Local systems in diffeology

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Rational homotopy theory (RHT) for topological spaces

The de Rham–Sullivan correspondence (Bousfield–Gugenheim '76) gives an equivalence between the homotopy categories

$$\mathsf{fN}\mathbb{Q}\operatorname{-Ho}(\mathsf{Top}) \xrightarrow[||]{Q \circ A_{PL}()} f\mathbb{Q}\operatorname{-Ho}(\mathsf{CDGA}^{op})$$

nilpotent rational connected spaces of finite \mathbb{Q} -type

cofibrant connected commutative differential graded algebras (CDGAs) of finite Q-type

Here Q denotes the cofibrant replacement; that is, one has a quasi-iso.

$$\wedge V_X = ((ext{poly. alg} \otimes ext{exterior alg}), d) \stackrel{\simeq}{ o} A_{PL}(X) := \mathsf{Set}^{\Delta^{\mathrm{op}}}(S(X), A_{PL})$$

for a space X (a Sullivan model for X). By using the equivalence of categories, we translate

- ullet data (informations) of a space to those of a CDGA $(\wedge V, d)$ and
- data of a continuous map f:X o Y to a morphism $Q\circ A_{PL}(f):(\wedge V_Y,d_Y) o (\wedge V_X,d_X)$ of CDGAs.

Some advantages of RHT

• (Numerical) topological invariants for rational spaces has algebraic descriptions (e.g., LS category, topological complexity, rational homotopy groups, their Whitehead products, loop products in string topology ...) In particular, if $\wedge V_X \xrightarrow{\simeq} A_{PL}(X)$ is a *minimal* Sullivan model for a simply-connected space X, then

 $V_X \cong \operatorname{Hom}(\pi_*(X), \mathbb{Q}).$

• An algebraic model for a fibration $F \xrightarrow{i} X \xrightarrow{\pi} K$, so-called the KS(Koszul–Sullivan)-extension

Theorem 1.1 (Halperin '83)

If $\pi_1(K)$ acts on $H^*(F; \mathbb{Q})$ nilpotently, then u is a quasi-isomorphism; that is, $\wedge W$ is a (minimal) Sullivan model for the fibre F.

RHT for nilpotent diffeological spaces

Theorem 1.2 (Kihara '21)

There exists a pair of Quillen equivalences

the category of simplicial sets with the usual (classical) model structure

$$\mathsf{Set}^{\Delta^{op}} \xrightarrow{| \ |_{D}} \mathsf{Diff}_{S^{D}(\neg)_{\bullet} := C^{\infty}(\Delta^{\bullet}, \ -)}$$

the category of diffeological spaces with the model structure due to Kihara

Theorem 1.3 (RHT for nilpotent diffeological spaces)

$$f\mathbb{Q}\text{-Ho}(\mathsf{CDGA}^{op}) \xrightarrow[Q \circ A_{PL}(S^{D}(\cdot))]{} f\mathbb{N}\mathbb{Q}\text{-Ho}(\mathsf{Diff})$$

We would like to develop RHT for diffeological spaces with *arbitrary* π_1 's in **Diff**!

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 Gómez-Tato, Halperin and Tanré [GHT] give an equivalence of homotopy categories

 $\operatorname{Ho}(\mathcal{M}_{\mathbb{Q}}) \xrightarrow{\simeq} \operatorname{fib}\mathbb{Q}\operatorname{-Ho}(\operatorname{Top}_{*}).$

- Extracting completely the simplicial argument from RHT for topological spaces in [GHT] and combining it with the pointed version of the model category on **Diff** due to Kihara [Ki21], we develop rational homotopy theory for diffeological spaces [K22].
- [GHT] A. Gómez-Tato, S. Halperin and D. Tanré, Rational homotopy theory for non-simply connected spaces, Transactions of AMS, 352 (2000), 1493– 1525.
- [Ki21] H. Kihara, Smooth homotopy of infinite-dimensional C[∞]-manifolds, to appear in Memoirs of the AMS, 2021, arXiv:2002.03618.
- [K22] K. Kuribayashi, Local systems in diffeology, preprint (2022). arXiv:2108.13084v2.

RHT for non-simply connected diffeological spaces

- A pointed connected Kan complex X is *fibrewise rational* if the universal cover \widetilde{X} of X is rational and finite \mathbb{Q} -type; that is, $H_i(\widetilde{X};\mathbb{Z})$ is a finite dimensional vector space over \mathbb{Q} for $i \geq 2$.
- We call a pointed connected diffeological space M <u>fibrewise rational</u> if so is the Kan complex $S^D(M)$. Our main result is described as follows.

Theorem 2.1 (K. '22)

Let $Ho(Diff_*)$ be the homotopy category of pointed diffeological spaces and $fibQ-Ho(Diff_*)$ the full subcategory of $Ho(Diff_*)$ consisting of fibrewise rational connected diffeological spaces. Then, there exists an equivalence of categories

 $\operatorname{Ho}(\mathcal{M}) \xrightarrow{\simeq} \operatorname{fib}\mathbb{Q}\operatorname{-Ho}(\operatorname{Diff}_*),$

where $Ho(\mathcal{M})$ is the homotopy category of minimal local systems introduced by Gómez-Tato, Halperin and Tanré (2000).

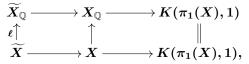
Fibrewise rational diffeological spaces

• Let M be a pointed connected diffeological space. Then, for the simplicial set $X := S^D(M)$, we have a fibration of the form

$$\widetilde{X} \to X \to K(\pi_1(X), 1)$$

in which \widetilde{X} is the universal cover of X.

• Let $X_{\mathbb{Q}}$ denote the *fibrewise rationalization* of X in the sense of Bousfield and Kan. By definition, the rationalization fits into the commutative diagram

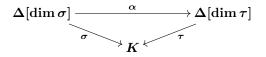


whose upper row sequence is also a fibration and l is the classical rationalization of the simply-connected simplicial set \widetilde{X} .

We call the realization $|S^D(M)_{\mathbb{Q}}|_D$ the <u>fibrewise rationalization</u> of M and denote it by $M_{\mathbb{Q}}$.

A local system on a simplicial set with values in CDGAs

• For a simplicial set K, we may regard K as a category whose objects are simplicial maps $\sigma : \Delta[n] \to K$ and whose morphisms $\alpha : \sigma \to \tau$ are the simplicial maps α



Definition 2.2 (Halperin '83)

(1) A local system E is a presheaf on a simplicial set K with values in CDGAs. (2) A morphism ψ : $E \rightarrow E'$ of local systems over K is a morphism of presheaves whose image ψ_{σ} : $E_{\sigma} \rightarrow E'_{\sigma}$ is a morphisms of CDGAs for each $\sigma \in K$. • Let E be a local system over a simplicial set K and $u: L \to K$ a simplicial map. Then, the *pullback* E^u of E is defined by $(E^u)_{\sigma} = E_{u \circ \sigma}$ for $\sigma \in L$.

Definition 2.3

A local system E is <u>extendable</u> if the restriction map

$$\Gamma(E^{\sigma}) \to \Gamma(E^{\sigma \circ i})$$

is surjective for any simplicial map $\sigma : \Delta[n] \to K$, where $i : \partial \Delta[n] \to \Delta[n]$ is the inclusion and $\Gamma : \mathsf{CDGA}^{K^{\mathrm{op}}} \to \mathsf{CDGA}$ is the <u>global section functor</u> defined by $\Gamma(E) = \mathsf{Set}^{K^{\mathrm{op}}}(1, E)$.

In what follows, A_● denotes the simplicial CDGA (A^{*}_{PL})_● of polynomial differential forms over Q.

$$(A_{PL})_n := \wedge (t_0,...,t_n,y_0,...,y_n) / (\sum t_i - 1,\sum y_i),$$

with $d(t_i) = y_i$, where deg $t_i = 0$ and deg $y_j = 1$.

Minimal local systems

Definition 2.4 (Gómez-Tato, Halperin and Tanré (2000))

An A-algebra $j_E: A_{\bullet} \to E$ is a morphism of local systems on a simplicial set K for which E is extendable and the system H(E) is locally constant; that is, $lpha^*: E_{ au} \stackrel{\simeq}{ o} E_{\sigma}$ for $lpha: \sigma o au$. An *A*-morphism is a morphism arphi: E o E' of local systems such that $\varphi \circ j_E = j_{E'}$.

 A_{\bullet} is a local system with $A_{\sigma} := A_{\dim \sigma}$.

Definition 2.5 (G-H-T '00)

(1) A local system ($\wedge Y, D_0$) with values in CDGAs is a 1-connected A^0 *minimal model* if there exists a 1-connected Sullivan minimal algebra $(\wedge Z, d)$ such that, as differential graded $(A^0_{\bullet})^{\sigma}$ -algebras, $(\wedge Y)^{\sigma} \cong (A^0_{\bullet})^{\sigma} \otimes (\wedge Z, d)$ for $\sigma \in K$.

(2) An
$$A$$
-algebra $(A_{ullet}\otimes_{A^0}\wedge Y, D=\sum_{i\geq 0}D_i)$ with

$$D_i: A^*_ullet\otimes_{A^0} \wedge Y o A^{*+i}_ullet\otimes_{A^0} \wedge Y$$

is a 1-connected A minimal model if $(\wedge Y, D_0)$ is a 1-connected A⁰ minimal model. (D_i decreases the degree of $\wedge Y$ by (i-1).)

Definition 2.6

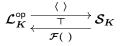
The category \mathcal{M} of minimal local systems:

- (1) Each object (E_K, K) in \mathcal{M} called a <u>minimal local system</u>, is a pair of a $K(\pi, 1)$ -simplicial set K in $\operatorname{Set}_{*}^{\Delta^{\operatorname{op}}}$ and a 1-connected A-minimal model E_K over K.
- (2) A morphism $(\varphi, u) : (E_K, K) \to (E'_{K'}, K')$ in \mathcal{M} is a pair of a based simplicial map $u : K \to K'$ and a morphism $\varphi : E_K \leftarrow (E'_{K'})^u$ of A-algebras over K (a A-morphism), where $(E'_{K'})^u$ denotes the pullback of $E'_{K'}$ along u.
 - The composition in \mathcal{M} is defined naturally with the functoriality of pullbacks of local systems along simplicial maps.
 - We may define the notion of *homotopy* in *M* with *cylinders*. That gives us the homotopy category

 $Ho(\mathcal{M})$

of minimal local systems.

Let L_K be the category of morphisms from A_• to local systems over a connected simplicial set K. Let S_K be the category of simplicial sets over K. We recall the adjoint functors



introduced in [G-H-T]. The realization functor $\langle \rangle$ is defined by

$$\langle (E,j) \rangle_n = \{ (\varphi,\sigma) \mid \sigma \in K_n, \varphi \in \mathcal{L}_{\Delta[n]}(E^{\sigma}, A_{\bullet}) \}$$
 (2)

for $j: A_{\bullet} \to E$, where E^{σ} denotes the pullback of the local system E over K along the map $\sigma: \Delta[n] \to K$. Observe that φ is a natural transformation.

ullet For a morphism p:X o K of simplicial sets, we define a local system $\mathcal{F}(X,p)$ to be

$$\mathcal{F}(X,p)_{\sigma} = A(X^{\sigma}) := \mathsf{Set}^{\Delta^{\circ \rho}}(X^{\sigma}, A_{\bullet})$$
(3)

for $\sigma \in K$, where X^σ is the pullback of p along $\sigma.$

Rational homotopy theory for diffeological spaces

Theorem 2.7 (K. '22)

The realization functor $\langle \rangle : Ho(\mathcal{M}) \to fibQ-Ho(Set_*^{\circ p})$ gives an equivalence of categories. Here $fibQ-Ho(Set_*^{\circ p})$ denotes the full subcategory of pointed connected fibrewise rational Kan complexes.

• We have a sequence containing equivalences of homotopy categories and an embedding

$$\mathsf{Ho}(\mathcal{M}) \xrightarrow{\langle \rangle}{\simeq} \mathsf{fib}\mathbb{Q} \operatorname{-}\mathsf{Ho}(\mathsf{Set}_*^{\Delta^{\mathsf{op}}}) \xrightarrow{i}{\subset} \mathsf{Ho}(\mathsf{Set}_*^{\Delta^{\mathsf{op}}}) \xrightarrow[]{i \mid D}{\underset{S^D(\cdot)}{\cong}} \mathsf{Ho}(\mathsf{Diff}_*).$$

Theorem 2.8 (K. '22 (RHT for diffeological spaces with arbitrary π_1 's)) One has an equivalence of categories

$$\operatorname{Ho}(\mathcal{M}) \xrightarrow{\simeq} \operatorname{fib}\mathbb{Q}\operatorname{-Ho}(\operatorname{Diff}_*).$$

Theorem 2.9 (comes from G-H-T '00 essentially)

Let M be a pointed connected diffeological space. Suppose that the cohomology of $A(\widetilde{S^D(M)}) := \operatorname{Set}^{\Delta^{\operatorname{op}}}(\widetilde{S^D(M)}, A_{\bullet})$ of the fibre $\widetilde{S^D(M)}$ of p below is of finite type. Then there is a 1-connected A minimal model

$$m: (A_{ullet}\otimes_{A^0_{ullet}}\wedge Y, D) \stackrel{\simeq}{ o} \mathcal{F}(S^D(M), p).$$

Moreover, the minimal model m gives rise to a fibrewise rationalization, called an A-localization, $ad(m) : S^D(M) \to \langle (A_{\bullet} \otimes_{A_{\bullet}^0} \wedge Y, D) \rangle$ which fits into the commutative diagram

$$\begin{array}{ccc} \widetilde{S^{D}(M)}_{A} & \longrightarrow \langle (A_{\bullet} \otimes_{A_{\bullet}^{0}} \wedge Y, D) \rangle \overset{\pi}{\longrightarrow} K(\pi_{1}(M), 1) \\ & \stackrel{\ell}{\frown} & \uparrow^{ad(m)} & & \\ & \widetilde{S^{D}(M)} & \longrightarrow S^{D}(M) \overset{p}{\longrightarrow} K(\pi_{1}(M), 1) \end{array}$$

consisting of two Kan fibrations π and p, where ℓ is the usual localization.

The fibrewise rationalization $M_{\mathbb{Q}}$ of a diffeological space M:

$$|\langle (A_{ullet}\otimes_{A^0_{ullet}}\wedge Y,D)
angle|_D=M_{\mathbb{Q}}$$

Examples

Comparatively tractable examples

• Let K be a simplicial set and $A(K) := \operatorname{Set}^{\Delta^{\operatorname{op}}}(K, A_{\bullet})$ the <u>polynomial</u> <u>de Rham complex</u> of a simplicila set K. In particular, for a diffeological space M, we have an isomorphism

 $H^*(A(S^D(M)))\cong H^*(M;\mathbb{Q})\,$ the singular cohomology of M

• We recall the local system R_* associated with a relative Sullivan algebra (KS extension) $A(K) \xrightarrow{i} R \to (T, d_T)$ in which T is simply connected. We observe that

$$R \cong A(K) \otimes T$$

as an algebra. For a simplex $\sigma:\Delta[n] o K$, we define a CDGA $(R_*)_\sigma$ by

$$A(n)\otimes_{e_{\sigma}}R=A(n)\otimes_{e_{\sigma},A(K)}(A(K)\otimes T),$$

where $e_{\sigma}: A(K) \to A(\Delta[n]) =: A(n)$ is the morphism of CDGAs induced by σ and the tensor product of CDGAs stands for the pushout of the diagram

$$A(n) \stackrel{e_\sigma}{\leftarrow} A(K) \stackrel{i}{
ightarrow} A(K) \otimes T = R.$$

Examples

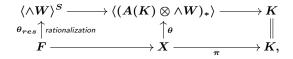
Lemma 2.10 (the $()_*$ -construction)

The natural map $j: A \to R_*$ induced by $j_{\sigma}: A(n) \to A(n) \otimes_{e_{\sigma}} R$ for $\sigma \in K_n$ is a 1-connected A minimal model.

Let $: F \to X \xrightarrow{\pi} K := K(\pi_1(X), 1)$ be a fibration with 1-connected fibre.

Proposition 2.11 (via the ()_{*}-construction in Lemma 2.10)

Suppose that $u: \wedge W \to A(F)$ is a quasi-isomorphism. Then there exists a commutative diagram



in which $\theta: X \to \langle (A(K) \otimes \wedge W)_* \rangle$ is the fibrewise rationalization.

Example 2.12 (A more concrete example)

Let M be a simply-connected manifold and $L^{\infty}M := \text{Diff}(S^1, M)$ the free loop space endowed with the functional diffeology. We construct the minimal local system model for $L^{\infty}M$ over \mathbb{Q} applying Proposition 2.11.

The strategy is as follows.

• Consider a fibration of the form

$$S^{D}(\widetilde{L^{\infty}M}) \longrightarrow S^{D}(L^{\infty}M) \xrightarrow[\pi]{} K(\pi_{1}(L^{\infty}M), 1) =: K$$

• We have a KS-extension via the "smoothing theorem" due to Kihara

• We have a minimal local system model $(A(K)\otimes\wedge W)_*$ for $L^\infty M.$

• For instance, let M be the complex projective space $\mathbb{C}P^n$. It is well-known that a minimal model for $C^0(S^1, \mathbb{C}P^n)$ is of the form

 $(\wedge(x,y)\otimes\wedge(\overline{x},\overline{y}),d)$

with $d(x) = d(\overline{x}) = 0$, $d(y) = x^{n+1}$ and $d(\overline{y}) = (n+1)\overline{x}x^n$, where $\deg x = 2$, $\deg y = 2(n+1)-1$, $\deg \overline{x} = 1$ and $\deg \overline{y} = 2(n+1)-2$.

• Then, we have a minimal local system model for $L^{\infty}\mathbb{C}P^n$ over $K:=K(\pi^D_1(L^{\infty}\mathbb{C}P^n),1)$ of the form

$$R_*:=(A(K)\otimes\wedge(x,y,\overline{y}))_*$$

for which there exist isomorphisms of CDGA's for each $\sigma \in K_n$

 $((R_*)_\sigma, d) \cong A(n) \otimes_{e_\sigma, A(K)} (A(K) \otimes \wedge (x, y, \overline{y})) \cong A(n) \otimes \wedge (x, y, \overline{y}),$ where d(x) = 0, $d(y) = x^{n+1}$ and

$$d(\overline{y}) = (n+1)(e_{\sigma} \circ v)(\overline{x}) \otimes x^{n}.$$

For \mathbb{R} -local (real) homotopy theory, we can use a simplicial CDGA

$$(A^*_{DR})_{\bullet}:=\{\Omega^*(\mathbb{A}^n)\}_{n\geq 0}$$

over \mathbb{R} , where $\Omega^*(\mathbb{A}^n)$ is the Souriau-de Rham complex of the affine space

$$\mathbb{A}^n := \left\{ (x_0,...,x_n) \in \mathbb{R}^{n+1} \middle| \left| \sum_{i=0}^n x_i = 1
ight\}$$

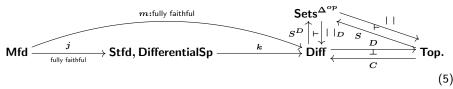
equipped with the sub-diffeology of the manifold \mathbb{R}^{n+1} . Observe that

$$A^*_{PL}(S^D(M))\otimes \mathbb{R}\simeq A^*_{DR}(S^D(M)) \stackrel{ ext{a morphism of CDGAs}}{\xleftarrow{\exists lpha: ext{ the factor map}}} \Omega^*(M).$$

We may investigate

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- Cartan-de Rham calculus (Calculus of differential forms) on Diff
- with Rational homotopy theory for diffeological spaces.



Local systems in deffeology

An application of local systems to a construction of a spectral sequence for an adjunction diffeological space

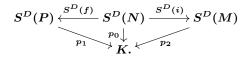
We have

- the Leray–Serre spectral sequence for the fibration in the sense of Chriatensen and Wu
- the Eilenberg–Moore spectral sequence for a pullback diagram
- and also a spectral sequence for an adjunction space in Diff.

Let $P \xleftarrow{f} N \xrightarrow{i} M$ be maps between connected diffeological spaces. These maps produce the diffeological adjunction space $P \cup_N M$ in **Diff** together with the quotient diffeology with respect to the projection $p: P \coprod M \to P \cup_N M$.

Theorem 3.1

Suppose that the map i is injective and that p_0 , p_1 and p_2 are Kan fibrations over a pointed connected simplicial set K in a commutative diagram



Then, there exists a first quadrant spectral sequence $\{E_r^{*,*}, d_r\}$ with

$$E_2^{*,*} \cong H^*(K, \mathcal{H}^*_{\mathcal{P}}) \Longrightarrow H^*(A(S^D(P) \cup_{S^D(N)} S^D(M)),$$

where $\mathcal{H}^*_{\mathcal{P}}$ is a local coefficients satisfies the condition that, for any $\sigma \in K$, one has an isomorphism $(\mathcal{H}^*_{\mathcal{P}})_{\sigma} \cong H^*(A(F_1) \times_{A(F_0)} A(F_2))$ with F_i the fibre of p_i for i = 0, 1 and 2.

Suppose that a sequence $S \xleftarrow{f} \partial W \xrightarrow{i} W$ of manifolds gives a *stratifold* $S \cup_f W$ in the sense of Kreck (for example, W has a collar). The SS converges to

$$H^*(\Omega^*(S \cup_f W)) \xrightarrow{\alpha}{\cong} H^*(A^*_{DR}(S^D(S \cup_f W))).$$