# A singular de Rham algebra and spectral sequences in diffeology

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## $\S1.$ A stratifold as a diffeological space

#### The embedding $C^\infty(\ ):\mathsf{Mfd} o \mathbb{R} ext{-}\mathsf{Alg}$

#### Definition 1.1 (Sikorski (1971).)

A differential space is a pair (S, C) consisting of a topological space S and an  $\mathbb{R}$ -subalgebra C of the  $\mathbb{R}$ -algebra  $C^0(S)$  of continuous real-valued functions on S, which is assumed to be *locally detectable* and  $C^{\infty}$ -closed.

Local detectability :  $f \in C$  if and only if for any  $x \in S$ , there exist an open neighborhood U of x and an element  $g \in C$  such that  $f|_U = g|_U$ .

 $\underline{C^{\infty}\text{-closedness}}$ : For each  $n \geq 1$ , each n-tuple  $(f_1, ..., f_n)$  of maps in  $\mathcal{C}$  and each smooth map  $g: \mathbb{R}^n \to \mathbb{R}$ , the composite  $h: S \to \mathbb{R}$  defined by  $h(x) = g(f_1(x), ..., f_n(x))$  belongs to  $\mathcal{C}$ .

For  $x \in S$ ,  $T_xS$  : the vector space consisting of derivations on the  $\mathbb{R}$ -algebra  $\mathcal{C}_x$  of the germs at x (*tangent space* ).

#### Definition 1.2 (Kreck (2010))

A *stratifold* is a differential space  $(S, \mathcal{C})$  such that the following four conditions hold:

- 1. old S is a locally compact Hausdorff space with countable basis;
- 2. the skeleta  $sk_k(S) := \{x \in S \mid \dim T_xS \leq k\}$  are closed in S;
- 3. for each  $x \in S$  and open neighborhood U of x in S, there exists a *bump function* at x subordinate to U
- 4. the strata  $S^k := sk_k(S) sk_{k-1}(S)$  are k-dimensional smooth manifolds such that restriction along  $i : S^k \hookrightarrow S$  induces an isomorphism of stalks  $i^* : \mathcal{C}_x \xrightarrow{\cong} C^{\infty}(S^k)_x$  for each  $x \in S^k$ .
- ► The 'cone' is a stratifold.
- A parametrized stratifold (p-stratifold) is constructed from a stratifold attaching other finite manifolds with boundaries.

A continuous map  $f : (S, C) \to (S', C')$  is a morphism of stratifolds if  $\phi \circ f \in C$  for any  $\phi \in C'$ . We denote by **Stfd** the category of stratifolds.

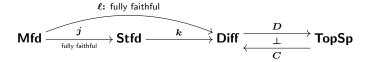
#### Proposition 1.3 (Aoki-K ('17))

There is a functor  $k : \mathsf{Stfd} \to \mathsf{Diff}$  defined by  $k(S, \mathcal{C}) = (S, \mathcal{D}_{\mathcal{C}})$  and k(f) = f for a morphism  $f : S \to S'$  of stratifolds, where

$$\mathcal{D}_{\mathcal{C}} := \left\{ \left. u: U 
ightarrow S \; \left| egin{array}{c} U: \textit{ open in } \mathbb{R}^q, q \geq 0, \ \phi \circ u \in C^\infty(U) \textit{ for any } \phi \in \mathcal{C} \end{array} 
ight. 
ight\},$$

Let M be a manifold and  $(S, \mathcal{C})$  a stratifold. Then the functor  $k : \mathsf{Stfd} \to \mathsf{Diff}$  induces a bijection

$$k_*: \operatorname{Hom}_{\operatorname{Stfd}}((M, C^{\infty}(M)), (S, \mathcal{C})) \xrightarrow{\cong} \operatorname{Hom}_{\operatorname{Diff}}((M, \mathcal{D}_{C^{\infty}(M)}), (S, \mathcal{D}_{\mathcal{C}})).$$



Observe that the functor k above is nothing but the functor  $\Pi$  in the sense of Batubenge, I-Zemmour, Karshon and Watts ('17).

## $\S2$ . The de Rham complex due to Souriau

For an open set U of  $\mathbb{R}^n$ , let  $\mathcal{D}^X(U)$  be the set of plots with U as the domain and  $\Omega^*_{de Rham}(U)$  the usual de Rham complex of U. Let **Open** be the category consisting of open sets of Euclidian spaces and smooth maps between them.

$$\Omega^p(X) := \left\{ egin{array}{c} \mathbf{Open}^{\mathrm{op}} \stackrel{\mathcal{D}^X}{\underset{\Omega^p_{\mathrm{de Rham}}}{\longrightarrow}} \mathsf{Sets} \ \end{array} \middle| \ \omega \ ext{is a natural transformation} \end{array} 
ight.$$

with the cochain algebra structure induced by that of  $\Omega^*_{{\scriptscriptstyle\mathsf{de}}\,\operatorname{\mathsf{Rham}}}(U).$ 

#### Remark 2.1

Let M be a manifold and  $\Omega^*_{\mathrm{deRham}}(M)$  the usual de Rham complex of M. Recall the *tautological map*  $\theta: \Omega^*_{\mathrm{deRham}}(M) \to \Omega^*(M)$  defined by

$$heta(\omega)=\{p^*\omega\}_{p\in\mathcal{D}^M}.$$

Then it follows that  $\theta$  is an isomorphism of cochain algebras.

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Iglesias-Zemmour (Canad. J. Math. 65 (2013)) has introduced an integration map of the form

$$\int^{lZ}: \Omega^*(X) \longrightarrow C^*_{\mathsf{cube}}(X)$$

to the cubic cochain complex.

For the irrational torus  $T_{\gamma}^2 = \mathbb{R}/(\mathbb{Z} + \gamma \mathbb{Z})$ , ( $\gamma$ : irrational ) with the quotient diffeology. We see that

$$\Omega^*(T^2_\gamma) \cong (\wedge^*(\mathbb{R}^1), d=0)$$

and then  $H^1(\Omega(T^2_\gamma))\cong \mathbb{R}^1.$ 

On the other hand, by the Hurewicz theorem in Diff enables us to conclude that

$$H^1(C^*_{\mathsf{cube}}(T^2_\gamma))\cong \mathbb{R}^2.$$

One might expect a new de Rham complex for which the 'de Rham theorem' holds. (For connecting de Rham calculus and homotopy theory.)

## §3. The singular de Rham complex

- The cubic de Rham complex (Iwase Izumida '19)
- The singular de Rham complex (K '20)

 $\mathbb{A}^n:=\{(x_0,...,x_n)\in\mathbb{R}^{n+1}\mid\sum_{i=0}^nx_i=1\}$  : a diff-space with subdiffeology of the manifold  $\mathbb{R}^{n+1}$ 

Define a simplicial DGA  $(A_{DR}^*)_{\bullet}$  as follows. For each  $n \geq 0$ ,  $(A_{DR}^*)_n := \Omega^*(\mathbb{A}^n)$  and define a simplicial set

$$S^D_ullet(X):=\{\{\sigma:\mathbb{A}^n o X\mid\sigma:C^\infty ext{-map}\}\}_{n\geq 0}$$

Moreover, we have a simplicial map

$$S^D_ullet(X) o S^D_ullet(X)_{\mathsf{sub}} := \{\{\sigma: \Delta^n_{\mathsf{sub}} o X \mid \sigma ext{ is a } C^\infty ext{-map}\}\}_{n\geq 0}$$

induced by the inclusion  $j:\Delta^n_{\mathsf{sub}} \to \mathbb{A}^n.$ 

Let  $\Delta$  be the category which has posets  $[n] := \{0, 1, ..., n\}$  for  $n \ge 0$  as objects and non-decreasing maps  $[n] \to [m]$  for  $n, m \ge 0$  as morphisms. By definition, a simplicial set is a contravariant functor from  $\Delta$  to **Sets** the category of sets.

$$A^*_{DR}(S^D_{\bullet}(X)) := \left\{ \begin{array}{c} \Delta^{\mathsf{op}} \underbrace{\overset{S^D_{\bullet}(X)}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})_{\bullet}}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{\overset{}}{\underset{(A^*_{DR})}{\overset{}}{\underset{(A^*_{DR})}{\overset{}}}}}}}}}} Sets} \right| \omega : a natural transformation} \right)$$

Definition 3.1 (For connecting new de Rham to the original one.) The factor map  $\alpha : \Omega^*(X) \to A^*_{DR}(S^D_{\bullet}(X))$  is defined by

$$lpha(\omega)(\sigma):=\sigma^*(\omega).$$

Variations of the singular de Rham complex  $A^*_{DR}(S^D_{ullet}(X))$  are considered.

The simplicial DGA  $(C^*_{PL})_{\bullet} := C^*(\Delta[\bullet])$ , where  $\Delta[n] = \hom_{\Delta}(\neg, [n])$  is the standard *n*-simplicial set.

 $\blacktriangleright$  We define an integration map  $\int_{\Delta^p} : (A^p_{DR})_p o \mathbb{R}$  by

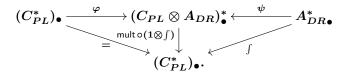
$$\int_{\Delta^p} \omega := \int_{\Delta^p} heta^{-1} \omega,$$

where  $\theta : \Omega^*_{deRham}(\mathbb{A}^p) \xrightarrow{\cong} \Omega^*(\mathbb{A}^p)$  is the tautological map mentioned above.

▶ Define a mor. of simpl. DG modules  $\int : (A_{DR}^*)_{\bullet} \to (C_{PL}^*)_{\bullet} = C^*(\Delta[\bullet])$  by

$$(\int \gamma)(\sigma) = \int_{\Delta^p} \sigma^* \gamma,$$

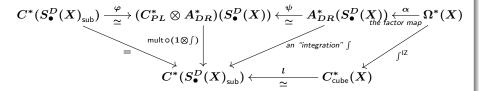
where  $\gamma \in (A_{DR}^p)_n$ ,  $\sigma : \mathbb{A}^p \to \mathbb{A}^n$  is the affine map induced by  $\sigma : [p] \to [n]$ . Then we have a commutative diagram of simplicial sets



## The de Rham theorem in diffeology

#### Theorem 3.2 (K (2020))

For a diffeological space  $(X, \mathcal{D}^X)$ , one has a homotopy commutative diagram



in which  $\varphi$  and  $\psi$  are quasi-isomorphisms of cochain algebras and the integration map  $\int$  is a morphism of cochain complexes.

Moreover, the factor map  $\alpha$  is a quasi-isomorphism if  $(X, \mathcal{D}^X)$  is a finite dimensional smooth CW-complex in the sense of lwase–lzumida, or stems from a p-stratifold via the functor k mentioned above.

## Chen's iterated integrals in diffeology

M: a diff-space,  $\omega_i \in \Omega^{p_i}(M)$  for each  $1 \leq i \leq k$  and  $q: U \to M^I$  a plot of the diff-space  $M^I$ .  $\widetilde{\omega_{iq}} := (id_U \times t_i)^* q_{\sharp}^* \omega_i$ , where  $q_{\sharp}: U \times I \to M$  is the adjoint to q and  $t_i: \Delta^k \to I$  denotes the projection in the *i*th factor.

$$(\int \omega_1 \cdots \omega_k)_q := \int_{\Delta^k} \widetilde{\omega_{1q}} \wedge \cdots \wedge \widetilde{\omega_{kq}}.$$

Then by definition, Chen's iterated integral  $\mathbf{lt}$  has the form

$$\mathsf{lt}(\omega_0[\omega_1|\cdots|\omega_k])=ev^*(\omega_0)\wedge\widetilde{\Delta^*}(\int\omega_1\cdots\omega_k),$$

where  $\widetilde{\Delta}: LM \to M^I$  is the lift of the diagonal map  $M \to M \times M$ . Theorem 3.3 (K (2020))

Let M be a simply-connected diff-space,  $\dim H^i(A_{DR}(S^D_{\bullet}(M))) < \infty$  for each  $i \geq 0$ . Suppose that the factor map for M is a quasi-isomorphism. Then

 $\alpha \circ \mathsf{lt}: \Omega^*(M) \otimes \overline{B}(A) \to \Omega^*(LM) \to A^*_{DR}(S^D_{\bullet}(LM))$ 

is a quasi-isomorphism of  $\Omega^*(M)$ -modules.

## The Leray–Serre spectral sequence in diffeology

Theorem 3.4 (K (2020),  $A^*(X) := A^*_{DR}(S^D(X))$  )

Let  $\pi:E\to M$  be a smooth map between path-connected diffeological spaces with path-connected fibre L which is

i) a fibration in the sense of Christensen and Wu or ii) the pullback of the evaluation map  $(\varepsilon_0, \varepsilon_1) : N^I \to N \times N$  for a connected diffeological space N along an embedding  $f : M \to N \times N$ .

Suppose further that in the case ii) the cohomology  $H(A^*(M))$  is of finite type. Then one has the Leary–Serre spectral sequence  $\{{}_{LS}E^{*,*}_r, d_r\}$  converging to  $H(A^*(E))$  as an algebra with an isomorphism

$${}_{LS}E_2^{*,*} \cong H^*(M,\mathcal{H}^*(L))$$

of bigraded algebras, where  $H^*(M, \mathcal{H}(L))$  is the cohomology with the local coefficients  $\mathcal{H}^*(L) = \{H(A^*(L_c))\}_{c \in S_0^D(M)}$ 

## The Eilenberg-Moore spectral sequence in diffeology

#### Theorem 3.5 (K (2020))

Let  $\pi : E \to M$  be the smooth map as in Theorem 3.4 with the same assumption,  $\varphi : X \to M$  a smooth map from a connected diffeological space Xfor which the cohomology  $H(A^*(X))$  is of finite type and  $E_{\varphi}$  the pullback of  $\pi$  along  $\varphi$ . Suppose further that M is simply connected in case of i) and N is

Then one has the Eilenberg–Moore spectral sequence  $\{_{EM}E_r^{*,*}, d_r\}$  converging to  $H(A^*(E_{\varphi}))$  as an algebra with an isomorphism

$${}_{EM}E_2^{*,*}\cong {\rm Tor}_{H(A^*(M))}^{*,*}(H(A^*(X)),H(A^*(E)))$$

of bigraded algebras.

On the proofs.

- For the case i), Dress' construction for the Leary-Serre spectral sequence is applicable to our setting.
- For the case ii), the spectral sequences are constructed by considering a smooth lifting problem with an appropriate homotopy pullback.

#### Definition 3.6 (Christensen-Wu (2014))

A morphism  $X \to Y$  in **Diff** is a *fibration* if  $S^D_{\bullet}(X) \to S^D_{\bullet}(Y)$  is a (Kan) fibration in **Sets**<sup> $\Delta^{op}$ </sup>.

FACT

- Any diffeological bundle (i.e. the pullback for every global plot is trivial) with fibrant fibre (for example, a diffeological group) is a fibration [C–W].
- For a diff-group G and a subgroup H with the sub-diffeology, the smooth map  $G \rightarrow G/H$  is a diffeological bundle with fibre H [Iglesias-Zemmour]. Then it is a fibration in the sense of C–W.

### Computational examples

$$T^2 := \{ (e^{2\pi i x}, e^{2\pi i y}) \mid (x, y) \in \mathbb{R}^2 \} \supset S_{\gamma} := \{ (e^{2\pi i t}, e^{2\pi i \gamma t}) \mid t \in \mathbb{R} \},$$

where  $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ . Then the *irrational torus*  $T_{\gamma}$  is defined by the quotient  $T^2/S_{\gamma}$  with the quotient diffeology.

In the category **Diff**,  $S_{\gamma} \to T^2 \xrightarrow{\pi} T_{\gamma}$ : a principal diffeological fibre bundle. By using the Leray–Serre s.s., we have

$$H^*(A(T_\gamma)) \xrightarrow{\pi^*}{\cong} H^*(A(T^2)) \stackrel{\mathsf{factor\ map}}{\xleftarrow{\cong}} H^*_{DR}(T^2) \cong \wedge (x_1, x_2)$$

Applications

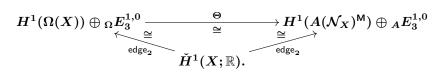
Recall the Čech-de Rham spectral sequence due to Zemmour:

A first quadrant spectral sequence

 $_{\Omega}E_{2}^{p,q} \cong H^{q}(HH^{p}(\mathbb{R}\mathsf{M}^{\mathsf{op}},\Omega^{*}(\mathcal{N}_{X})),d_{\Omega}),$ 

 $_{\Omega}E_{r}^{*,*} \Longrightarrow H^{*}(\operatorname{Tot} C^{*,*}) \cong HH^{*}(\mathbb{R}\mathsf{M}^{\operatorname{op}},\operatorname{map}(\mathcal{G},\mathbb{R})) =: \check{H}(X)$ 

• Comparing the spectral sequences for  $\Omega(X)$  and A(X), we have a commutative diagram



In particular, we see

$$\Theta: H^1(\Omega(T_\gamma)) \oplus {}_{\Omega}E_2^{1,0} \stackrel{\cong}{ o} H^1(A(T_\gamma))$$

#### Corollary 3.7 (K '21)

There exists an isomorphism  $H^*(A(T_{\gamma})) \cong \wedge(\Theta(t), \Theta(\xi))$  of algebras, where  $t \in H^*(\Omega(T_{\gamma})) \cong \wedge(t)$  is a generator and  $\xi \in \mathsf{Fl}^{\bullet}(T_{\gamma}) \cong \mathbb{R}$  is a flow bundle over  $T_{\gamma}$  with a connection 1-form, which is a generator of the group  $\mathsf{Fl}^{\bullet}(T_{\gamma})$ .

- Let  $f: M \to T_{\gamma}$  be a smooth map from a diffeological space M. Then via the pullback construction along the map f, (\*) :  $S_{\gamma} \to M \times_{T_{\gamma}} T^2 \xrightarrow{\pi'} M$ : a principal diffeological bundle
- Then the Leray–Serre spectral sequence in Theorem 3.4 for the fibration (\*) allows us to deduce that

$$(\pi')^*: H^*(A^*(M)) \stackrel{\cong}{\longrightarrow} H^*(A^*(M imes_{T_{\gamma}} T^2))$$

of algebras, where  $A^*(-):=A^*_{DR}(S^D_{ullet}(-)).$ 

Suppose further that M is simply connected. Then the comparison of the EMSS's in Theorem 3.5 for LM and  $L(M \times_{T_{\gamma}} T^2)$  allows us to obtain an algebra isomorphism

$$(L\pi')^*: H^*(A^*(LM)) \xrightarrow{\cong} H^*(A^*(L(M imes_{T_{\gamma}} T^2)).$$

By Theorem 3.3 (On the composite  $\alpha \circ \mathbf{lt}$ ), we have

#### Assertion 3.8

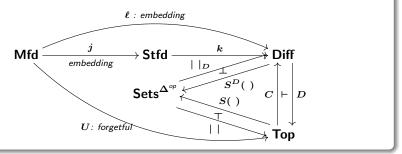
If  $H^*(A^*(M)) \cong H^*(A^*(S^{2k+1}))$  as an algebra with  $k \ge 1$  and the factor map for M is a quasi-isomorphism, then

 $H^*(A^*(L(M \times_{T_{\gamma}} T^2))) \cong \wedge (\alpha \circ \mathsf{lt}((\pi')^*(\omega))) \otimes \mathbb{R}[\alpha \circ \mathsf{lt}(1 \otimes (\pi')^*(\omega))]$ 

as an  $H^*(A^*(M))$ -algebra, where lpha is the factor map and  $\omega$  denotes the volume form of M.

## $\S4$ . With functors and a model structure on **Diff**

Assertion 4.1 (With the simplicial DGA  $(A_{DR}^*)_{\bullet} = \Omega^*(\mathbb{A}^{\bullet}))$ )



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