Interleavings of spaces over the classifying space of the circle

(joint work with T. Naito, S. Wakatsuki and T. Yamaguchi)

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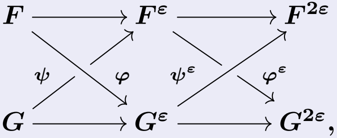
§1. Interleaving distances

Consider the functor category $\mathcal{C}^{(\mathbb{R},\leq)}$ for a category $\mathcal{C}$. For a real number $\varepsilon \geq 0$, define a functor $T_\varepsilon : (\mathbb{R},\leq) \to (\mathbb{R},\leq)$ by $T_\varepsilon(a) = a + \varepsilon$. Moreover, the *shift functor* $(\ )^\varepsilon : \mathcal{C}^{(\mathbb{R},\leq)} \to \mathcal{C}^{(\mathbb{R},\leq)}$ is defined by $(\ )^\varepsilon(F) = F^\varepsilon := FT_\varepsilon$.

Definition 1.1

Objects F and G in $\mathcal{C}^{(\mathbb{R},\leq)}$ are *ε-interleaved* if there exist natural transformations φ and ψ which fit in the commutative diagrams



(1)

where horizontal arrows are the natural transformations defined by the structure maps of F and G. Such a pair (φ,ψ) is called an *ε-interleaving* of F and G.

The natural transformations in Definition 1.1 give the commutative diagrams

$$
\begin{array}{ccc}
F(i) & \xrightarrow{F(i\to i+2\varepsilon)} & F(i+2\varepsilon) \\
& {\scriptstyle \varphi(i)}\searrow \quad \nearrow{\scriptstyle \psi(i+\varepsilon)} & \\
& G(i+\varepsilon) &
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
& F(i+\varepsilon) & \\
& {\scriptstyle \psi(i)}\nearrow \quad \searrow{\scriptstyle \varphi(i+\varepsilon)} & \\
G(i) & \xrightarrow[G(i\to i+2\varepsilon)]{} & G(i+2\varepsilon)
\end{array}
\qquad (2)
$$

for all $i \in \mathbb{R}$. We observe that F is isomorphic to G in $\mathcal{C}^{(\mathbb{R},\leq)}$ if and only if F and G are 0-interleaved.

Definition 1.2

For objects F and G in $\mathcal{C}^{(\mathbb{R},\leq)}$, the *interleaving distance* $d_{\mathrm{I}}(F,G)$ between F and G is defined by

$$d_{\mathrm{I}}(F,G) := \inf\{\varepsilon \geq 0 \mid F \text{ and } G \text{ are } \varepsilon\text{-interleaved}\}.$$

Proposition 1.3 (Bubenik–Scott '14)

The interleaving distance d_{I} is an extended pseudometric on the class of objects of $\mathcal{C}^{(\mathbb{R},\leq)}$.

For an interval J, we define a persistence module χ_J by

$$\chi_J(x) = \begin{cases} \mathbb{K} & \text{if } x \in J, \\ \mathbf{0} & \text{otherwise,} \end{cases} \qquad \chi_J(x \le y) = \begin{cases} id_{\mathbb{K}} & \text{if } x, y \in J, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then, the barcode $\mathcal{B}_M$ associated with a graded $\mathbb{K}[t]$-module M gives rise to the object $\chi(\mathcal{B}_M)$ in $\mathbf{Mod}_{\mathbb{K}}^{(\mathbb{R},\le)}$ defined by $\bigoplus_{J \in B_M} \chi_J$.

Definition 1.4

Let S and T be two barcodes, namely, multisets of intervals. Define the *bottleneck distance* between S and T (with matchings $S \leftrightarrow T$ between them) by

$$d_{\mathrm{B}}(S, T) := \inf_{f : S \leftrightarrow T} \sup_{I \in \mathrm{dom}(f)} d_{\mathrm{I}}(\chi_I, \chi_{f(I)}),$$

where $\chi_{\mathbb{R}}$ and $\chi_{\emptyset}$ denote the constant functors $\mathbb{K}$ and $\mathbf{0}$, respectively.

Theorem 1.5 ((The isometry theorem) Chazal–Cohen-Steiner–Glisse–Guibas–Oudot '09, Bubenik–Scott '14)

For locally finite $\mathbb{K}[t]$-modules M and N (persistence modules), one has

$$d_{\mathrm{I}}(\chi(\mathcal{B}_M), \chi(\mathcal{B}_N)) = d_{\mathrm{B}}(\mathcal{B}_M, \mathcal{B}_N).$$

1.2 Interleavings up to homotopy

Let $\mathcal{M}$ be a cofibrantly generated model category (e.g. **Top**, $\mathbf{Ch}_{\mathbb{K}}$) and $\mathcal{M}^{(\mathbb{R},\leq)}$ the model category endowed with the *projective model structure*.

Definition 1.6 (Blumberg–Lesnick '23, Lanari–Scoccola '23)

(1) For objects X and Y in $\mathcal{M}^{(\mathbb{R},\leq)}$, we say that X and Y are ε-*homotopy interleaved* if there exist $X \simeq X'$ and $Y \simeq Y'$ such that X' and Y' are ε-interleaved in $\mathcal{M}^{(\mathbb{R},\leq)}$. Here $W \simeq W'$ means that there is a zigzag of weak equivalences connecting W and W'.

(2) We say that objects X and Y in $\mathcal{M}^{(\mathbb{R},\leq)}$ are ε-*interleaved in the homotopy category* if they are ε-interleaved in $\mathrm{Ho}(\mathcal{M}^{(\mathbb{R},\leq)})$.

(3) Let $q_* : \mathcal{M}^{(\mathbb{R},\leq)} \to \mathrm{Ho}(\mathcal{M})^{(\mathbb{R},\leq)}$ be the functor induced by the localization functor. We say that X and Y in $\mathcal{M}^{(\mathbb{R},\leq)}$ are ε-*homotopy commutative interleaved* if q_*X and q_*Y are ε-interleaved in $\mathrm{Ho}(\mathcal{M})^{(\mathbb{R},\leq)}$

Let X and Y be objects in $\mathcal{M}^{(\mathbb{R},\leq)}$. Blumberg and Lesnick ('23) introduce the *homotopy interleaving distance* and the *homotopy commutative interleaving distance* defined by

$$d_{\mathsf{HI}}(X,Y) := \inf\{\varepsilon \geq 0 \mid X,\ Y \text{ are } \varepsilon\text{-homotopy interleaved}\} \quad \text{and}$$

$$d_{\mathsf{HC}}(X,Y) := \inf\{\varepsilon \geq 0 \mid X,\ Y \text{ are } \varepsilon\text{-homotopy commutative interleaved}\},$$

respectively. Moreover, Lanari and Scoccola ('23) introduce the *interleaving distance in the homotopy category* define by

$$d_{\mathsf{IHC}}(X,Y) := \inf\{\varepsilon \geq 0 \mid X,\ Y \text{ are } \varepsilon\text{-interleaved in the homotopy category}\}.$$

$$\mathrm{Ho}(\mathcal{M})^{(\mathbb{R},\leq)} \xleftarrow{\ \theta\ } \mathrm{Ho}(\mathcal{M}^{(\mathbb{R},\leq)}) \xleftarrow{\ \pi\ } \mathcal{M}^{(\mathbb{R},\leq)}, \qquad q_* : \mathcal{M}^{(\mathbb{R},\leq)} \to \mathrm{Ho}(\mathcal{M})^{(\mathbb{R},\leq)}$$

$$d_{\mathsf{HC}} \leq d_{\mathsf{IHC}} \leq d_{\mathsf{HI}} \leq 2d_{\mathsf{HC}}.$$

[Lanari–Sococcola]

Theorem 1.7 (Lanari–Scoccola, '23)

Let **Top** *be the category of topological spaces. Suppose that $d_{\mathsf{HI}} \leq c d_{\mathsf{HC}}$ on* $\mathbf{Top}^{(\mathbb{R},\leq)}$*. Then $c \geq \frac{3}{2}$. Hence $d_{\mathsf{HI}} \neq d_{\mathsf{HC}}$ in general.*

Let $\mathbf{Ch}_{\mathbb{K}}$ be the category of dg modules over a field $\mathbb{K}$ of arbitrary characteristic. We say that a persistence object X in $\mathbf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$ is *bounded below* if there exists an integer k such that $\bigoplus_{i+n\leq k} X(i)^n = 0$.

Theorem 1.8 (K. -Naito-Wakatsuki-Yamaguchi)

One has the equalities $d_{\mathsf{HC}} = d_{\mathsf{IHC}} = d_{\mathsf{HI}}$ *in the class of the objects bounded below in* $\mathbf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$.

Proof.

Let $\eta^k(H)_* : \mathbf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)} \to \mathbf{Mod}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ be the functor defined by the homology functor.

$$\begin{aligned} d_{\mathsf{HI}}(X,Y) &= d_{\mathsf{HI}}(H(X),H(Y)) \\ &\qquad (\because X,\ Y \text{ are formal as cochain complexes of } \mathbb{K}[t]\text{-modules}) \\ &\leq \sup\{d_{\mathsf{I}}(\eta^k(H)_*(X),\eta^k(H)_*(Y)) \mid k\in\mathbb{Z}\} \\ &\leq d_{\mathsf{HC}}(X,Y) \leq d_{\mathsf{IHC}}(X,Y) \leq d_{\mathsf{HI}}(X,Y). \end{aligned}$$

□

§2. Interleavings of DG $\mathbb{K}[u]$-modules ($\deg u = 2$)

For a dg $\mathbb{K}[u]$-module $K = \{K^l, \partial\}$, that is, $\oplus_l K^l$ is a $\mathbb{K}[u]$-module and $(u\times) \circ \partial = \partial \circ (u\times)$.

Define a functor $C : \mathbb{K}[u]$-**Ch** $\to$ $\mathbf{Ch}_{\mathbb{K}}^{(\mathbb{Z},\leq)}$ by $C(\{K^l, \partial\})(i) = \Sigma^{2i}K$ and

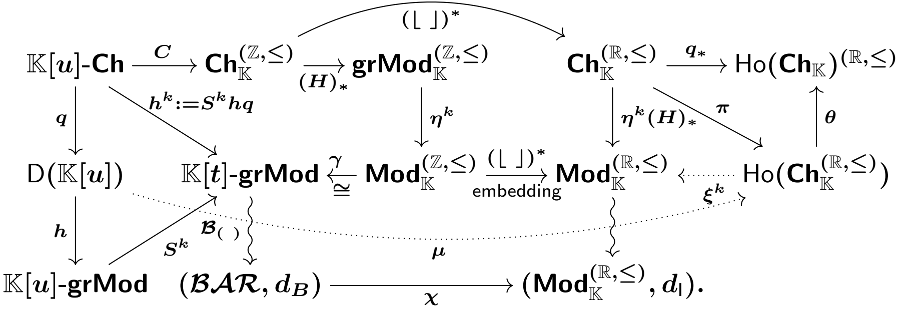
$$C(\{K^l,\partial\})(i \to i+1) : C(\{K^l,\partial\})(i) \xrightarrow{\times u} C(\{K^l,\partial\})(i+1).$$

$$\cdots \Sigma^{-2}K \xrightarrow{\times u} K \xrightarrow{\times u} \Sigma^{2}K \xrightarrow{\times u} \Sigma^{4}K \quad \cdots$$

$$\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & & \\
 & \uparrow & \uparrow & \uparrow & & \\
 & K^{0} & K^{2} & K^{4} & \vdots & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
 & K^{-1} & K^{1} & K^{3} & K^{5} & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
\cdots & K^{-2} & K^{0} & K^{2} & K^{4} & \cdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
 & \vdots & K^{-1} & K^{1} & K^{3} &
\end{array}$$

Interleavings up to homotopy, barcodes, dg $\mathbb{K}[u]$-modules

We have a commutative diagram



Here η^k is defined by $(\eta^k)(V)(i) = V(i)^k$, $\mathrm{D}(\mathbb{K}[u])$ denotes the derived category of dg $\mathbb{K}[u]$-modules, q is the localization, h is the homology functor, S^k is the functor defined by $S^0(K) = \oplus_i K^{2i}$ and $S^1(K) = \oplus_i K^{2i+1}$.

Let $\alpha : \mathbb{K}[u]\text{-}\mathbf{Ch} \to \mathbf{Ch}_{\mathbb{K}}^{(\mathbb{R},\leq)}$ be the functor $(\lfloor\ \rfloor)^* \circ C$.

Definition 2.1 (K. -Naito-Wakatsuki-Yamaguchi)

Let M and N be dg $\mathbb{K}[u]$-modules. The *even cohomology interleaving distance* $d^0_{\mathsf{CohI}}(M, N)$ and the *odd cohomology interleaving distance* $d^1_{\mathsf{CohI}}(M, N)$ are defined by

$$d_{\mathsf{I}}(\eta^0(H_*)\alpha(M), \eta^0(H_*)\alpha(N)) \quad \text{and} \quad d_{\mathsf{I}}(\eta^1(H_*)\alpha(M), \eta^1(H_*)\alpha(N)),$$

respectively.

Theorem 2.2 (K. -Naito-Wakatsuki-Yamaguchi)

The equalities

$$\begin{aligned} d_{\mathsf{HC}}(\alpha M, \alpha N) &= d_{\mathsf{IHC}}(\alpha M, \alpha N) = d_{\mathsf{HI}}(\alpha M, \alpha N) \\ &= \max\{d^k_{\mathsf{CohI}}(M, N) \mid k = 0, 1\} \end{aligned}$$

hold for formal dg $\mathbb{K}[u]$-modules M and N. In particular, for dg $\mathbb{K}[u]$-modules M and N whose gradings are bounded below, one has the equalities.

§3. The cohomology interleaving of spaces over BS^1

- For spaces $p : X \to BS^1$ and $q : Y \to BS^1$ over BS^1, the cohomology groups $H^*(X;\mathbb{K})$ and $H^*(Y;\mathbb{K})$ are regarded as $\mathbb{K}[u] = H^*(BS^1;\mathbb{K})$-modules with the maps p^* and q^*, respectively.
- In fact, the map $X \to BS^1$ gives rise to the morphism $\mathbb{K}[u] \to C^*(X;\mathbb{K})$ of DGA's, where $\deg u = 2$. Then, the cochain complex $C^*(X;\mathbb{K})$ is considered a dg $\mathbb{K}[u]$-module.

Definition 3.1 (The cohomology interleaving distance between spaces X and Y over BS^1)

$d^k_{\mathrm{CohI},\mathbb{K}}(X,Y) := d^k_{\mathrm{CohI}}(C^*(X;\mathbb{K}), C^*(Y;\mathbb{K}))$ for $k = 0$ and $k = 1$,

$d_{\mathrm{CohI},\mathbb{K}}(X,Y) := \max\{d^k_{\mathrm{CohI}}(M,N) \mid k = 0,1\}$.

The *cup-length* $\mathrm{cup}(f)_{\mathbb{K}}$ of a map $f : X \to Y$
:= the length of the longest non-zero product in the image of the homomorphism $f^* : \widetilde{H}^*(Y;\mathbb{K}) \to \widetilde{H}^*(X;\mathbb{K})$.

Proposition 3.2

Let $v_1 : X \to BS^1$ and $v_2 : Y \to BS^1$ be spaces over BS^1. Then, it holds that for $k = 0$ and 1,

$$d^k_{\mathrm{CohI},\mathbb{K}}(X,Y) \leq \frac{1}{2}\max\{\mathrm{cup}(v_1)_{\mathbb{K}} + 1, \mathrm{cup}(v_2)_{\mathbb{K}} + 1\}.$$

Proposition 3.3 (A shriek map gives rise to an interleaving)

Let $u : X \to BS^1$ and $v : Y \to BS^1$ be connected closed oriented manifolds over BS^1. Suppose that there exists a continuous map $f : X \to Y$ with $v \circ f = u$. Then

(i) $d_{\mathrm{CohI},\mathbb{K}}(X,Y) \leq \frac{1}{2}(\dim Y - \dim X)$ *if* $\dim X$ *and* $\dim Y$ *are even and* $\dim Y \geq 2\dim X$, *and*

(ii) $d_{\mathrm{CohI},\mathbb{K}}(X,Y) < \frac{1}{2}(\dim Y - \dim X)$ *if* $\dim X$ *and* $\dim Y$ *are odd and* $\dim Y > 2\dim X$.

§4. Toy examples

Proposition 3.4

Let X and Y be formal spaces, more general BV-exact spaces (KNWY '24). Then, it holds that for $k = 0$ and 1,

$$d^k_{\mathsf{CohI},\mathbb{Q}}((LX)_{hS^1}, (LY)_{hS^1}) = \begin{cases} 0 & \text{if } h^k C^*((LX)_{hS^1}, \mathbb{Q}) \\ & \cong h^k C^*((LY)_{hS^1}, \mathbb{Q}) \text{ as a } \mathbb{Q}[t]\text{-mod.,} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

In particular, $d_{\mathsf{CohI},\mathbb{Q}}((LX)_{hS^1}, (LY)_{hS^1}) = 0$ if and only if $C^*((LX)_{hS^1};\mathbb{Q}) \cong C^*((LY)_{hS^1};\mathbb{Q})$ in $\mathrm{D}(\mathbb{Q}[\mathrm{u}])$.

Proposition 3.5

Let $f_n : \mathbb{C}P^n \to BS^1$ be a map which represents 1 in $[\mathbb{C}P^n, BS^1] \cong H^2(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$. Then,

$$d^0_{\mathsf{CohI},\mathbb{K}}((\mathbb{C}P^n, f_n), (\mathbb{C}P^m, f_m)) = \min\left\{|n-m|, \max\left\{\frac{m+1}{2}, \frac{n+1}{2}\right\}\right\}$$

Proposition 3.6

For $j = 0, 1$, let $v_j : M_j \to BS^1$ be a space over BS^1 whose relative Sullivan model has the form $(\wedge u, 0) \to (\wedge(x, y, z, u), d)$ with $dz = jxyu + u^4$ and $dx = 0 = dy$, where $\deg x = \deg y = 3$, $\deg z = 7$ and $\deg u = 2$. Then, one has

$$d^0_{\mathrm{CohI},\mathbb{Q}}(M_0, M_1) = 3 \quad \textit{and} \quad d^1_{\mathrm{CohI},\mathbb{Q}}(M_0, M_1) = 0.$$

Remark 3.7

Let $\iota : (\wedge(u), 0) \to (\wedge W \otimes \wedge(u), d)$ be a relative Sullivan algebra. We have a fibration $|\iota| : |(\wedge W \otimes \wedge(u), d)| \to |(\wedge(u), 0)|$. The pullback of the fibration along the rationalization map $\ell : BS^1 \to |(\wedge(u), 0)|$ gives

$$\begin{array}{ccccc}
F & \xrightarrow{\simeq} & X' & \longrightarrow & ES^1 \\
\| & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
F & \longrightarrow & X & \longrightarrow & BS^1 \\
\| & & \downarrow & & \downarrow{\scriptstyle \ell} \\
|(\wedge W, \bar{d})| & \longrightarrow & |(\wedge W \otimes \wedge(u), d)| & \xrightarrow[|\iota|]{} & |(\wedge(u), 0)|
\end{array}$$

in which p is the universal S^1-bundle and the right-hand upper squares is also pullback.

Remark 3.8

Let X and Y be spaces over BS^1. Then, the triangle inequality of the interleaving distance allows us to deduce an inequality

$$\left| \, d^k_{\mathsf{CohI},\mathbb{K}}(X, \mathbb{C}P^n) - d^k_{\mathsf{CohI},\mathbb{K}}(Y, \mathbb{C}P^n) \, \right| \leq d^k_{\mathsf{CohI},\mathbb{K}}(X, Y)$$

for each $n \geq 1$, $k = 0$ and 1.

Assertion 3.9

Let $v_j : M_j \to BS^1$ be the space over BS^1 in Proposition 3.6 for each $j = 0$ and 1. Then, cup$(v_0)_{\mathbb{Q}} = 3$ and cup$(v_1)_{\mathbb{Q}} = 6$.

The equalities in Proposition 3.2 and Remark 3.8 do not hold in general. We have

$$\begin{aligned} & \left| \, d^0_{\mathsf{CohI},\mathbb{Q}}(M_0, \mathbb{C}P^6) - d^0_{\mathsf{CohI},\mathbb{Q}}(M_1, \mathbb{C}P^6) \right| \\ = \;& 3 - \frac{1}{2} \\ < \;& d^0_{\mathsf{CohI},\mathbb{Q}}(M_0, M_1) = 3 \\ < \;& \frac{7}{2} = \frac{1}{2}\max\{\mathrm{cup}(v_0)_{\mathbb{Q}} + 1, \mathrm{cup}(v_1)_{\mathbb{Q}} + 1\}. \end{aligned}$$

Future work and perspective

Applying applied topology, we develop methods for computing the cohomology interleaving distances between spaces over $\boldsymbol{BS^1}$.

We consider

- the interleaving distance in $(\textbf{CDGA}^{\textbf{op}})^{(\mathbb{R},\leq)}$ (Hess, Lavenir and Maggs '24) in order to deal with the *rational* homotopy interleaving distance of $\mathbb{R}$-spaces,
- multiparameter persistence theory and spaces over $\boldsymbol{BT^n}$, such as appearing in toric topology, from the view point of persistence theory.

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