*(X) \to A^*_{DR}(S^D_\bullet(X))$ is defined by

$$\alpha(\omega)(\sigma) := \sigma^*(\omega).$$

Variations of the *singular de Rham complex* $A^*_{DR}(S^D_\bullet(X))$ are considered.

The simplicial DGA $(C^*_{PL})_\bullet := C^*(\Delta[\bullet])$, where $\Delta[n] = \mathrm{hom}_\Delta(-, [n])$ is the standard n-simplicial set.

- We define an integration map $\int_{\Delta^p} : (A^p_{DR})_p \to \mathbb{R}$ by
$$\int_{\Delta^p} \omega := \int_{\Delta^p} \theta^{-1}\omega,$$
where $\theta : \Omega^*_{\text{deRham}}(\mathbb{A}^p) \xrightarrow{\cong} \Omega^*(\mathbb{A}^p)$ is the tautological map mentioned above.
- Define a mor. of simpl. DG modules $\int : (A^*_{DR})_\bullet \to (C^*_{PL})_\bullet = C^*(\Delta[\bullet])$ by
$$(\int \gamma)(\sigma) = \int_{\Delta^p} \sigma^*\gamma,$$
where $\gamma \in (A^p_{DR})_n$, $\sigma : \mathbb{A}^p \to \mathbb{A}^n$ is the affine map induced by $\sigma : [p] \to [n]$. Then we have a commutative diagram of simplicial sets
$$\begin{tikzcd}
(C^*_{PL})_\bullet \arrow[r, "\varphi"] \arrow[dr, "="'] & (C_{PL} \otimes A_{DR})^*_\bullet \arrow[d, "\mathrm{mult}\circ(1\otimes\int)"'] & A^*_{DR\bullet} \arrow[l, "\psi"'] \arrow[dl, "\int"] \\
& (C^*_{PL})_\bullet . &
\end{tikzcd}$$

The de Rham theorem in diffeology

Theorem 3.2 (K (2020))

For a diffeological space $(X, \mathcal{D}^X)$, one has a homotopy commutative diagram

$$\begin{tikzcd}
C^*(S^D_\bullet(X)_{\mathrm{sub}}) \arrow[r, "\varphi", "\simeq"'] \arrow[dr, "="'] & (C^*_{PL} \otimes A^*_{DR})(S^D_\bullet(X)) \arrow[d, "\mathrm{mult}\circ(1\otimes\int)"'] & A^*_{DR}(S^D_\bullet(X)) \arrow[l, "\psi"', "\simeq"] \arrow[dl, "\text{an ``integration'' } \int"] & \Omega^*(X) \arrow[l, "\alpha"', "\text{the factor map}"] \arrow[d, "\int^{\mathrm{IZ}}"] \\
 & C^*(S^D_\bullet(X)_{\mathrm{sub}}) & & C^*_{\mathrm{cube}}(X) \arrow[ll, "l"', "\simeq"]
\end{tikzcd}$$

in which φ and ψ are quasi-isomorphisms of cochain algebras and the integration map $\int$ is a morphism of cochain complexes.

Moreover, the factor map α is a quasi-isomorphism if $(X, \mathcal{D}^X)$ is a finite dimensional smooth CW-complex in the sense of Iwase–Izumida, or stems from a p-stratifold via the functor k mentioned above.

Chen's iterated integrals in diffeology

M : a diff-space, $\omega_i \in \Omega^{p_i}(M)$ for each $1 \leq i \leq k$ and $q : U \to M^I$ a plot of the diff-space M^I. $\widetilde{\omega_{iq}} := (id_U \times t_i)^* q_\sharp^* \omega_i$, where $q_\sharp : U \times I \to M$ is the adjoint to q and $t_i : \Delta^k \to I$ denotes the projection in the ith factor.

$$(\int \omega_1 \cdots \omega_k)_q := \int_{\Delta^k} \widetilde{\omega_{1q}} \wedge \cdots \wedge \widetilde{\omega_{kq}}.$$

Then by definition, Chen's iterated integral **It** has the form

$$\mathbf{It}(\omega_0[\omega_1|\cdots|\omega_k]) = ev^*(\omega_0) \wedge \widetilde{\Delta^*}(\int \omega_1 \cdots \omega_k),$$

where $\widetilde{\Delta} : LM \to M^I$ is the lift of the diagonal map $M \to M \times M$.

Theorem 3.3 (K (2020))

Let M be a simply-connected diff-space, $\dim H^i(A_{DR}(S^D_\bullet(M))) < \infty$ for each $i \geq 0$. Suppose that the factor map for M is a quasi-isomorphism. Then

$$\alpha \circ \mathbf{It} : \Omega^*(M) \otimes \overline{B}(A) \to \Omega^*(LM) \to A^*_{DR}(S^D_\bullet(LM))$$

is a quasi-isomorphism of $\Omega^*(M)$-modules.

The Leray–Serre spectral sequence in diffeology

Theorem 3.4 (K (2020), $A^*(X) := A^*_{DR}(S^D(X))$)

Let $\pi : E \to M$ *be a smooth map between path-connected diffeological spaces with path-connected fibre* L *which is*

i) *a fibration in the sense of Christensen and Wu or*
ii) *the pullback of the evaluation map* $(\varepsilon_0, \varepsilon_1) : N^I \to N \times N$ *for a connected diffeological space* N *along an embedding* $f : M \to N \times N$.

Suppose further that in the case ii) *the cohomology* $H(A^*(M))$ *is of finite type. Then one has the Leary–Serre spectral sequence* $\{{}_{LS}E_r^{*,*}, d_r\}$ *converging to* $H(A^*(E))$ *as an algebra with an isomorphism*

$${}_{LS}E_2^{*,*} \cong H^*(M, \mathcal{H}^*(L))$$

of bigraded algebras, where $H^*(M, \mathcal{H}(L))$ *is the cohomology with the local coefficients* $\mathcal{H}^*(L) = \{H(A^*(L_c))\}_{c \in S_0^D(M)}$

The Eilenberg–Moore spectral sequence in diffeology

Theorem 3.5 (K (2020))

Let $\pi : E \to M$ be the smooth map as in Theorem 3.4 with the same assumption, $\varphi : X \to M$ a smooth map from a connected diffeological space X for which the cohomology $H(A^*(X))$ is of finite type and E_φ the pullback of π along φ. Suppose further that M is simply connected in case of i) *and N is simply connected in case of* ii).

$$\begin{tikzcd}
E_\varphi \arrow[r] \arrow[d] & E \arrow[r, "\widetilde{f}"] \arrow[d, "\pi"'] & N^I \arrow[d, "(\varepsilon_0, \varepsilon_1)"] \\
X \arrow[r, "\varphi"'] & M \arrow[r, "f"'] & N \times N
\end{tikzcd}$$

Then one has the Eilenberg–Moore spectral sequence $\{{}_{EM}E_r^{*,*}, d_r\}$ converging to $H(A^*(E_\varphi))$ as an algebra with an isomorphism

$$ {}_{EM}E_2^{*,*} \cong \mathrm{Tor}^{*,*}_{H(A^*(M))}(H(A^*(X)), H(A^*(E))) $$

of bigraded algebras.

On the proofs.

- For the case i), Dress' construction for the Leary-Serre spectral sequence is applicable to our setting.
- For the case ii), the spectral sequences are constructed by considering a *smooth lifting problem* with an appropriate *homotopy* pullback.

Definition 3.6 (Christensen–Wu (2014))

A morphism $X \to Y$ in **Diff** is a *fibration* if $S^D_\bullet(X) \to S^D_\bullet(Y)$ is a (Kan) fibration in $\mathbf{Sets}^{\Delta^{op}}$.

FACT

- Any diffeological bundle (i.e. the pullback for every global plot is trivial) with fibrant fibre (for example, a diffeological group) is a fibration [C–W].
- For a diff-group G and a subgroup H with the sub-diffeology, the smooth map $G \to G/H$ is a diffeological bundle with fibre H [Iglesias-Zemmour]. Then it is a fibration in the sense of C–W.

Computational examples

$$T^2 := \{(e^{2\pi i x}, e^{2\pi i y}) \mid (x, y) \in \mathbb{R}^2\} \supset S_\gamma := \{(e^{2\pi i t}, e^{2\pi i \gamma t}) \mid t \in \mathbb{R}\},$$

where $\gamma \in \mathbb{R}\backslash\mathbb{Q}$. Then the *irrational torus* T_γ is defined by the quotient T^2/S_γ with the quotient diffeology.

In the category **Diff**, $S_\gamma \to T^2 \xrightarrow{\pi} T_\gamma$: a principal diffeological fibre bundle. By using the Leray–Serre s.s., we have

$$H^*(A(T_\gamma)) \xrightarrow[\cong]{\pi^*} H^*(A(T^2)) \xleftarrow[\cong]{\text{factor map}} H^*_{DR}(T^2) \cong \wedge(x_1, x_2)$$

Recall the Čech-de Rham spectral sequence due to Zemmour:

- A first quadrant spectral sequence

$$ {}_{\Omega}E_2^{p,q} \cong H^q(HH^p(\mathbb{R}\mathsf{M}^{\mathrm{op}}, \Omega^*(\mathcal{N}_X)), d_{\Omega}), $$

$$ {}_{\Omega}E_r^{*,*} \Longrightarrow H^*(\mathrm{Tot}\ C^{*,*}) \cong HH^*(\mathbb{R}\mathsf{M}^{\mathrm{op}}, \mathrm{map}(\mathcal{G}, \mathbb{R})) =: \check{H}(X) $$

- Comparing the spectral sequences for $\Omega(X)$ and $A(X)$, we have a commutative diagram

$$\begin{tikzcd}
H^1(\Omega(X)) \oplus {}_{\Omega}E_3^{1,0} \arrow[rr, "\Theta", "\cong"'] & & H^1(A(\mathcal{N}_X)^{\mathsf{M}}) \oplus {}_{A}E_3^{1,0} \\
& \check{H}^1(X; \mathbb{R}). \arrow[ul, "\cong"', "\mathrm{edge}_2"] \arrow[ur, "\cong", "\mathrm{edge}_2"'] &
\end{tikzcd}$$

In particular, we see

$$ \Theta : H^1(\Omega(T_\gamma)) \oplus {}_{\Omega}E_2^{1,0} \overset{\cong}{\to} H^1(A(T_\gamma)) $$

Corollary 3.7 (K '21)

There exists an isomorphism $H^*(A(T_\gamma)) \cong \wedge(\Theta(t), \Theta(\xi))$ *of algebras, where* $t \in H^*(\Omega(T_\gamma)) \cong \wedge(t)$ *is a generator and* $\xi \in \mathsf{Fl}^\bullet(T_\gamma) \cong \mathbb{R}$ *is a flow bundle over* T_γ *with a connection* 1*-form, which is a generator of the group* $\mathsf{Fl}^\bullet(T_\gamma)$*.*

- Let $f : M \to T_\gamma$ be a smooth map from a diffeological space M. Then via the pullback construction along the map f, (*) : $S_\gamma \to M \times_{T_\gamma} T^2 \xrightarrow{\pi'} M$: a principal diffeological bundle
- Then the Leray–Serre spectral sequence in Theorem 3.4 for the fibration (*) allows us to deduce that

$$(\pi')^* : H^*(A^*(M)) \xrightarrow{\cong} H^*(A^*(M \times_{T_\gamma} T^2))$$

of algebras, where $A^*(-) := A^*_{DR}(S^D_\bullet(-))$.

- Suppose further that M is simply connected. Then the comparison of the EMSS's in Theorem 3.5 for LM and $L(M \times_{T_\gamma} T^2)$ allows us to obtain an algebra isomorphism

$$(L\pi')^* : H^*(A^*(LM)) \xrightarrow{\cong} H^*(A^*(L(M \times_{T_\gamma} T^2)).$$

By Theorem 3.3 (On the composite $\alpha \circ \mathsf{It}$), we have

Assertion 3.8

If $H^*(A^*(M)) \cong H^*(A^*(S^{2k+1}))$ as an algebra with $k \geq 1$ and the factor map for M is a quasi-isomorphism, then

$$H^*(A^*(L(M \times_{T_\gamma} T^2))) \cong \wedge(\alpha \circ \mathsf{It}((\pi')^*(\omega))) \otimes \mathbb{R}[\alpha \circ \mathsf{It}(1 \otimes (\pi')^*(\omega))]$$

as an $H^*(A^*(M))$-algebra, where α is the factor map and ω denotes the volume form of M.

§4. With functors and a model structure on **Diff**

Assertion 4.1 (With the simplicial DGA $(A^*_{DR})_\bullet = \Omega^*(\mathbb{A}^\bullet)$)

$$\begin{tikzcd}
\mathbf{Mfd} \arrow[r, "j", "embedding"'] \arrow[rr, bend left, "\ell : embedding"] \arrow[ddrr, bend right, "U : forgetful"'] & \mathbf{Stfd} \arrow[r, "k"] & \mathbf{Diff} \arrow[dl, "S^D(\ )"] \arrow[dd, shift left, "D"] \\
& \mathbf{Sets}^{\Delta^{op}} \arrow[ur, "|\ |_D", "\perp"'] \arrow[dr, "|\ |"', "\top"] & \\
& & \mathbf{Top} \arrow[ul, "S(\ )"'] \arrow[uu, shift left, "C"] \arrow[uu, phantom, "\vdash"]
\end{tikzcd}$$

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