Local systems in diffeology

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Rational homotopy theory (RHT) for topological spaces

The de Rham–Sullivan correspondence (Bousfield–Gugenheim '76) gives an equivalence between the homotopy categories

$$\text{fN}\mathbb{Q}\text{-Ho}(\mathbf{Top}) \underset{|\ |}{\overset{Q\circ A_{PL}(\ )}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \text{f}\mathbb{Q}\text{-Ho}(\mathbf{CDGA}^{op})$$

nilpotent rational connected spaces of finite $\mathbb{Q}$-type

cofibrant connected commutative differential graded algebras (CDGAs) of finite $\mathbb{Q}$-type

Here Q denotes the cofibrant replacement; that is, one has a quasi-iso.

$$\wedge V_X = ((\text{poly. alg} \otimes \text{exterior alg}), d) \xrightarrow{\simeq} A_{PL}(X) := \mathsf{Set}^{\Delta^{\text{op}}}(S(X), A_{PL})$$

for a space X (a *Sullivan model* for X). By using the equivalence of categories, we translate

- data (informations) of a space to those of a CDGA $(\wedge V, d)$ and
- data of a continuous map $f : X \to Y$ to a morphism $Q \circ A_{PL}(f) : (\wedge V_Y, d_Y) \to (\wedge V_X, d_X)$ of CDGAs.

Some advantages of RHT

- (Numerical) topological invariants for rational spaces has algebraic descriptions (e.g., LS category, topological complexity, rational homotopy groups, their Whitehead products, loop products in string topology ...)
  In particular, if $\wedge V_X \xrightarrow{\simeq} A_{PL}(X)$ is a *minimal* Sullivan model for a simply-connected space X, then
  $$V_X \cong \mathrm{Hom}(\pi_*(X), \mathbb{Q}).$$
- An algebraic model for a fibration $F \xrightarrow{i} X \xrightarrow{\pi} K$, so-called the KS(Koszul–Sullivan)-extension

$$\begin{tikzcd}
\wedge V \arrow[r, "j"] \arrow[d, "\simeq"', "v"] & \wedge V \otimes \wedge W \arrow[r, "\rho"] \arrow[d, "\simeq"', "\alpha"] & \wedge W \arrow[d, "u"] \\
A_{PL}(K) \arrow[r, "\pi^*"'] & A_{PL}(X) \arrow[r, "i^*"'] & A_{PL}(F)
\end{tikzcd} \tag{1}$$

Theorem 1.1 (Halperin '83)

If $\pi_1(K)$ acts on $H^*(F;\mathbb{Q})$ nilpotently, then u is a quasi-isomorphism; that is, $\wedge W$ is a (minimal) Sullivan model for the fibre F.

RHT for nilpotent diffeological spaces

Theorem 1.2 (Kihara '21)

There exists a pair of Quillen equivalences

the category of simplicial sets with the usual (classical) model structure

$$\mathbf{Set}^{\Delta^{op}} \underset{S^D(-)_\bullet := C^\infty(\Delta^\bullet,\ -)}{\overset{|\ |_D}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Diff}$$

the category of diffeological spaces with the model structure due to Kihara

Theorem 1.3 (RHT for nilpotent diffeological spaces)

$$\text{f}\mathbb{Q}\text{-Ho}(\mathbf{CDGA}^{op}) \underset{Q\circ A_{PL}(S^D(\ ))}{\overset{|\langle\ \rangle|_D}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \text{fN}\mathbb{Q}\text{-Ho}(\mathbf{Diff})$$

We would like to develop RHT for diffeological spaces with *arbitrary* π_1's in **Diff**!

- Gómez-Tato, Halperin and Tanré [GHT] give an equivalence of homotopy categories

$$\mathrm{Ho}(\mathcal{M}_{\mathbb{Q}}) \xrightarrow{\simeq} \mathrm{fib}\mathbb{Q}\text{-}\mathrm{Ho}(\mathbf{Top}_*).$$

- Extracting completely the simplicial argument from RHT for topological spaces in [GHT] and combining it with the pointed version of the model category on **Diff** due to Kihara [Ki21], we develop rational homotopy theory for diffeological spaces [K22].

[GHT] A. Gómez-Tato, S. Halperin and D. Tanré, Rational homotopy theory for non-simply connected spaces, Transactions of AMS, **352** (2000), 1493–1525.

[Ki21] H. Kihara, Smooth homotopy of infinite-dimensional C^∞-manifolds, to appear in Memoirs of the AMS, 2021, `arXiv:2002.03618`.

[K22] K. Kuribayashi, Local systems in diffeology, preprint (2022). `arXiv:2108.13084v2`.

RHT for non-simply connected diffeological spaces

- A pointed connected Kan complex X is *fibrewise rational* if the universal cover $\widetilde{X}$ of X is rational and finite $\mathbb{Q}$-type; that is, $H_i(\widetilde{X};\mathbb{Z})$ is a finite dimensional vector space over $\mathbb{Q}$ for $i \geq 2$.
- We call a pointed connected diffeological space M *fibrewise rational* if so is the Kan complex $S^D(M)$. Our main result is described as follows.

Theorem 2.1 (K. '22)

Let $\mathrm{Ho}(\mathbf{Diff}_*)$ *be the homotopy category of pointed diffeological spaces and* fib$\mathbb{Q}$-$\mathrm{Ho}(\mathbf{Diff}_*)$ *the full subcategory of* $\mathrm{Ho}(\mathbf{Diff}_*)$ *consisting of fibrewise rational connected diffeological spaces. Then, there exists an equivalence of categories*

$$\mathrm{Ho}(\mathcal{M}) \xrightarrow{\simeq} \mathrm{fib}\mathbb{Q}\text{-}\mathrm{Ho}(\mathbf{Diff}_*),$$

where $\mathrm{Ho}(\mathcal{M})$ *is the homotopy category of* minimal local systems *introduced by Gómez-Tato, Halperin and Tanré (2000).*

Fibrewise rational diffeological spaces

- Let M be a pointed connected diffeological space. Then, for the simplicial set $X := S^D(M)$, we have a fibration of the form

$$\widetilde{X} \to X \to K(\pi_1(X), 1)$$

in which $\widetilde{X}$ is the universal cover of X.

- Let $X_{\mathbb{Q}}$ denote the *fibrewise rationalization* of X in the sense of Bousfield and Kan. By definition, the rationalization fits into the commutative diagram

$$\begin{tikzcd} \widetilde{X}_{\mathbb{Q}} \arrow[r] & X_{\mathbb{Q}} \arrow[r] & K(\pi_1(X), 1) \\ \widetilde{X} \arrow[u, "\ell"] \arrow[r] & X \arrow[u] \arrow[r] & K(\pi_1(X), 1), \arrow[u, equal] \end{tikzcd}$$

whose upper row sequence is also a fibration and l is the classical rationalization of the simply-connected simplicial set $\widetilde{X}$.

We call the realization $|S^D(M)_{\mathbb{Q}}|_D$ the *fibrewise rationalization* of M and denote it by $M_{\mathbb{Q}}$.

A local system on a simplicial set with values in CDGAs

- For a simplicial set K, we may regard K as a category whose objects are simplicial maps $\sigma : \Delta[n] \to K$ and whose morphisms $\alpha : \sigma \to \tau$ are the simplicial maps α

$$\begin{tikzcd} \Delta[\dim \sigma] \arrow[rr, "\alpha"] \arrow[dr, "\sigma"'] & & \Delta[\dim \tau] \arrow[dl, "\tau"] \\ & K & \end{tikzcd}$$

Definition 2.2 (Halperin '83)

(1) A *local system* E is a presheaf on a simplicial set K with values in CDGAs.
(2) A morphism $\psi : E \to E'$ of local systems over K is a morphism of presheaves whose image $\psi_\sigma : E_\sigma \to E'_\sigma$ is a morphisms of CDGAs for each $\sigma \in K$.

- Let E be a local system over a simplicial set K and $u : L \to K$ a simplicial map. Then, the *pullback* E^u of E is defined by $(E^u)_\sigma = E_{u\circ\sigma}$ for $\sigma \in L$.

Definition 2.3

A local system E is *extendable* if the restriction map

$$\Gamma(E^\sigma) \to \Gamma(E^{\sigma\circ i})$$

is surjective for any simplicial map $\sigma : \Delta[n] \to K$, where $i : \partial\Delta[n] \to \Delta[n]$ is the inclusion and $\Gamma : \mathsf{CDGA}^{K^{\mathrm{op}}} \to \mathsf{CDGA}$ is the *global section functor* defined by $\Gamma(E) = \mathsf{Set}^{K^{\mathrm{op}}}(1, E)$.

- In what follows, $A_\bullet$ denotes the simplicial CDGA $(A^*_{PL})_\bullet$ of polynomial differential forms over $\mathbb{Q}$.

$$(A_{PL})_n := \wedge(t_0, ...t_n, y_0, ..., y_n)/(\sum t_i - 1, \sum y_i),$$

with $d(t_i) = y_i$, where $\deg t_i = 0$ and $\deg y_j = 1$.

Definition 2.4 (Gómez-Tato, Halperin and Tanré (2000))

An *A-algebra* $j_E : A_\bullet \to E$ is a morphism of local systems on a simplicial set K for which E is extendable and the system $H(E)$ is locally constant; that is, $\alpha^* : E_\tau \overset{\simeq}{\to} E_\sigma$ for $\alpha : \sigma \to \tau$. An *A-morphism* is a morphism $\varphi : E \to E'$ of local systems such that $\varphi \circ j_E = j_{E'}$.

$A_\bullet$ is a local system with $A_\sigma := A_{\dim \sigma}$.

Definition 2.5 (G-H-T '00)

(1) A local system $(\wedge Y, D_0)$ with values in CDGAs is a *1-connected A^0 minimal model* if there exists a 1-connected Sullivan minimal algebra $(\wedge Z, d)$ such that, as differential graded $(A^0_\bullet)^\sigma$-algebras, $(\wedge Y)^\sigma \cong (A^0_\bullet)^\sigma \otimes (\wedge Z, d)$ for $\sigma \in K$.

(2) An A-algebra $(A_\bullet \otimes_{A^0} \wedge Y, D = \sum_{i \geq 0} D_i)$ with

$$D_i : A^*_\bullet \otimes_{A^0} \wedge Y \to A^{*+i}_\bullet \otimes_{A^0} \wedge Y$$

is a 1-*connected* A *minimal model* if $(\wedge Y, D_0)$ is a 1-connected A^0 minimal model. (D_i decreases the degree of $\wedge Y$ by $(i-1)$.)

Definition 2.6

The category $\mathcal{M}$ of minimal local systems:

(1) Each object (E_K, K) in $\mathcal{M}$ called a <u>*minimal local system*</u>, is a pair of a $K(\pi, 1)$-simplicial set K in $\mathbf{Set}_*^{\Delta^{\mathrm{op}}}$ and a 1-connected A-minimal model E_K over K.

(2) A morphism $(\varphi, u) : (E_K, K) \to (E'_{K'}, K')$ in $\mathcal{M}$ is a pair of a based simplicial map $u : K \to K'$ and a morphism $\varphi : E_K \leftarrow (E'_{K'})^u$ of A-algebras over K (a A-morphism), where $(E'_{K'})^u$ denotes the pullback of $E'_{K'}$ along u.

- The composition in $\mathcal{M}$ is defined naturally with the functoriality of pullbacks of local systems along simplicial maps.
- We may define the notion of *homotopy* in $\mathcal{M}$ with *cylinders*. That gives us the homotopy category

$$\mathrm{Ho}(\mathcal{M})$$

of minimal local systems.

- Let $\mathcal{L}_K$ be the category of morphisms from $A_\bullet$ to local systems over a connected simplicial set K. Let $\mathcal{S}_K$ be the category of simplicial sets over K. We recall the adjoint functors

$$\begin{tikzcd} \mathcal{L}_K^{\mathrm{op}} \arrow[r, shift left, "\langle\ \rangle"] \arrow[r, phantom, "\top"] & \mathcal{S}_K \arrow[l, shift left, "\mathcal{F}(\ )"] \end{tikzcd}$$

introduced in [G-H-T]. The realization functor $\langle\ \rangle$ is defined by

$$\langle (E, j)\rangle_n = \{(\varphi, \sigma) \mid \sigma \in K_n, \varphi \in \mathcal{L}_{\Delta[n]}(E^\sigma, A_\bullet)\} \tag{2}$$

for $j : A_\bullet \to E$, where E^σ denotes the pullback of the local system E over K along the map $\sigma : \Delta[n] \to K$. Observe that φ is a natural transformation.

- For a morphism $p : X \to K$ of simplicial sets, we define a local system $\mathcal{F}(X, p)$ to be

$$\mathcal{F}(X, p)_\sigma = A(X^\sigma) := \mathbf{Set}^{\Delta^{\mathrm{op}}}(X^\sigma, A_\bullet) \tag{3}$$

for $\sigma \in K$, where X^σ is the pullback of p along σ.

Rational homotopy theory for diffeological spaces

Theorem 2.7 (K. '22)

The realization functor $\langle\ \rangle : \mathrm{Ho}(\mathcal{M}) \to \mathrm{fib}\mathbb{Q}\text{-}\mathrm{Ho}(\mathbf{Set}_*^{\mathbf{\Delta}^{op}})$ *gives an equivalence of categories. Here* $\mathrm{fib}\mathbb{Q}\text{-}\mathrm{Ho}(\mathbf{Set}_*^{\mathbf{\Delta}^{op}})$ *denotes the full subcategory of pointed connected fibrewise rational Kan complexes.*

- We have a sequence containing equivalences of homotopy categories and an embedding

$$\mathrm{Ho}(\mathcal{M}) \xrightarrow[\simeq]{\langle\ \rangle} \mathrm{fib}\mathbb{Q}\text{-}\mathrm{Ho}(\mathbf{Set}_*^{\mathbf{\Delta}^{\mathrm{op}}}) \xrightarrow[\subset]{i} \mathrm{Ho}(\mathbf{Set}_*^{\mathbf{\Delta}^{\mathrm{op}}}) \underset{S^D(\ )}{\overset{|\ |_D}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathrm{Ho}(\mathbf{Diff}_*).$$

Theorem 2.8 (K. '22 (RHT for diffeological spaces with arbitrary π_1's))

One has an equivalence of categories

$$\mathrm{Ho}(\mathcal{M}) \xrightarrow{\simeq} \mathrm{fib}\mathbb{Q}\text{-}\mathrm{Ho}(\mathbf{Diff}_*).$$

Theorem 2.9 (comes from G-H-T '00 essentially)

Let M be a pointed connected diffeological space. Suppose that the cohomology of $A(\widetilde{S^D(M)}) := \mathbf{Set}^{\Delta^{op}}(\widetilde{S^D(M)}, A_\bullet)$ of the fibre $\widetilde{S^D(M)}$ of p below is of finite type. Then there is a 1-connected A minimal model

$$m : (A_\bullet \otimes_{A^0_\bullet} \wedge Y, D) \xrightarrow{\simeq} \mathcal{F}(S^D(M), p).$$

Moreover, the minimal model m gives rise to a fibrewise rationalization, called an A-localization, $ad(m) : S^D(M) \to \langle (A_\bullet \otimes_{A^0_\bullet} \wedge Y, D)\rangle$ which fits into the commutative diagram

$$\begin{tikzcd}
\widetilde{S^D(M)}_A \arrow[r] & \langle (A_\bullet \otimes_{A^0_\bullet} \wedge Y, D)\rangle \arrow[r, "\pi"] & K(\pi_1(M), 1) \arrow[d, equal] \\
\widetilde{S^D(M)} \arrow[u, "\ell"] \arrow[r] & S^D(M) \arrow[u, "ad(m)"'] \arrow[r, "p"'] & K(\pi_1(M), 1)
\end{tikzcd}$$

consisting of two Kan fibrations π and p, where ℓ is the usual localization.

The fibrewise rationalization $M_{\mathbb{Q}}$ of a diffeological space M:

$$|\langle (A_\bullet \otimes_{A^0_\bullet} \wedge Y, D)\rangle|_D = M_{\mathbb{Q}}$$

Comparatively tractable examples

- Let K be a simplicial set and $A(K) := \mathbf{Set}^{\Delta^{\mathrm{op}}}(K, A_\bullet)$ the *polynomial de Rham complex* of a simplicila set K. In particular, for a diffeological space M, we have an isomorphism

$$H^*(A(S^D(M))) \cong H^*(M;\mathbb{Q}) \text{ the singular cohomology of } M$$

- We recall the local system R_* associated with a relative Sullivan algebra (KS extension) $A(K) \xrightarrow{i} R \to (T, d_T)$ in which T is simply connected. We observe that

$$R \cong A(K) \otimes T$$

as an algebra. For a simplex $\sigma : \Delta[n] \to K$, we define a CDGA $(R_*)_\sigma$ by

$$A(n) \otimes_{e_\sigma} R = A(n) \otimes_{e_\sigma, A(K)} (A(K) \otimes T),$$

where $e_\sigma : A(K) \to A(\Delta[n]) =: A(n)$ is the morphism of CDGAs induced by σ and the tensor product of CDGAs stands for the pushout of the diagram

$$A(n) \xleftarrow{e_\sigma} A(K) \xrightarrow{i} A(K) \otimes T = R.$$

Lemma 2.10 (the $(\ )_*$-construction)

The natural map $j : A \to R_*$ induced by $j_\sigma : A(n) \to A(n) \otimes_{e_\sigma} R$ for $\sigma \in K_n$ is a 1-connected A minimal model.

Let : $F \to X \xrightarrow{\pi} K := K(\pi_1(X), 1)$ be a fibration with 1-connected fibre.

$$\begin{tikzcd} \wedge V \arrow[r, "i"] \arrow[d, "v", "\simeq"'] & \wedge V \otimes \wedge W \arrow[r, "\rho"] \arrow[d, "\alpha", "\simeq"'] & \wedge W \arrow[d, "u"] \\ A(K) \arrow[r, "\pi^*"'] & A(X) \arrow[r] & A(F), \end{tikzcd} \tag{4}$$

Proposition 2.11 (via the $(\ )_*$-construction in Lemma 2.10)

Suppose that $u : \wedge W \to A(F)$ is a quasi-isomorphism. Then there exists a commutative diagram

$$\begin{tikzcd} \langle \wedge W \rangle^S \arrow[r] & \langle (A(K) \otimes \wedge W)_* \rangle \arrow[r] & K \arrow[d, equal] \\ F \arrow[u, "\theta_{res}", "\text{rationalization}"'] \arrow[r] & X \arrow[u, "\theta"'] \arrow[r, "\pi"'] & K, \end{tikzcd}$$

in which $\theta : X \to \langle (A(K) \otimes \wedge W)_* \rangle$ is the fibrewise rationalization.

Example 2.12 (A more concrete example)

Let M be a simply-connected manifold and $L^\infty M := \mathbf{Diff}(S^1, M)$ the free loop space endowed with the functional diffeology. We construct the minimal local system model for $L^\infty M$ over $\mathbb{Q}$ applying Proposition 2.11.

The strategy is as follows.

- Consider a fibration of the form

$$S^D(\widetilde{L^\infty M}) \longrightarrow S^D(L^\infty M) \xrightarrow[\pi]{} K(\pi_1(L^\infty M), 1) =: K$$

- We have a KS-extension via the *"smoothing theorem"* due to Kihara

$$\begin{tikzcd} \wedge V \arrow[r, "i"] \arrow[d, "\simeq"', "v"] & \wedge V \otimes \wedge W \arrow[r, "j"] \arrow[d, "\simeq"', "\alpha"] & \wedge W \arrow[d, "\simeq"', "u"] \\ A(K) \arrow[r, "\pi^*"'] & A(S^D(L^\infty M)) \arrow[r] & A(S^D(\widetilde{L^\infty M})). \end{tikzcd}$$

- We have a minimal local system model $(A(K) \otimes \wedge W)_*$ for $L^\infty M$.

- For instance, let M be the complex projective space $\mathbb{C}P^n$. It is well-known that a minimal model for $C^0(S^1, \mathbb{C}P^n)$ is of the form

$$(\wedge(x, y) \otimes \wedge(\overline{x}, \overline{y}), d)$$

with $d(x) = d(\overline{x}) = 0$, $d(y) = x^{n+1}$ and $d(\overline{y}) = (n+1)\overline{x}x^n$, where $\deg x = 2$, $\deg y = 2(n+1)-1$, $\deg \overline{x} = 1$ and $\deg \overline{y} = 2(n+1)-2$.

- Then, we have a minimal local system model for $L^\infty \mathbb{C}P^n$ over $K := K(\pi_1^D(L^\infty \mathbb{C}P^n), 1)$ of the form

$$R_* := (A(K) \otimes \wedge(x, y, \overline{y}))_*$$

for which there exist isomorphisms of CDGA's for each $\sigma \in K_n$

$$((R_*)_\sigma, d) \cong A(n) \otimes_{e_\sigma, A(K)} (A(K) \otimes \wedge(x, y, \overline{y})) \cong A(n) \otimes \wedge(x, y, \overline{y}),$$

where $d(x) = 0$, $d(y) = x^{n+1}$ and

$$d(\overline{y}) = (n+1)(e_\sigma \circ v)(\overline{x}) \otimes x^n.$$

For $\mathbb{R}$-local (real) homotopy theory, we can use a simplicial CDGA

$$(A^*_{DR})_\bullet := \{\Omega^*(\mathbb{A}^n)\}_{n\geq 0}$$

over $\mathbb{R}$, where $\Omega^*(\mathbb{A}^n)$ is the Souriau-de Rham complex of the affine space

$$\mathbb{A}^n := \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} x_i = 1 \right\}$$

equipped with the sub-diffeology of the manifold $\mathbb{R}^{n+1}$. Observe that

$$A^*_{PL}(S^D(M)) \otimes \mathbb{R} \simeq A^*_{DR}(S^D(M)) \xleftarrow[\exists \alpha\text{: the factor map}]{\text{a morphism of CDGAs}} \Omega^*(M).$$

We may investigate

- Cartan–de Rham calculus (Calculus of differential forms) on **Diff**
- with Rational homotopy theory for diffeological spaces.

$$\begin{tikzcd}
& & \mathbf{Sets}^{\Delta^{op}} \arrow[d, shift left, "|\ |_D"] \arrow[d, phantom, "\vdash" near start] \arrow[dr, shift left, "|\ |"] & \\
\mathbf{Mfd} \arrow[r, "j", "\text{fully faithful}"'] \arrow[rr, bend left, "m\text{:fully faithful}"] & \mathbf{Stfd, DifferentialSp} \arrow[r, "k"] & \mathbf{Diff} \arrow[u, shift left, "S^D"] \arrow[r, shift left, "D"] \arrow[r, phantom, "\perp"] & \mathbf{Top.} \arrow[l, shift left, "C"] \arrow[ul, shift left, "S"] \arrow[ul, phantom, "\vdash"]
\end{tikzcd} \tag{5}$$

An application of local systems to a construction of a spectral sequence for an adjunction diffeological space

We have

- the Leray–Serre spectral sequence for the fibration in the sense of Chriatensen and Wu
- the Eilenberg–Moore spectral sequence for a pullback diagram
- and also a spectral sequence for an adjunction space in **Diff**.

Let $P \xleftarrow{f} N \xrightarrow{i} M$ be maps between connected diffeological spaces. These maps produce the diffeological adjunction space $P \cup_N M$ in **Diff** together with the quotient diffeology with respect to the projection $p : P \coprod M \to P \cup_N M$.

Theorem 3.1

Suppose that the map i is injective and that p_0, p_1 and p_2 are Kan fibrations over a pointed connected simplicial set K in a commutative diagram

$$\begin{tikzcd} S^D(P) \arrow[dr, "p_1"'] & S^D(N) \arrow[l, "S^D(f)"'] \arrow[r, "S^D(i)"] \arrow[d, "p_0"] & S^D(M) \arrow[dl, "p_2"] \\ & K. & \end{tikzcd}$$

Then, there exists a first quadrant spectral sequence $\{E_r^{*,*}, d_r\}$ with

$$E_2^{*,*} \cong H^*(K, \mathcal{H}^*_{\mathcal{P}}) \Longrightarrow H^*(A(S^D(P) \cup_{S^D(N)} S^D(M)),$$

where $\mathcal{H}^*_{\mathcal{P}}$ is a local coefficients satisfies the condition that, for any $\sigma \in K$, one has an isomorphism $(\mathcal{H}^*_{\mathcal{P}})_\sigma \cong H^*(A(F_1) \times_{A(F_0)} A(F_2))$ with F_i the fibre of p_i for $i = 0, 1$ and 2.

Suppose that a sequence $S \xleftarrow{f} \partial W \xrightarrow{i} W$ of manifolds gives a *stratifold* $S \cup_f W$ in the sense of Kreck (for example, W has a collar). The SS converges to

$$H^*(\Omega^*(S \cup_f W)) \xrightarrow[\cong]{\alpha} H^*(A^*_{DR}(S^D(S \cup_f W))).$$