On the cohomology algebras of the free loop space of the real projective space and its diffeological version

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17 May 2023
Global Diffeology Seminar
Online on Zoom

arXiv:2301.09827

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§1. A multiplicative spectral sequence for the nerve of a topological category

Let $\mathcal{C} = [C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0]$ be a topological category. The nerve functor gives rise to a cosimplicial cohain complex

$$n \mapsto C^*(\mathrm{Nerve}_n\mathcal{C}, \mathbb{K}) =: C^{n,*}$$

and then this induces a cosimplicial abelian group

$$n \mapsto H^q(\mathrm{Nerve}_n\mathcal{C}, \mathbb{K})$$

for any q, where $\mathbb{K}$ is a field.
Let B$\mathcal{C}$ be the classifying space, namely, B$\mathcal{C} = ||\mathrm{Nerve}_\bullet\mathcal{C}||$ which is the fat geometric realization of the simplicial space $\mathrm{Nerve}_\bullet\mathcal{C}$

$$\omega \cup_T \eta := (-1)^{qp'}(d^h_{p+1} \cdots d^h_{p+p'})^*\omega \cup (d^h_0 \cdots d^h_{p-1})^*\eta$$

for $\omega \in C^{p,q}$ and $\eta \in C^{p',q'}$.

- A multiplication on $\mathrm{Tot}C^*(\mathrm{Nerve}_\bullet\mathcal{C}, \mathbb{K})$

Theorem 1.1

Let $\mathcal{C} = [C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0]$ be a category internal to **Top**. *Then there exists a spectral sequence $\{E_r^{*,*}, d_r\}$ converging to $H^*(\mathrm{B}\mathcal{C}; \mathbb{K})$ as an algebra with*

$$E_2^{p,q} \cong H^p(H^q(\mathrm{Nerve}_\bullet\mathcal{C}; \mathbb{K})).$$

- The cohomology of the $\mathrm{Tot}C^{\bullet,*} \cong H^*(B\mathcal{C})$ by a *method of acyclic models*.

Theorem 1.2 (Gugenheim–May, '74: A torsion functor version)

Let G be a topological group and X a G-space. Then there exists a spectral sequence converging to the Borel cohomology

$$H^*_G(X;\mathbb{K}) := H^*(EG \times_G X;\mathbb{K})$$

as an algebra with $E_2^{p,q} \cong H^p(H^q(\mathrm{Nerve}_\bullet \mathcal{G};\mathbb{K}))$. *Here* $\mathcal{G} := [G \times X \underset{t}{\overset{s}{\rightrightarrows}} X]$ *denotes the transformation groupoid associated to the G-space X. In particular, one has an isomorphism*

$$E_2^{p,q} \cong \mathrm{Cotor}^{p,q}_{H^*(G;\mathbb{K})}(\mathbb{K}, H^*(X;\mathbb{K}))$$

provided $H^*(G;\mathbb{K})$ and $H^*(X;\mathbb{K})$ are locally finite.

C : a comodule over $\mathbb{K}$ a field and M, N : right C-comodule and a left one, respactively.

$$M \square_C N := \mathrm{Ker}\, \nabla \longrightarrow M \otimes N \xrightarrow{\nabla := \nabla_M \otimes 1 - 1 \otimes \nabla_N} M \otimes C \otimes N$$

§2. The computation of the cohomology algebra of the free loop space of the real projective space

Let G be a *discrete* group acting on a topological space M. For $g \in G$, we define

$$\mathcal{P}_g(M) := \{\gamma : [0,1] \to M \mid \gamma(1) = g\gamma(0)\}$$

which is a subspace of the space of continuous paths $M^{[0,1]}$ from the interval $[0,1]$ to M. Moreover, we put

$$\mathcal{P}_G(M) := \coprod_{g \in G} (\mathcal{P}_g(M) \times \{g\}). \tag{1}$$

Then, $\mathcal{P}_G(M)$ admits a G-action defined by $h \cdot (\gamma, g) = ({}_h\gamma, hgh^{-1})$, where ${}_h\gamma(t) = h \cdot \gamma(t)$. For a space X, let LX denote the free loop space of X, namely, the space of continuous maps from the circle S^1 to X.

Let $G \to M \xrightarrow{p} M/G$ be a principal G-bundle.

To (1) in Section 3, page 18

Proposition 2.1 (cf. Behrend, Ginot, Noohi and Xu '12)

The map $\overline{p} : EG \times_G \mathcal{P}_G(M) \to L(M/G)$ *induced naturally by the projection* $p : M \to M/G$ *is a weak homotopy equivalence.*

Sketch of proof. For each $m \in M$, a fibration

$$\mathcal{P}_G^m(M) \longrightarrow \mathcal{P}_G(M) \xrightarrow{q := \coprod q_g} M$$

is constructed, where $\mathcal{P}_G^m(M) = \coprod_{g \in G} \mathcal{P}_g^m(M)$ and $\mathcal{P}_g^m(M) := \{\gamma : [0,1] \to M \mid \gamma(0) = m, \gamma(1) = g\gamma(0) = gm\}$. Moreover, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{P}_G^m(M) & \xrightarrow{\overline{p}} & \Omega_{[m]}(M/G) \\ \downarrow & & \downarrow \\ EG \times_G \mathcal{P}_G(M) & \xrightarrow{\overline{p}} & L(M/G) \\ {\scriptstyle 1 \times_G ev_0} \downarrow & & \downarrow {\scriptstyle ev_0} \\ EG \times_G M & \underset{\widetilde{\pi}}{\xrightarrow{\simeq}} & M/G \end{array}$$

in which two vertical sequences are fibrations. We can prove $\overline{p}$ on the fibre is a weak homotopy equivalence. □

- On the cohomology of $\mathcal{P}_g(M)$ for a simply connected G-space M

For each $g \in G$, in order to compute $H^*(\mathcal{P}_g(M);\mathbb{K})$, we may use the Eilenberg–Moore spectral sequence (henceforth EMSS) for the pullback diagram

$$\begin{array}{ccc} \mathcal{P}_g(M) & \longrightarrow & M^{[0,1]} \\ \downarrow & & \downarrow{\scriptstyle \varepsilon_0\times\varepsilon_1} \\ M & \xrightarrow[1\times g]{} & M\times M, \end{array} \tag{2}$$

where ε_i is the evaluation map at i for $i = 0, 1$ and g denotes the map induced by the action on M with the element g. We observe that the EMSS $\{E_r^{*,*}, d_r\}$ converges to $H^*(\mathcal{P}_g(M);\mathbb{K})$ as an *algebra* with

$$E_2^{*,*} \cong \mathrm{Tor}^{*,*}_{H^*(M;\mathbb{K})\otimes H^*(M;\mathbb{K})}(H^*(M;\mathbb{K})_g, H^*(M;\mathbb{K}))$$

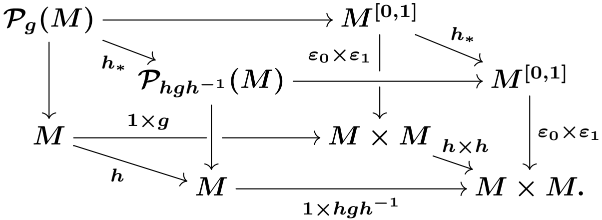
as a *bigraded algebra*. Here $H^*(M;\mathbb{K})_g$ is the cohomology algebra $H^*(M;\mathbb{K})$ endowed with the right $H^*(M;\mathbb{K})\otimes H^*(M;\mathbb{K})$-action defined by

$$a\cdot(\lambda\otimes\lambda') = a(\lambda g^*(\lambda'))$$

for $a \in H^*(M;\mathbb{K})_g$ and $\lambda, \lambda' \in H^*(M;\mathbb{K})$.

- On the G-action on $\mathcal{P}_G(M)$.

For $h \in G$, the G-action on $\mathcal{P}_G(M)$ induces $h_* : \mathcal{P}_g(M) \to \mathcal{P}_{hgh^{-1}}(M)$ which fits in the commutative diagram



(3)

Then, the naturality of the EMSS gives rise to a morphism of spectral sequences which is compatible with the map
$(h_*)^* : H^*(\mathcal{P}_{hgh^{-1}}(M); \mathbb{K}) \to H^*(\mathcal{P}_g(M); \mathbb{K})$.

Theorem 2.2 (K. 2023)

Let p be an odd prime or 0, then as algebras,

$$\begin{aligned} H^*(L\mathbb{R}P^{2m+1};\mathbb{Z}/p) &\cong H^*(LS^{2m+1};\mathbb{Z}/p)\oplus H^*(LS^{2m+1};\mathbb{Z}/p) \\ &\cong (\wedge(y)\otimes\Gamma[\overline{y}])^{\oplus 2} \quad \textit{and} \\ H^*(L\mathbb{R}P^{2m};\mathbb{Z}/p) &\cong (\wedge(x\otimes u)\otimes\Gamma[w])\oplus\mathbb{Z}/p, \end{aligned}$$

where $\deg y = 2m+1$, $\deg\overline{y} = 2m$, $\deg(x\otimes u) = 4m-1$, $\deg w = 4m-2$ *and* $\mathbb{Z}/0 := \mathbb{Q}$.

Sketch of Proof. $\mathbb{R}P^n = S^n/(\mathbb{Z}/2)$, $G := \mathbb{Z}/2$. By applying Theorem 1.2 to the groupoid $[G\times\mathcal{P}_G(S^n) \rightrightarrows \mathcal{P}_G(S^n)]$, we have a spectral sequence $\{E_r^{*,*}, d_r\}$ converging to $H^*(L\mathbb{R}P^n;\mathbb{Z}/p)$ with

$$E_2^{*,*} \cong \mathrm{Cotor}^{*,*}_{H^*(G)}(\mathbb{Z}/p, H^*(\mathcal{P}_G(S^n)))$$

as an algebra.

Since G is abelian, it follows that the G action on $\mathcal{P}_G(S^n)$ is restricted to each $\mathcal{P}_g(S^n)$ for $g \in G$. Then, we see that

$$L(\mathbb{R}P^n) \simeq_w EG \times_G \mathcal{P}_G(S^n) = \coprod_{g \in G} \big(EG \times_G \mathcal{P}_g(S^n)\big) \text{ and}$$

$$\mathrm{Cotor}^{*,*}_{H^*(G)}(\mathbb{Z}/p, H^*(\mathcal{P}_G(S^n))) = \oplus_{g\in G}\mathrm{Cotor}^{*,*}_{H^*(G)}(\mathbb{Z}/p, H^*(\mathcal{P}_g(S^n))).$$

We compute the cotorsion functor with the nomalized cobar complex

$$\big(\mathbb{Z}/p\{\tau^*\}^{\otimes k} \otimes H^*(\mathcal{P}_g(S^n)), \partial_k = \nabla_G \otimes 1 + (-1)^{k+1} 1 \otimes \nabla_{\tau^*}\big)_{k \geq 0},$$

where $\tau \in G$ denotes the nontrivial element,

$$\nabla_{\tau^*} : H^*(\mathcal{P}_g(S^n)) \to \widetilde{H}^0(G) \otimes H^*(\mathcal{P}_g(S^n)) = \mathbb{Z}/p\{\tau^*\} \otimes H^*(\mathcal{P}_g(S^n))$$

is the coaction induced by the G-action on $\mathcal{P}_g(S^n)$ and the projection $G \times \mathcal{P}_g(S^n) \to \mathcal{P}_g(S^n)$ gives rise to the map ∇_G. Observe that the complex is nothing but the E_1-term of the spectral sequence $\{E_r^{*,*}, d_r\}$. □

- A generalization of the computation above

Let G be a finite group and N a finite dimensional left $\mathbb{K}[G]^\vee$-comodule. Then the module $N^\vee$ is regarded as a left $\mathbb{K}[G]$-module via the natural map

$$\nabla^\vee : \mathbb{K}[G] \otimes N^\vee \to N^\vee$$

induced by the left comodule structure ∇ on N. Thus, by using the isomorphism $N \cong (N^\vee)^\vee$, we consider N a right $\mathbb{K}[G]$-module.

Lemma 2.3 (cf. Abrams and Weibel, '02)

Under the setup above, there are isomorphisms of vector spaces

$$\mathrm{Cotor}^*_{\mathbb{K}[G]^\vee}(\mathbb{K}, N) \cong \mathrm{HH}^*(\mathbb{K}[G], \mathbb{K}\otimes N) \cong \mathrm{Ext}^*_{\mathbb{K}[G]}(\mathbb{K}, \mathbb{K}\otimes N) = H^*(G, N).$$

Here the module N in the Hochschild cohomology is regarded as a right $\mathbb{K}[G]$-module mentioned above.

- Comments

- $\{E_r^{*,*}, d_r\}$: the SS in Theorem 1.2 converging to

$$H^*(EG \times_G \mathcal{P}_G(M); \mathbb{K}) \cong H^*(L(EG \times_G M)); \mathbb{K}).$$

Corollary 2.4 (cf. Lupercio, Uribe and Xicotencatl, '08)

Let G be a finite group acting on a space M and $\mathcal{P}_G(M)$ the space defined in (1). Suppose that $H^*(\mathcal{P}_G(M); \mathbb{K})$ is locally finite and $(\mathrm{ch}(\mathbb{K}), |G|) = 1$. Then as an algebra

$$H^*(L(EG \times_G M); \mathbb{K}) \cong \mathbb{K} \square_{\mathbb{K}[G]^\vee} H^*(\mathcal{P}_G(M); \mathbb{K}).$$

Proposition 2.5 (Benson '91)

Let G be a finite group. Then as an algebra,

$$H^*(LBG; \mathbb{K}) \cong \mathrm{Cotor}^*_{\mathbb{K}[G]^\vee}(\mathbb{K}, (\mathbb{K}[G]_{\mathrm{ad}})^\vee),$$

where $\mathbb{K}[G]_{\mathrm{ad}}$ denotes the adjoint representation of G. Especially, if G is abelian, then $H^*(LBG; \mathbb{K}) \cong H^*(G; \mathbb{K})^{\oplus |G|}$ as an algebra.

Since the trivial map $v : \mathcal{P}_g(M) \to *$ gives rise to a G-equivariant map $H_0(\mathcal{P}_G(M); \mathbb{K}) \to \mathbb{K}[G]_{\mathrm{ad}}$, it follows from Proposition 2.5 that the horizontal edge (p-axis) $E_2^{*,0}$ is a module over the algebra $H^*(LBG; \mathbb{K})$ via the morphism

$$v^* : H^*(LBG; \mathbb{K}) \to \mathrm{Cotor}^{p,0}_{\mathbb{K}[G]^\vee}(\mathbb{K}, H^*(\mathcal{P}_G(M); \mathbb{K}))$$

of algebras induced by v. Thus, the spectral sequence admits an $H^*(LBG; \mathbb{K})$-module structure on the spectral sequence;

$$\bullet : H^p(LBG; \mathbb{K}) \otimes E_r^{*,*} \xrightarrow{v^* \otimes 1} E_2^{p,0} \otimes E_r^{*,*} \overset{p_r \otimes 1}{\twoheadrightarrow} E_r^{p,0} \otimes E_r^{*,*} \xrightarrow{m} E_r^{*+p,*},$$

where p_r is the canonical projection and m is the product structure on the E_r-term. Thus we see that $d_r(a \bullet x) = (-1)^p a \bullet d_r(x)$ for $a \in H^p(LBG; \mathbb{K})$ and $x \in E_r^{*,*}$.

- Diagram

§3. A diffeological counterpart of a result on the cohomology of the free loop space of a Borel construction

We recall

- the singular de Rham complex (K '20)
- the factor map
- Chen's iterated integral map in diffeology

$\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1, x_i \geq 0\}$: the standard simplex in the sense of Kihara: In particular,

$$\Lambda_k^n : \{(x_0, \ldots, x_n) \in \Delta^n \mid x_i = 0 \text{ for some } i \neq k\} \hookrightarrow \Delta^n$$

has a smooth retraction for $n \geq 1$ and $0 \leq k \leq n$.

Define a simplicial DGA $(A^*_{DR})_\bullet$ as follows.
For each $n \geq 0$, $(A^*_{DR})_n := \Omega^*(\mathbb{A}^n)$ and define a simplicial set

$$S^D_\bullet(X) := \{\{\sigma : \Delta^n \to X \mid \sigma : C^\infty\text{–map}\}\}_{n\geq 0}$$

$\boldsymbol{\Delta}$: objects $[n] := \{0, 1, \dots, n\}$ for $n \geq 0$,
morphisms : non-decreasing maps $[n] \to [m]$ for $n, m \geq 0$

By definition, a simplicial set is a contravariant functor from $\boldsymbol{\Delta}$ to **Sets** the category of sets.

$$A^*_{DR}(S^D_\bullet(X)) := \left\{ \boldsymbol{\Delta}^{\mathrm{op}} \underset{(A^*_{DR})_\bullet}{\overset{S^D_\bullet(X)}{\Downarrow \eta}} \textbf{Sets} \;\middle|\; \eta : \text{a natural transformation} \right\}$$

Definition 3.1 (Connecting the singular de Rham to the original one.)

The *factor map* $\alpha : \Omega^*(X) \to A^*_{DR}(S^D_\bullet(X)_{\text{aff}})$ is defined by

$$\alpha(\omega)(\sigma) := \sigma^*(\omega),$$

where $S^D_\bullet(X)_{\text{aff}} := \{\{\sigma : \mathbb{A}^n \to X \mid \sigma : C^\infty\text{–map}\}\}_{n \geq 0}$

The inclusion $\iota : \Delta^n \to \mathbb{A}^n$, $(\iota^*)^* : A^*_{DR}(S^D_\bullet(X)) \overset{\simeq}{\leftarrow} A^*_{DR}(S^D_\bullet(X)_{\text{aff}})$.

M : a diff-space, $\omega_i \in \Omega^{p_i}(M)$ for each $1 \leq i \leq k$ and $q : U \to M^I$ a plot of the diff-space M^I. $\widetilde{\omega_{iq}} := (id_U \times t_i)^* q_\sharp^* \omega_i$, where $q_\sharp : U \times \mathbb{R} \to M$ is (the composite of a cut-off function and) the adjoint to q, and $t_i : \mathbb{R}^k \to \mathbb{R}$ denotes the projection in the ith factor.

$$(\int \omega_1 \cdots \omega_k)_q := \int_{\Delta^k} \widetilde{\omega_{1q}} \wedge \cdots \wedge \widetilde{\omega_{kq}}.$$

Then by definition, Chen's iterated integral **It** has the form

$$\mathsf{It}(\omega_0[\omega_1|\cdots|\omega_k]) = ev^*(\omega_0) \wedge \widetilde{\Delta}^*(\int \omega_1 \cdots \omega_k),$$

where $\widetilde{\Delta} : L^\infty M \to M^I$ is the lift of the diagonal map $M \to M \times M$.

Theorem 3.2 (K. '20)

Let M be a simply-connected diff-space, $\dim H^i(A_{DR}(S^D_\bullet(M))) < \infty$ for each $i \geq 0$. Suppose that the factor map for M is a quasi-isomorphism. Then

$$\alpha \circ \mathsf{It} : \Omega^*(M) \otimes \overline{B}(A) \to \Omega^*(L^\infty M) \to A^*_{DR}(S^D_\bullet(L^\infty M))$$

is a quasi-isomorphism of $\Omega^*(M)$-modules and algebras

Let G be a finite group acting freely and smoothly on a manifold M. Then, we have the principal G-bundle $G \to M \xrightarrow{p} M/G$ in the category of manifolds.

Let $\mathcal{P}^{\infty}_{G}(M)$ be the diffeological space obtained by applying the construction (1) in **Diff**. We consider a smooth map

$$\widetilde{p} : \mathcal{P}^{\infty}_{G}(M) \to L^{\infty}(M/G)$$

defined by $\widetilde{p}((\gamma, g)) = p \circ \gamma$, where $L^{\infty}(M/G)$ denotes the diffeological free loop space on M/G.

For a diffeological space X, we may write $H^*_{DR}(X)$ for the singular de Rham cohomology $H^*(A_{DR}(S^D_\bullet(X)))$.

Theorem 3.3 (K. '23)

Under the same setting as above, suppose further that M is simply connected. Then, the smooth map $\widetilde{p}$ gives rise to a well-defined isomorphism

$$\widetilde{p}^* : H^*_{DR}(L^{\infty}(M/G)) \xrightarrow{\cong} \mathbb{R}\square_{\mathbb{R}[G]^\vee} H^*_{DR}(\mathcal{P}^{\infty}_{G}(M))$$

of algebras.

From (1) in Section 1, page 6

- The functor $D : \mathbf{Diff} \to \mathbf{Top}$; (D-topology)

Let M and N be diffeological spaces and $C^\infty(M, N)$ the space of smooth maps from M to N with the functional diffeology.

Moreover, the inclusion $i : D(C^\infty(M, N)) \to C^0(DM, DN)$ is contiunuous (Christensen, G. Sinnamon and E. Wu, '14). Thus, it follows that the functor D induces a morphism

$$\xi : S^D_\bullet(C^\infty(M, N)) \stackrel{D(\ )}{\to} \mathrm{Sing}_\bullet(DC^0(M, N)) \stackrel{i_*}{\to} \mathrm{Sing}_\bullet(C^0(DM, DN))$$

of simplicial sets.

Theorem 3.4 (Smoothing theorem, Kihara '21)

Let M and N be finite dimensional manifolds. Then, the well-defined map $\xi : S^D_\bullet(C^\infty(M, N)) \to \mathrm{Sing}_\bullet(C^0(DM, DN))$ is a weak homotopy equivalence.

For a manifold M, we consider the composite

$$\lambda := i \circ Dj : D(\mathcal{P}_g^\infty(M)) \longrightarrow D(C^\infty([0,1], M)) \longrightarrow C^0(D[0,1], DM),$$

where j is the smooth inclusion. Since $DM = M$ and $D[0,1]$ is the subspace I of $\mathbb{R}$, it follows that $\lambda : D(\mathcal{P}_g^\infty(M)) \to \mathcal{P}_g(M)$ is continuous. Therefore, the composite

$$\xi' := \lambda_* \circ D(\ ) : S_\bullet^D(\mathcal{P}_g^\infty(M)) \to \mathrm{Sing}_\bullet(\mathcal{P}_g(M))$$

is a morphism of simplicial sets.

Lemma 3.5

Let M be a simply-connected manifold. Then, one has a sequence of quasi-isomorphisms

$$A_{PL}(\mathrm{Sing}_\bullet(\mathcal{P}_g(M))) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow[\simeq]{(\xi')^*} A_{PL}(S_\bullet^D(\mathcal{P}_g^\infty(M))) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow[\simeq]{\zeta} A_{DR}(S_\bullet^D(\mathcal{P}_g^\infty(M))).$$

Sketch of Proof. Consider the following commutative diagram.

$$\begin{array}{ccc}
H^*(A_{DR}(S^D_\bullet(\mathcal{P}^\infty_g(M))) & \overset{EM_1}{\underset{\cong}{\longleftarrow}} & \mathrm{Tor}^*_{A_{DR}(S^D(M^{\times 2}))}(A_{DR}(S^D(M)), A_{DR}(S^D(M^I))) \\
\cong \uparrow H(\zeta) & & \cong \uparrow \mathrm{Tor}(\zeta,\zeta) \\
H^*(A_{PL}(S^D_\bullet(\mathcal{P}^\infty_g(M))))_{\mathbb{R}} & \overset{EM_2}{\longleftarrow} & \mathrm{Tor}^*_{A_{PL}(S^D(M^{\times 2}))}(A_{PL}(S^D(M)), A_{PL}(S^D(M^I)))_{\mathbb{R}} \\
\uparrow H((\xi')^*) & & \uparrow \mathrm{Tor}(\xi^*,\xi^*) \\
H^*(A_{PL}(\mathrm{Sing}_\bullet(\mathcal{P}_g(M))))_{\mathbb{R}} & \overset{EM_3}{\underset{\cong}{\longleftarrow}} & \mathrm{Tor}^*_{A_{PL}(M^{\times 2})}(A_{PL}(\mathrm{Sing}_\bullet(M)), A_{PL}(\mathrm{Sing}_\bullet(M^I))_{\mathbb{R}}
\end{array}$$

- EM_i ; the Eilenberg-Moore map
  EM_3 : iso. (the oroginal one due to Eilenberg-Moore), EM_1 :iso. (K. '20)
- $\zeta : A_{PL}(-) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\simeq} A_{DR}(-)$ (K. '20)

We define a quasi-isomorphism $\widetilde{\xi_1} := \zeta \circ (\xi')^*$ the composite in Lemma 3.5.

Sketch of the proof of Theorem 3.3. For the translation groupoid $[G \times \mathcal{P}^\infty_G(M) \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{P}^\infty_G(M)]$, we see that $s \circ \widetilde{p} = t \circ \widetilde{p}$. Then the map $\widetilde{p}^* : H^*_{DR}(L^\infty(M/G)) \to H^*_{DR}(\mathcal{P}^\infty_G(M))$ induced by $\widetilde{p}$ factors through $\mathbb{R}\square_{\mathbb{R}[G]^\vee} H^*_{DR}(\mathcal{P}^\infty_G(M))$. We show that the morphism $\widetilde{p}^*$ of algebras in the theorem is an isomorphism. Consider a commutative diagram

$$\begin{array}{ccc}
H^*(C^0(S^1, M/G); \mathbb{R}) & \xrightarrow[\cong]{(\widetilde{\xi})^*} & H^*_{DR}(C^\infty(S^1, M/G)) \\
{\scriptstyle (q^*)^*} \uparrow \cong & & \cong \uparrow {\scriptstyle (q^*)^*} \\
H^*(L(M/G); \mathbb{R}) & \xrightarrow{(\widetilde{\xi})^*} & H^*_{DR}(L^\infty(M/G)) \\
{\scriptstyle \widetilde{p}^*} \downarrow \cong & & \downarrow {\scriptstyle \widetilde{p}^*} \\
\mathbb{R}\square_{\mathbb{R}[G]^\vee} H^*(\mathcal{P}_G(M); \mathbb{R}) & \xrightarrow[(\widetilde{\xi_1})^*]{} & \mathbb{R}\square_{\mathbb{R}[G]^\vee} H^*_{DR}(\mathcal{P}^\infty_G(M))
\end{array}$$

in which each $\widetilde{\xi}$ is the composite of the morphism induced by ξ in the smooth theorem and the quasi-isomorphism $\zeta : A_{PL}(K) \otimes_{\mathbb{Q}} \mathbb{R} \to A_{DR}(K)$ for a simplicial set K mentioned above. □

Theorem 3.6 (K. '23)

One has sequences

$$H^*_{DR}(L^\infty \mathbb{R}P^{2m+1}) \xrightarrow[\cong]{\widetilde{p}^*} \mathbb{R}\Box_{\mathbb{R}[G]^\vee} H^*_{DR}(\mathcal{P}^\infty_G(S^{2m+1}))$$

$$\uparrow \cong$$

$$\Big(\wedge (\alpha \circ \mathsf{It}(v_{2m+1})) \otimes \mathbb{R}[\alpha \circ \mathsf{It}([v_{2m+1})]\Big)^{\oplus 2} \quad \textit{and}$$

$$H^*_{DR}(L^\infty \mathbb{R}P^{2m}) \xrightarrow[\cong]{\widetilde{p}^*} \mathbb{R}\Box_{\mathbb{R}[G]^\vee} H^*_{DR}(\mathcal{P}^\infty_G(S^{2m}))$$

$$\uparrow \cong$$

$$\Big(\wedge (\alpha \circ \mathsf{It}(v_{2m}[v_{2m}])) \otimes \mathbb{R}[\alpha \circ \mathsf{It}(1[v_{2m}|v_{2m}])\Big) \oplus \mathbb{R}$$

of isomorphisms of algebras, where v_n denotes the volume form on $H^*_{DR}(S^n)$, **It** *and α are Chen's iterated integral map and the factor map, respectively.*

Note: Chen's iterated integral map does not work for a non-simply connected manifold M! (when considering $H^*(LM)$)