The non-triviality of the whistle cobordism operation associated with string topology for classifying spaces

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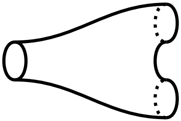
String topology and its variants

- String topology for orientable manifolds, Chas and Sullivan (1999)

New algebra structures (e.g. the loop product μ) in the singular homology of the free loop space $LM := \mathrm{map}(S^1, M)$ are investigated.

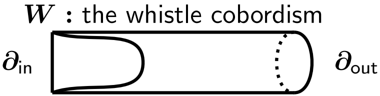
- Cohen and Godin (2004), A 2-dim. closed topological quantum field theory (TQFT) structure on string topology for manifolds

$\mu =$ the TQFT operation for  the pair of pants cobordism

- Félix and Thomas (2009), String topology for Gorenstein spaces (BG, Poincaré duality spaces, Borel constructions $EG \times_G M$).
- Chataur and Menichi (2012), String topology for classifying spaces and its TQFT and HCFT (Homological conformal field theory) structures

Introduction –The main assertion in this talk –

Assertion: The whistle cobordism operation in the labeled $\mathbf{2}$-dimensional topological quantum field theory (TQFT) for the classifying space of a Lie group in the sense of Guldberg is non-trivial in general.



1. A labeled 2-dimensional open-closed TQFT
2. The labeled open-closed TQFT for classifying spaces due to Guldberg
3. The non-triviality of the whistle cobordism operation

A labeled 2-dimensional open-closed TQFT

The category **oc-cob**(S) of open-closed strings labeled by a set S:

Objects: $Y = \coprod^{\text{finite}} S^1 \amalg \coprod^{\text{finite}}_{K,H} I^K_H$, where I^K_H denotes the interval labeled by elements H and K of S at 0 and 1, respectively.

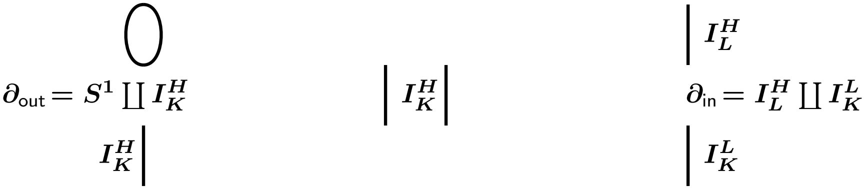
Morphisms: 2- dim. cobordisms from Y_0 to Y_1, namely 2-dim orientable manifolds Σ with $\partial\Sigma = \partial_{\text{in}} \cup \partial_{\text{out}} \cup \partial_{\text{free}}\Sigma$ in which $Y_0 = \partial_{\text{in}}$ and $Y_1 = \partial_{\text{out}}$. Moreover, $\partial_{\text{free}}\Sigma$ is a 1-dim. cobordism between ∂Y_0 and ∂Y_1 and that is labeled by elements in S which are compatible with labels of ∂Y_0 and ∂Y_1.

$$\Sigma = (\Sigma, \{\Sigma^H\}_{H\in S' : \text{finite subset} \subset S}), \ \partial_{\text{free}}\Sigma = \coprod_{H\in S'} \Sigma^H,$$

Composites are given by gluing cobordisms (keeping the labelings).

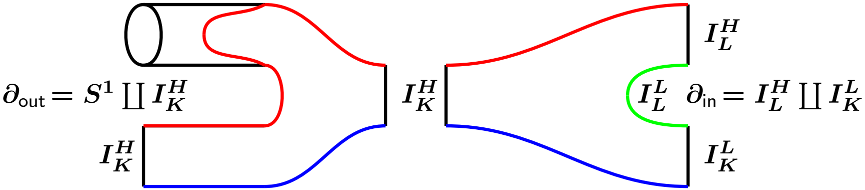
More precisely, components in an object are ordered, the order are also preserved when the gluing is made and morphisms are diffeo. classes of cobordisms.

Example –Gluing cobordisms–



Example –Gluing cobordisms–

The red parts are free boundaries labeled by $\boldsymbol{H}$.



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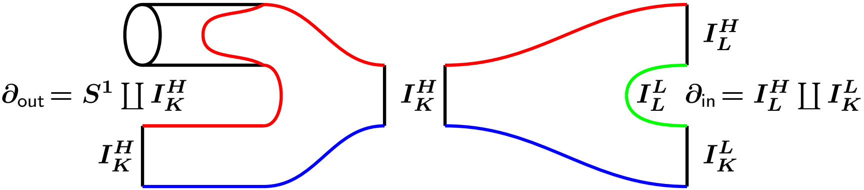
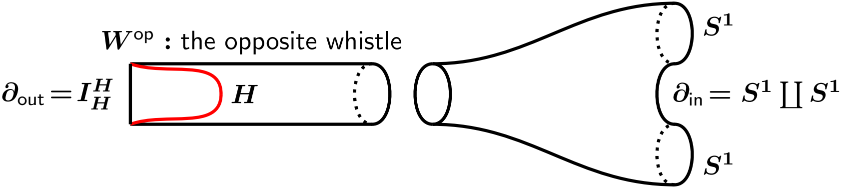




Definition 1.1

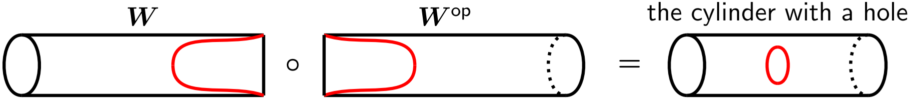
A labeled open-closed TQFT is a monoidal functor

$$\mu : (\textbf{oc-cob}(S), \coprod) \to (\mathbb{K}\textbf{-Vect}, \otimes),$$

where $\coprod$ denotes the disjoint union operator of cobordisms. In particular,

$$\mu_{\Sigma_1 \circ \Sigma_2} = \mu_{\Sigma_1} \circ \mu_{\Sigma_2}.$$

$$\mu_{\Sigma_1 \coprod \Sigma_2} = \mu_{\Sigma_1} \otimes \mu_{\Sigma_2}.$$



$$\mu_{(\text{the cylinder with a hole})} = \mu_{W \circ W^{\mathrm{op}}} = \mu_W \circ \mu_{W^{\mathrm{op}}}$$

The labeled open-closed TQFT for classifying spaces due to Guldberg

Setup:

- G : a connected compact Lie group and BG the classifying space of G.
- $\mathcal{B}$: a set of closed connected subgroups of G.
- $\Sigma := (\Sigma, \{\Sigma^H\}_{H \in \mathcal{B}'})$: a two dimensional labeled cobordism with incoming boundary ∂_{in} and outgoing boundary ∂_{out}.

We define a space $\mathcal{M}(\Sigma)$, which is called the *M-construction*, by the pullback diagram

$$\begin{tikzcd}
\mathcal{M}(\Sigma) \arrow[r] \arrow[d] & \mathrm{map}(\Sigma, BG) \arrow[d, "i^*"] \\
\coprod_H \mathrm{map}(\Sigma^H, BH) \arrow[r, "B\iota_*"'] & \coprod_H \mathrm{map}(\Sigma^H, BG),
\end{tikzcd}$$

where $\iota : H \to G$ is the inclusion and $i : \coprod_H \Sigma^H = \partial_{\text{free}}\Sigma \to \Sigma$ denotes the embedding.

The M-construction is functorial.

$$\begin{tikzcd}
\mathcal{M}(\Sigma) \arrow[rr] \arrow[d] & & \mathrm{map}(\Sigma, BG) \arrow[d, "i^*"] \\
\prod_H \mathrm{map}(\Sigma^H, BH) \arrow[rr, "B\iota_*"'] & & \prod_H \mathrm{map}(\Sigma^H, BG).
\end{tikzcd}$$

Moreover, we have the maps (*) $\mathcal{M}(\partial_{\mathrm{in}}) \xleftarrow{in^*} \mathcal{M}(\Sigma) \xrightarrow{out^*} \mathcal{M}(\partial_{\mathrm{out}})$ induced by inclusions $\partial_{\mathrm{in}} \xrightarrow[in]{} \Sigma \xleftarrow[out]{} \partial_{\mathrm{out}}$.

Remark 2.1

The map in^* in (*) is an orientable fibration whose fibre is the products of H's , G/K's and the total space of a fibration of the form $L \to E \to G/L$, where K, L and H are in $\mathcal{B}$. We define a map

$$\mu_\Sigma : H_*(\mathcal{M}(\partial_{\mathrm{in}})) \xrightarrow{(in^*)!} H_*(\mathcal{M}(\Sigma)) \xrightarrow{(out^*)_*} H_*(\mathcal{M}(\partial_{\mathrm{out}}))$$

with the *integration along the fibre* $(in^*)!$, where $H_*(\quad)$ denotes the homology with coefficients in a field $\mathbb{K}$.

Theorem 2.2 (Guldberg (2011))

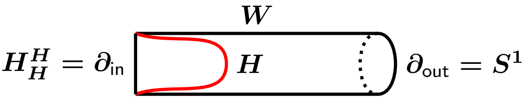
The operations μ_Σ for labeled cobordisms give rise to a 2-dimensional labeled open closed TQFT for the classifying space BG.

Remark 2.3

In the string topology for a manifold, the cobordism operation for a surface with boundaries and the genus ≥ 1 is trivial. (Tamanoi, 2010)

Is the open-closed labeled theory non-trivial?

We consider the problem with the whistle cobordism W and the opposite W^{op}.



$$\mu_W := (out)^* \circ (in^*)! : H_*(\mathcal{M}(I^H_H)) \to H_{*+\dim H}(LBG)$$

$$\mu_{W^{\mathrm{op}}} := (in)^* \circ (out^*)! : H_*(LBG) \to H_{*+\dim G/H}(\mathcal{M}(I^H_H))$$

The non-triviality of the whistle cobordism operation

Setup:

- G : a connected compact Lie group and H a connected closed subgroup of maximal rank.
- Suppose that the integral homology groups of G and H are p-torsion free, where p is the characteristic of $\mathbb{K}$.

Theorem 3.1 (K, 2019)

With the assumption above, the operations μ_W and $\mu_{W^{\mathrm{op}}}$ associated to the whistle cobordisms $(W, \{W^H\})$ and $(W^{\mathrm{op}}, \{(W^{\mathrm{op}})^H\})$ are non-trivial. Moreover, the composite operation

$$\mu_W \circ \mu_{W^{\mathrm{op}}} = \mu_{W \circ W^{\mathrm{op}}} = \mu_{(\text{the cylinder with a hole})}$$

is also non-trivial if $(\deg(B\iota)^*(x_i), p) = 1$ for any $i = 1, \ldots, l$, where $B\iota : BH \to BG$ stands for the map between classifying spaces induced by the inclusion $\iota : H \to G$ and $x_1, \ldots, x_l$ are generators of $H^*(BG; \mathbb{K})$.

Outline of the proof (The non-triviality of μ_W)

Say $H^*(BG) \cong \mathbb{K}[x_1, ..., x_l]$ and $H^*(BH) \cong \mathbb{K}[u_1, ..., u_l]$.

The Eilenberg–Moore spectral sequence argument gives a commutative diagram

$$
\begin{tikzcd}
H^*(\mathcal{M}(\partial_{\text{out}})) \cong H^*(LBG) \arrow[d, "(out^*)^*"] \arrow[dd, bend right=60, swap, "D\mu_W"] & H^*(BG) \otimes \wedge(y_1, ..., y_l) \arrow[l, swap, "\cong"] \arrow[d, "(B\iota)^* \otimes 1"] \\
H^*(\mathcal{M}(\Sigma)) \arrow[d, dotted, bend right, swap, "(in^*)^!"] & H^*(BH) \otimes \wedge(y_1, ..., y_l) \arrow[l, swap, "\cong"] \\
H^*(\mathcal{M}(\partial_{\text{in}})) \arrow[u, swap, "(in^*)^*"] & \dfrac{H^*(BH) \otimes H^*(BH)}{((B\iota)^* x_i \otimes 1 - 1 \otimes (B\iota)^* x_i)} \arrow[l, swap, "\cong"] \arrow[u, swap, "\mathfrak{m}"]
\end{tikzcd},
$$

where $\deg y_i = \deg x_i - 1$.

The integration along the fibre $(in^*)^!$ is defined by using the Leray–Serre spectral sequence $\{E_r^{*,*}, d_r\}$ for the fibration $H \to \mathcal{M}(\Sigma) \stackrel{in^*}{\to} \mathcal{M}(\partial_{\text{in}})$.

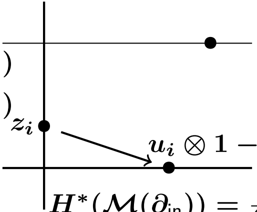
We can write, in $H^*(BH) \otimes H^*(BH)$,

$$(B\iota)^*x_i \otimes 1 - 1 \otimes (B\iota)^*x_i = \sum_{j=1}^{l} \zeta_{ij}(u_j \otimes 1 - 1 \otimes u_j)$$

with elements ζ_{ij} which satisfy the condition that $\mathfrak{m}(\zeta_{ij}) = \frac{\partial (B\iota)^*x_i}{\partial u_j}$.

$\mathrm{Tot}E_\infty^{*,*} \cong$
$H^*(BH) \otimes \wedge(y_1, ..., y_l)$

$H^*(H) = \wedge(z_1, .., z_l)$



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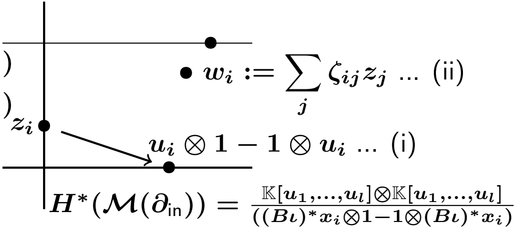

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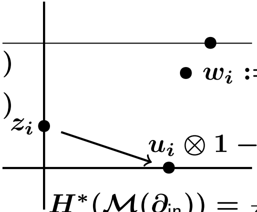

$\bullet\ w_i := \sum_j \zeta_{ij} z_j$ ... (ii)

$u_i \otimes 1 - 1 \otimes u_i$ ... (i)

$H^*(\mathcal{M}(\partial_{\mathrm{in}})) = \frac{\mathbb{K}[u_1, ..., u_l] \otimes \mathbb{K}[u_1, ..., u_l]}{((B\iota)^* x_i \otimes 1 - 1 \otimes (B\iota)^* x_i)}$

$\mathbb{K}\{w_1, ..., w_l\} \cong Q(\mathrm{Tot} E_\infty^{*,*})^{\mathrm{odd}} \cong \mathbb{K}\{y_1, ..., y_l\}$ ... (iii)

We can write, in $H^*(BH) \otimes H^*(BH)$,

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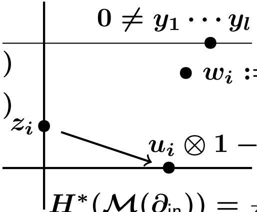

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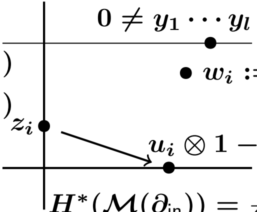


$\mathrm{Tot} E_\infty^{*,*} \cong$
$H^*(BH) \otimes \wedge(y_1, ..., y_l)$
$H^*(H) = \wedge(z_1, .., z_l)$

$0 \neq y_1 \cdots y_l = \det(\zeta_{ij}) z_1 \cdots z_l$ ... (iv)

$w_i := \sum_j \zeta_{ij} z_j$ ... (ii)

z_i

$u_i \otimes 1 - 1 \otimes u_i$ ... (i)

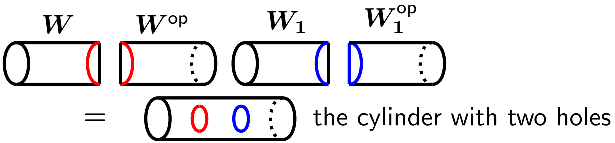
$H^*(\mathcal{M}(\partial_{\mathrm{in}})) = \frac{\mathbb{K}[u_1, \dots, u_l] \otimes \mathbb{K}[u_1, \dots, u_l]}{((B\iota)^* x_i \otimes 1 - 1 \otimes (B\iota)^* x_i)}$

For $1 \otimes y_1 \cdots y_l \in H^*(\mathcal{M}(\partial_{\mathrm{out}})) \cong H^*(BG) \otimes \wedge(y_1, ..., y_l)$,

$$\begin{aligned} D\mu_W(1 \otimes y_1 \cdots y_l) &= (in^*)^!(out^*)^*(1 \otimes y_1 \cdots y_l) \\ &= (in^*)^!(\det(\zeta_{ij}) z_1 \cdots z_l) = \det(\zeta_{ij}) \neq 0 \end{aligned}$$

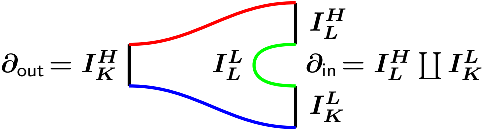
Remark 3.2

While $\mu_{(W\circ W^{\rm op})}$ is non-trivial in general, $\mu_{(W^{\rm op}\circ W_1)} \equiv \mathbf{0}$ for which the label of the whistle W_1 is not necessarily the same as that of W. In consequence, $\mu_{(\text{the cylinder with two holes})} \equiv \mathbf{0}$.



Theorem 3.3 (A result in the *open* TQFT (K. 2019))

Let Υ be the basic cobordism from two labeled intervals I_L^H and I_K^L to one labeled interval I_K^H, which is pictured below. Under the setup mentioned above, the cobordism operation μ_Υ is trivial but not $\mu_{\Upsilon^{\rm op}}$ in general. More precisely, the operation $\mu_{\Upsilon^{\rm op}}$ is injective.



Conclusions

Assertion 3.4 (in the rational case)

Let $\mathcal{B}$ be the set of connected closed subgroup of G of maximal rank. Then one can make a calculation of each of the dual operations for the labeled TQFT μ : $(\textbf{oc-Cobor}(\mathcal{B}), \coprod) \to (\mathbb{Q}\text{-}\textbf{Vect}, \otimes)$ introduced by Guldberg up to multiplication by non-zero scalar with the cohomology algebras and their generators.

(Non-rational case) With the results in

K. Kuribayashi and L. Menichi, The Batalin-Vilkovisky algebra in the string topology of classifying spaces, Canadian Journal of Math.,**71** (2019), 843-889,

we see that, under the same assumption as in Theorem 3.1,

- the cobordism operation μ_Σ is trivial if Σ contains the cylinder with two holes or a surface with genus one as a component,
- (the Batalin–Vilkovisky op.) $\circ\, \mu_W$ is non-trivial in general, and
- the closed TQFT and the open TQFT are not separated.