Cartan calculi on the free loop spaces

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Models for spaces and maps in rational homotopy theory

The de Rham–Sullivan correspondence gives an equivalence between the homotopy category of nilpotent rational connected spaces of finite $\mathbb{Q}$-type and that of cofibrant connected commutative differential graded algebras of finite $\mathbb{Q}$-type.

$$\mathrm{fN}\mathbb{Q}\text{-}\mathrm{Ho}(\mathbf{Top}) \underset{|\ |}{\overset{Q\circ A_{PL}(\ )}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathrm{f}\mathbb{Q}\text{-}\mathrm{Ho}(\mathbf{CDGA}^{op})$$

Here Q denotes the cofibrant replacement. As a consequence, we have a quasi-iso. $(\wedge V = (\text{poly. alg} \otimes \text{exterior alg}), d) \xrightarrow{\simeq} A_{PL}(X)$ for a space X.

- The CDGA $(\wedge V, d)$ is called a *Sullivan (rational) model* for X.
- The morphism $Q \circ A_{PL}(f) : (\wedge V_Y, d_Y) \to (\wedge V_X, d_X)$ for a map $f : X \to Y$ is called the *Sullivan representative* for f (or a *model* for f).

Motivated results

$$\Delta : H_*(LM;\mathbb{Q}) \xrightarrow{-\times[S^1]} H_{*+1}(LM \times S^1;\mathbb{Q}) \xrightarrow{\text{rotation action}_*} H_{*+1}(LM;\mathbb{Q})$$

Definition 1.1 (K, Naito, Wakatsuki, Yamaguchi '21 (KNWY21))

A manifold M is *Batalin–Vilkovisky (BV) exact* if $\mathrm{Im}\,\widetilde{\Delta} = \mathrm{Ker}\,\widetilde{\Delta}$ for the reduced BV operator $\widetilde{\Delta} : \widetilde{H}_*(LM;\mathbb{Q}) \to \widetilde{H}_{*+1}(LM;\mathbb{Q})$.

Theorem 1.2 (KNWY21)

Let M be a simply-connected closed manifold. Assume further that M is BV exact. Then there exists a commutative diagram

$$\begin{tikzcd}
H_*^{S^1}(LM;\mathbb{Q})^{\otimes 2} \arrow[r, "\Psi\otimes\Psi", "\cong"'] \arrow[d, "{[\ ,\ ]}"', "\textit{string bracket}"] & (\mathrm{Ker}\,\widetilde{\Delta} \oplus \mathbb{Q}[u])^{\otimes 2} \arrow[d, dotted, "\textit{Gerstenhaber bracket}"] \arrow[r, "inc.\oplus 0"] & H_*(LM;\mathbb{Q})^{\otimes 2} \arrow[d, "\textit{loop product}"', "\bullet"] \\
H_*^{S^1}(LM;\mathbb{Q}) \arrow[r, "\Psi", "\cong"'] & (\mathrm{Ker}\,\widetilde{\Delta} \oplus \mathbb{Q}[u]) & H_*(LM;\mathbb{Q}). \arrow[l, "\Delta"]
\end{tikzcd}$$

The new homotopy invariant, the BV exactness, is related to traditional ones.

Theorem 1.3 (KNWY21)

A simply-connected space X admitting positive weights is BV exact. In particular, a formal space is BV exact.

The proof uses linear maps

$$L, e : \mathbf{Der}(\text{the minimal model for } X) \to \mathbf{End}(\text{the minimal model for } LM),$$

which satisfies the Cartan magic formular

$$L_\theta = [B, e_\theta]$$

for $\theta \in \mathbf{Der}(\text{the minimal model for } X)$. Here B is a map which induces the BV operator.

This inspires us to consider algebraic and topological backgrounds for such a Lie representation L and a linear map e.

The original Cartan calculus

The Lie derivative L_X and the interior product (contraction) ι_X for each vector field X on a manifold M are incorporated in the framework of a *Cartan calculus*

$$\mathrm{Der}(C^\infty(M)) \xrightarrow[\iota_{(\ )}]{L_{(\ )}} \big(\mathrm{Der}(\Omega^*(M)), d\big) = \big(\mathrm{Der}(HH^*_{conti}(C^\infty(M))), B\big)$$

in the sense that $L_{(\ )}$ is a Lie algebra representation and $\iota_{(\ )}$ is a linear map which satisfy the *formula* $L_X = [d, \iota_X]$ for any vector field X.

$\exists$? a *second stage* of a Cartan calculus

$$H\big(\mathrm{Der}(\Omega^*(M)), [d, -]\ \big)? \to ??$$

In the Hochschild and cyclic theory, we consider such a *second stage* for $\Omega^*(M)$ and the rational de Rham complex $A_{PL}(X)$ for a space X.

A guiding principle for the second stage

Let A be a unital algebra over a commutative ring k. For a derivation D on A, we define a map L_D on the Hochschild complex $C_*(A)$ by

$$L_D(a_0, ..., a_n) = \sum_{i \geq 0} (a_0, .., a_{i-1}, Da_i, a_{i+1}, ..., a_n).$$

We also recall the Hochschild cohomology $HH^*(A, A)$ of A. In particular, the first cohomology $HH^1(A, A)$ is isomorphic to $\mathrm{Der}(A)/\{\text{inner derivations}\}$ as a k-module. Then, we have

Proposition 1.4 (Loday, 4.1.6 Corollary)

There are well-defined homomorphisms of Lie algebras $[D] \mapsto L_D$:

$$HH^1(A, A) \to \mathrm{End}_k(HH_n(A)) \quad \textit{and} \quad HH^1(A, A) \to \mathrm{End}_k(HC_n(A)).$$

Assertion 1.5

We have

- *an algebraic construction of a Cartan calculus* $({}_{HH}L_{(\ )}, {}_{HH}e_{(\ )})$ *of the André–Quillen cohomology of the de Rham complex* $\Omega^*(M)$ *with values in the endomorphism ring* $\mathrm{End}(HH_*(\Omega^*(M)))$ *and*
- *a geometric construction of a Cartan calculus* $(L_{(\ )}, \iota_{(\ )})$ *of the real homotopy group* $\pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{R}$ *of the monoid of self-homotopy equivalences of* M *with values in the derivation ring* $\mathrm{Der}(H^*(LM : \mathbb{R}))$.

There exists a commutative diagram

$$
\begin{tikzcd}
H^*_{AQ}(\Omega^*(M)) \arrow[r, "{}_{HH}L_{(\ )}", "{}_{HH}e_{(\ )}"'] & \big(\mathrm{End}(HH_*(\Omega^*(M))), B\big) \\
\pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{R} \arrow[u, "\text{Sullivan's isp. } \Phi", "\cong"'] \arrow[r, "L_{(\ )}", "(-1)^*\iota_{(\ )}"'] & \big(\mathrm{Der}(H^*(LM : \mathbb{R})), \Delta\big) \arrow[u, "\text{a monic map}"']
\end{tikzcd}
$$

such that each sequence is a Cartan calculus, where $* \geq 2$, Δ *is the BV operator on the loop cohomology.*

§2 A homotopy Cartan calculus in the sense of Fiorenza and Kowalzig

Let (M, d, B) be a *mixed complex*; $d\colon M \to M$ is a differential of degree 1, $B\colon M \to M$ of degree -1 with $B^2 = 0$ and $[d, B] = dB + Bd = 0$.

Definition 2.1 (FK20, Definition 3.1)

Let $(\mathfrak{g}, \delta, [\,,\,])$ be a dg Lie algebra. A *homotopy pre-Cartan calculus* of $\mathfrak{g}$ on M consists of linear maps $e\colon \mathfrak{g} \to \mathrm{End}(M)$ of degree 1, $L\colon \mathfrak{g} \to \mathrm{End}(M)$ of degree 0 and $S\colon \mathfrak{g} \to \mathrm{End}(M)$ of degree (-1) such that

$$L_\theta = [B, e_\theta] + [d, S_\theta] + S_{\delta\theta}, \tag{1}$$

$$[d, e_\theta] + e_{\delta\theta} = 0, \tag{2}$$

$$[B, S_\theta] = 0 \tag{3}$$

for any $\theta \in \mathfrak{g}$. Here e is called a *contraction operator* (or *cap product*) and L a *Lie derivative*.

(1) and (2) imply that e and L are chain maps of degree 1 and 0, respectively.

Definition 2.2 (FK20, Definition 3.7, Remark 3.8)

A *homotopy Cartan calculus* on M is a homotopy pre-Cartan calculus $(\mathfrak{g}, e, L, S)$ endowed with a linear map $T\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}(M)$ of degree 0 such that

$$[e_\theta, L_\rho] - e_{[\theta,\rho]} = [d, T_{\theta,\rho}] - T_{\delta\theta,\rho} - (-1)^{\deg\theta} T_{\theta,\delta\rho} \tag{4}$$
$$[S_\theta, L_\rho] - S_{[\theta,\rho]} = [B, T_{\theta,\rho}] \tag{5}$$

for any $\theta, \rho \in \mathfrak{g}$.

- Let $(\wedge V, d)$ be a Sullivan algebra with $V^1 = 0$. Then we define a mixed complex $(\wedge V \otimes \wedge \overline{V}, d, s)$, where $(\overline{V})^i = V^{i+1}$, s is the derivation of degree (-1) defined by $sv = \bar{v}$ and $s\bar{v} = 0$, and d is the unique extension of $d\colon \wedge V \to \wedge V$ satisfying $[d, s] = 0$.
- Observe that $\wedge V \otimes \wedge \overline{V}$ is a model for the free loop space LM of a simply-connected space M if $\wedge V$ is a Sullivan model for M. In particular, we have

$$H^*(\wedge V \otimes \wedge \overline{V}, d) \cong H^*(LM; \mathbb{Q}).$$

Definition 2.3

For a derivation θ on $\wedge V$, we define derivations ${}_aL_\theta$ and ${}_ae_\theta$ on $\wedge V \otimes \wedge \overline{V}$ by

$$ {}_aL_\theta v = \theta v, \quad {}_aL_\theta \bar{v} = (-1)^{\deg \theta} s\theta v, \tag{6} $$

$$ {}_ae_\theta v = 0, \quad {}_ae_\theta \bar{v} = (-1)^{\deg \theta} \theta v \tag{7} $$

for $v \in V$. This defines linear maps ${}_aL\colon \operatorname{Der}(\wedge V) \to \operatorname{End}(\wedge V \otimes \wedge \overline{V})$ of degree 0 and ${}_ae\colon \operatorname{Der}(\wedge V) \to \operatorname{End}(\wedge V \otimes \wedge \overline{V})$ of degree 1.

Proposition 2.4 (KNWY22)

The above maps give a homotopy Cartan calculus

$$ ((\operatorname{Der}(\wedge V), {}_ae, {}_aL, S = 0, T = 0) $$

on $(\wedge V \otimes \wedge \overline{V}, d, s)$.

Let A be a DGA and $C_*(A)$ the Hochschild complex. For a derivation θ on A, we define ${}_{HH}L_\theta : C_*(A) \to C_*(A)$ by ${}_{HH}L_\theta = \sum_i L_{\theta,i}$ and

$$L_{\theta,i}(a_0[a_1|\cdots|a_n]) = \begin{cases} \theta(a_0)[a_1|a_2|\cdots|a_n] & (i=0) \\ (-1)^{|\theta|(\varepsilon_i+1)} a_0[a_1|\cdots|\theta(a_i)|\cdots|a_n] & (1 \le i \le n). \end{cases}$$

We also define ${}_{HH}e_\theta : C_*(A) \to C_*(A)$ by ${}_{HH}e_\theta|_A = 0$ and

$${}_{HH}e_\theta(a_0[a_1|\cdots|a_n]) = (-1)^{|\theta||a_0|+|\theta|+|a_0|} a_0\theta(a_1)[a_2|\cdots|a_n].$$

Moreover, we define $S_\theta : C_*(A) \to C_*(A)$ by $S_\theta|_A = 0$ and, for $n \ge 1$,

$$S_\theta|_{A\otimes T^n(s\bar{A})} = \sum_{j=1}^{n} \left(\sum_{k=0}^{n-j} s \circ t_n^k \right) \circ L_{\theta,j}.$$

Proposition 2.5 (KNWY22)

The morphisms described above give a homotopy Cartan calculus

$$(\mathrm{Der}(A), {}_{HH}L, {}_{HH}e, S, T = 0)$$

on the mixed complex $(C_*(A), d, B)$.

§3. Geometric descriptions of L and e

Given $\theta \in \pi_n(\mathrm{aut}_1(X))$, let $\mathbf{ad}(\theta) : S^n \times X \to X$ be the adjoint of θ and consider the map between the free loop spaces $L(\mathbf{ad}(\theta)) : LS^n \times LX \to LX$ induced by the 'loop construction'. Define
$L : \pi_n(\mathrm{aut}_1(X)) \to \mathbf{End}(H^*(LX))$ by the composite

$$L_\theta : H^*(LX) \xrightarrow{L(\mathbf{ad}(\theta))^*} H^*(LS^n \times LX) \xrightarrow{\int_{[S^n]}} H^*(LX), \tag{8}$$

where $\int_{[S^n]}$ denotes the integration along the image of the fundamental class via the map $H^n(S^n) \xrightarrow{ev_0^*} H^n(LS^n)$. Moreover, we define
$e : \pi_n(\mathrm{aut}_1(X)) \to \mathbf{End}(H^*(LX))$ by the composite

$$e_\theta : H^*(LX) \xrightarrow{L(\mathbf{ad}(\theta))^*} H^*(LS^n \times LX) \xrightarrow{\int_{\overline{[S^n]}}} H^*(LX). \tag{9}$$

Here $\overline{[S^n]}$ is the cohomology class in $H^{n-1}(LS^n)$ which is the image of the fundamental class of S^n induced by the composite

$$H^n(S^n) \xrightarrow{ev_0^*} H^n(LS^n) \xrightarrow{\Delta} H^{n-1}(LS^n).$$

Sulllivan's isomorphism of Lie algebras

We have the following sequence of the homotopy sets

$$\pi_n(\mathrm{aut}_1(X)) \xrightarrow{k} [S^n \times X, X] \xrightarrow{\mu} [\mathcal{M}_X, \mathcal{M}_{S^n \times X}],$$

where $\mathcal{M}_Y$ denotes a minimal Sullivan model for a space Y and μ assigns a map f a Sullivan representative for f. We may replace $\mathcal{M}_{S^n \times X}$ with the DGA $H^*(S^n) \otimes \mathcal{M}_X$.

We write

$$(\mu \circ k)(\theta) = 1 \otimes 1_{\mathcal{M}_X} + \iota \otimes \theta',$$

where ι is the generator of $H^n(S^n)$. Then, Sulllivan's isomorphism of Lie algebras

$$\Phi : \pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q} \xrightarrow{\cong} H^*_{AQ}(A^*_{PL}(X)) = H^*(\mathrm{Der}(\mathcal{M}_X), [d, -])$$

is defined by $\Phi(\theta) = \theta'$. Here, the homotopy group $\pi_*(\mathrm{aut}_1(M))$ is regarded as a Lie algebra endowed with the Samelson product.

Theorem 3.1 (KNWY22)

One has a commutative diagram

$$\begin{tikzcd} H^*_{AQ}(\Omega^*(M)) \arrow[r, "{}_{HH}L_{(\ )}", "{}_{HH}e_{(\ )}"'] & \big(\mathrm{End}(HH_*(\Omega^*(M))), B\big) \\ \pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{R} \arrow[u, "\text{Sullivan's iso. } \Phi", "\cong"'] \arrow[r, "L_{(\ )}", "(-1)^*\iota_{(\ )}"'] & \big(\mathrm{Der}(H^*(LM : \mathbb{R})), \Delta\big) \arrow[u, "\text{a monic map}"'] \end{tikzcd}$$

in which each sequence is a Cartan calculus, where $* \geq 2$, Δ *is the BV operator on the loop cohomology. In particular, the calculi give the formulae*

$${}_{HH}L_\eta = [B, {}_{HH}e_\eta] \ \text{ and } \ L_\theta = [\Delta, \pm\iota_\theta]$$

for $\eta \in H^*_{AQ}(\Omega^*(M))$ *and* $\theta \in \pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{R}$*. Moreover, The right vertical map is a monic linear map induced by the isomorphism between the loop cohomology and the Hochshcild homology preserving operators* Δ *and* B*.*

Sketch of the proof.

Over the rational, we use a model $\mathcal{L} = (\wedge V \otimes \wedge \overline{V}, d)$ for LM described in Section 2.
(i) One has a commutative diagram

$$\begin{tikzcd}
H^*(\mathrm{Der}(\wedge V)) \arrow[r, "L_{(\ )}", "(\text{resp.} e_{(\ )})"'] \arrow[dr, "{}_aL_{(\ )}", "(\text{resp.} e_{(\ )})"'] & \mathrm{End}^{-n}(HH_*(\wedge V)) \\
 & \mathrm{End}^{-n}(H_*(\mathcal{L})). \arrow[u, "\text{inclusion}"']
\end{tikzcd}$$

(ii) Let X be a simply-connected space of finite type and $\wedge V$ the minimal model for X. Then there exists a commutative diagram

$$\begin{tikzcd}
\pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q} \arrow[r, "L_{(\ )}", "(-1)^* e_{(\ )}"'] \arrow[d, "\Phi", "\cong"'] & \mathrm{End}^{-*}(H^*(LX)) \arrow[d, "\cong"] \\
H_*(\mathrm{Der}(\wedge V)) \arrow[r, "{}_aL_{(\ )}", "e_{(\ )}"'] & \mathrm{End}^{-*}(H_*(\mathcal{L})).
\end{tikzcd}$$

□

A geometric description of Sullivan's iso. Φ

Félix and Thomas ([FT '04]) define a morphism Γ_1 by the composite

$$\begin{array}{ccc} \pi_n(\Omega\mathrm{aut}_1(M)_0)\otimes\mathbb{Q} & \xrightarrow{\text{Hurewicz map}} & H_n(\Omega\mathrm{aut}_1(M)_0;\mathbb{Q}) \\ & & \Big\downarrow \times[M] \\ & & H_{n+m}(\Omega\mathrm{aut}_1(M)_0\times M;\mathbb{Q}) \xrightarrow{g_*} H_{n+m}(LM;\mathbb{Q}) \end{array}$$

for $n\geq 1$, where $g:\Omega\mathrm{aut}_1(M)_0\times M\to LM$ is defined by $g(\gamma,x)(t)=\gamma(t)(x)$ for $\gamma\in\Omega\mathrm{aut}_1(M)_0$, $x\in M$ and $t\in S^1$.

Theorem 3.2 (KNWY22)

There exists a commutative diagram

$$\begin{tikzcd} \pi_n(\mathrm{aut}_1(M))\otimes\mathbb{Q} \arrow[rr, "\Phi:\text{Sullivan's iso.}", "\cong"'] & & H_n(\mathrm{Der}(\wedge V)) \arrow[d, "\lambda", "\cong"'] \\ \pi_{n-1}(\Omega\mathrm{aut}_1(M)_0)\otimes\mathbb{Q} \arrow[u, "\partial", "\cong"'] \arrow[r, "\Gamma_1\cong", "\text{[FT'04]}"'] & H^{(1)}_{n+m-1}(LM) \arrow[r, "\mathrm{PD}^{-1}", "\cong"'] & HH^{-n+1}_{(1)}(\wedge V), \end{tikzcd}$$

where ∂ is the adjoint map.

The strategy of the proof.

Lannes' division functor $(\wedge V : B)$;

$$\mathbf{CDGA}((\wedge V : B), C) \cong \mathbf{CDGA}(\wedge V, B \otimes C)$$

- $A_{PL}(X) \xleftarrow{\simeq} (\wedge V, d)$: a minimal model for a rational space X
- $A_{PL}(U) \xleftarrow{\simeq} (B, d_B)$: a commutative model for a connected space U

We use *twice* the Haefliger ('82) (Bousfield–Peterson–Smith ('89), Brown–Szczarba ('97)) model for a function space of the form

$$\left| (\text{Sullivan model for } ((\wedge V : B)) \middle/ M_u \right| \simeq \text{a component of } \mathcal{F}(U, X)$$

to construct a model for $g : \Omega\mathrm{aut}_1(M)_0 \times M \to LM$ □

Proposition 3.3 (Loday, 4.1.6 Corollary)

There are well-defined homomorphisms of Lie algebras $[D] \mapsto L_D$*:*

$HH^1(A, A) \to \mathrm{End}_k(HH_n(A))$ *and* $HH^1(A, A) \to \mathrm{End}_k(HC_n(A))$.

§4 An equivariant cohomology (cyclic homology) version of the Lie representation L and its geometrical description

Let X be a simply-connected space. For an element θ in the homotopy group $\pi_n(\mathrm{aut}_1(X))$ for $n > 1$, we define a map u_θ by the composite

$$u_\theta := L(\ ) \circ inc \circ \theta : S^n \longrightarrow \mathrm{aut}_1(X) \longrightarrow \mathrm{map}(X, X) \xrightarrow{L} \mathrm{map}(LX, LX),$$

where inc denotes the inclusion and L is the map which assigns $Lf : LX \to LX$ defined by $Lf(l) = f \circ l$ to a map $f : X \to X$. Then, the adjoint map $ad(u_\theta) : S^n \times LX \to LX$ gives rise to the derivation

$$L_\theta : H^*(LX; \mathbb{K}) \xrightarrow{(ad(u_\theta))^*} H^*(S^n) \otimes H^*(LX; \mathbb{K}) \xrightarrow{\int_{S^n}} H^{*-n}(LX; \mathbb{K})$$

on the cohomology $H^*(LX; \mathbb{K})$ with coefficients in a field $\mathbb{K}$ of arbitrary characteristic, where $\int_{S^n}$ denotes the integration along the fibre. We see that the definition of L_θ coincides with that in (8).

Observe that the adjoint map $ad(u_\theta) : S^n \times LM \to LM$ is an S^1-equivariant map, where the S^1-action on S^n is defined to be trivial. Thus, we have a map $ad(u_\theta) \times_{S^1} 1 : (S^n \times LX) \times_{S^1} ES^1 \to LX \times_{S^1} ES^1$ between the Borel constructions. Therefore, the same construction as that of L_θ with the integration enables us to obtain a derivation

$$\overline{L}_\theta : H^*_{S^1}(LX; \mathbb{K}) \longrightarrow H^{*-n}_{S^1}(LX; \mathbb{K})$$

of degree $-n$.

Theorem 4.1 (KNWY22)

The map

$$\overline{L}_{(\ )} : \pi_*(\mathrm{aut}_1(X)) \to \mathrm{Der}_*(H^*_{S^1}(LX; \mathbb{K})) \tag{10}$$

is a morphism of Lie algebras.

- Over the rational

Let $\wedge V$ be a Sullivan model for X. For a derivation $\theta \in (\mathrm{Der}(\wedge V))$, we define a derivation ${}_a\overline{L}_{(\theta)}$ on $\mathcal{E} := \wedge V \otimes \wedge \overline{V} \otimes \mathbb{Q}[u]$, which is a Sullivan model for $LX \times_{S^1} ES^1$, by ${}_a\overline{L}_{(\theta)} = {}_aL_{(\theta)} \otimes 1_{\mathbb{Q}[u]}$.

Theorem 4.2 (KNWY22)

(i) There exists a commutative diagram

$$\begin{tikzcd} \pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q} \arrow[r, "\overline{L}_{(\ )}"] \arrow[d, "\Phi", "\cong"'] & \mathrm{Der}_*(H^*_{S^1}(LX)) \arrow[d, "\cong"] \\ H_*(\mathrm{Der}(\wedge V)) \arrow[r, "{}_a\overline{L}_{(\ )}"'] & \mathrm{Der}_*(H_*(\mathcal{E})) \end{tikzcd}$$

modulo the filtration of the EMSS in the sense that $({}_a\overline{L}_\theta - \overline{L}_\theta)(F^p) \subset F^{p+1}$ *for* θ *in* $\pi_*(\mathit{aut}_1(X)) \otimes \mathbb{Q}$ *and* $p \geq 0$, *where* $\{F^p\}$ *is the filtration of* $H^*_{S^1}(LX)$ *associated with the EMSS converges to the* S^1*-equivariant cohomology* $H^*_{S^1}(LM;\mathbb{Q}) \cong HC^-_*(\wedge V)$ *with*

$$E_2^{*,*} \cong \mathrm{Cotor}^{*,*}_{H^*(S^1;\mathbb{Q})}(H^*(LM;\mathbb{Q}), \mathbb{Q}).$$

(ii) Suppose that X *is BV exact, then the diagram above is indeed commutative.*

Theorem 4.3

There are the following implications concerning rational homotopy invariants for a simply-connected space X.

$$
\begin{tikzcd}
\boxed{X \text{ is formal}} \arrow[r, Rightarrow, "\text{[H–S'79]}"] & \boxed{\begin{array}{c} X \text{ admits} \\ \text{positive weights} \\ \text{[Body-Douglas'80]} \end{array}} \arrow[dl, Rightarrow, "\text{[KNWY21]}"'] \arrow[rr, Leftrightarrow, "\text{[Scheerer'84]}", "(*)"'] & & \boxed{\begin{array}{c} X \text{ is } p\text{-universal} \\ \text{[Mimura-O'Neil-Toda'71]} \end{array}} & \\
\boxed{X \text{ is (1-)BV exact}} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow, "\text{[KNWY21]}"'] & \boxed{X \text{ is 2-BV exact}} \arrow[r, Rightarrow] & \cdots \arrow[r, Rightarrow] & \boxed{X \text{ is } r\text{-BV exact}} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow, "\text{[KNWY21]}"'] & \cdots \\
\boxed{\begin{array}{l} \text{The } S\text{-action on } \widetilde{H}^*_{S^1}(LX;\mathbb{Q}) \\ \text{is trivial} \end{array}} \arrow[r, Rightarrow] & \cdots \arrow[rr, Rightarrow] & & \boxed{\begin{array}{l} \text{The } r \text{ times } S\text{-action on} \\ \widetilde{H}^*_{S^1}(LX;\mathbb{Q}) \text{ is trivial} \end{array}} &
\end{tikzcd}
$$

Here the S-action on $\widetilde{H}^*_{S^1}(LX;\mathbb{Q})$ is defined by the multiplication of the generator of $\widetilde{H}^*(BS^1;\mathbb{Q})$ with the map induced by the projection q of the fibration $LX \to ES^1 \times_{S^1} LX \xrightarrow{q} BS^1$. Observe that the equivalence (*) holds if X has the homotopy type of a finite CW complex.