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On Félix–Tanré rational models for polyhedral products

by

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Abstract. The Félix–Tanré rational model for the polyhedral product of a fibre inclusion is considered. In particular, we investigate the rational model for the polyhedral product of a pair of Lie groups corresponding to on arbitrary simplicial complex and the rational homotopy group of the polyhedral product. Furthermore, it is proved that for a partial quotient N associated with a toric manifold M, the following conditions are equivalent: (i) $N = M$. (ii) The odd-degree rational cohomology of N is trivial. (iii) The torus bundle map from N to the Davis–Januszkiewicz space is formalizable.

1. Introduction. Toric varieties are fascinating objects in the study of algebraic geometry, combinatorics, symplectic geometry and topology. Nonsingular toric varieties, so-called toric manifolds, are given as the quotient of a moment-angle manifold by a torus action with Cox's construction. By generalizing the construction of moment-angle manifolds, we obtain *moment-angle complexes* and more general *polyhedral products* [1, 12, 15], which are defined to be the colimits of spaces with gluing data obtained from a simplicial complex.

Félix and Tanré [10] have given a rational model for a polyhedral product of a tuple of spaces corresponding to an arbitrary simplicial complex. One of the aims of this article is to construct a tractable rational model for a polyhedral product by refining the model due to Félix and Tanré. By applying the construction to a polyhedral product for a pair of Lie groups, we obtain a result on the rational homotopy groups of the polyhedral product; see Theorem 1.2 and Proposition 4.1.

Moreover, the formality of a toric manifold and the nonformalizability for a partial quotient which is not a toric manifold are discussed, with their

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models induced by the Félix and Tanré rational models for polyhedral products; see Theorems 1.6, 5.1 and 5.5 for more details.

Throughout this article, each space X is assumed to be connected and ($\mathbb{Q}$-)locally finite, that is, the rational cohomology group $H^i(X;\mathbb{Q})$ is of finite dimension for $i \geq 0$. In the rest of this section, we describe our main results more precisely. Following Kishimoto and Levi [15], we define a polyhedral product with the homotopy colimit instead of the colimit; see also [16] for the study of the Davis–Januszkiewicz space with the homotopy colimit functor.

Definition 1.1 ([15, Definition 1.2]). Let

$$(\underline{X}, \underline{A}) := ((X_1, A_1), \dots, (X_m, A_m))$$

be a tuple of spaces with $A_i \subset X_i$ for each i, and K a simplicial complex with m vertices. The *polyhedral product* $(\underline{X}, \underline{A})^K$ of the tuple $(\underline{X}, \underline{A})$ corresponding to K is defined by

$$(\underline{X}, \underline{A})^K := \mathrm{hocolim}_{\sigma\in K}(\underline{X}, \underline{A})^\sigma,$$

where $(\underline{X}, \underline{A})^\sigma = Y_1 \times \cdots \times Y_m$ with

$$Y_i = \begin{cases} A_i, & i \notin \sigma, \\ X_i, & i \in \sigma. \end{cases}$$

We write $(X, A)^K$ for $(\underline{X}, \underline{A})^K$ if there is a space X and a subspace A such that $X_i = X$ and $A_i = A$ for each i.

In what follows, we assume that a simplicial complex K has no ghost vertices unless otherwise specified.

Suppose that each (X_i, A_i) is a pair of CW-complexes. Then the natural map $\mathrm{colim}_{\tau\in\partial(\sigma)}(\underline{X}, \underline{A})^\tau \to (\underline{X}, \underline{A})^\sigma$ is a cofibration. Thus, in view of [17, §2 and Proposition 4.8] and also [5, Proposition 8.1.1], we have a weak homotopy equivalence

$$(\underline{X}, \underline{A})^K \xrightarrow{\simeq_w} \mathrm{colim}_{\sigma\in K}(\underline{X}, \underline{A})^\sigma = \bigcup_{\sigma\in K}(\underline{X}, \underline{A})^\sigma =: \mathcal{Z}_K(\underline{X}, \underline{A}).$$

In particular, by definition, the moment-angle complex $\mathcal{Z}_K(D^2, S^1)$ corresponding to a simplicial complex K is the colimit $\bigcup_{\sigma\in K}(D^2, S^1)^\sigma$ and so it is weak homotopy equivalent to the polyhedral product $(D^2, S^1)^K$.

Our first result concerns the rational homotopy groups of a polyhedral product of a pair of Lie groups. We denote by $\pi_*(X)_\mathbb{Q}$ the rational homotopy group $\pi_*(X)\otimes\mathbb{Q}$ for a pointed connected space X whose fundamental group is abelian.

Theorem 1.2. *Let G be a connected compact Lie group and $i : H \to G$ the inclusion of a maximal rank subgroup. Suppose that G/H is simply-connected and $(Bi)^*(x_k)$ is decomposable in $H^*(BH;\mathbb{Q})$ for each generator*

x_k of $H^*(BG;\mathbb{Q})$. Then one has a short exact sequence of rational homotopy groups

$$0 \to \pi_*((G,H)^K)_{\mathbb{Q}} \xrightarrow{q_*} \pi_*((G/H,*)^K)_{\mathbb{Q}} \xrightarrow{\partial_*} \pi_{*-1}(\textstyle\prod^m H)_{\mathbb{Q}} \to 0$$

for an arbitrary simplicial complex K with m vertices, where ∂_* denotes the connecting homomorphism of the homotopy exact sequence of the middle vertical sequence in (1.1) *below.*

We stress that the exactness in the theorem above does not depend on any property of the given simplicial complex K.

REMARK 1.3. While we do not pursue questions about the cohomology $H^*((G,H)^K;\mathbb{K})$ with coefficients in an arbitrary field $\mathbb{K}$, in order to compute the cohomology algebra, we may use the commutative diagram

$$\begin{array}{ccccc} (H,H)^K & = & (H,H)^K & = & \prod^m H \\ \downarrow & & \downarrow & & \downarrow \\ (EG,H)^K & \longleftarrow & (G,H)^K & \longrightarrow & \prod^m G \\ \downarrow & & \downarrow{\scriptstyle q} & & \downarrow \\ (BH,*)^K & \xleftarrow{j} & (G/H,*)^K & \longrightarrow & \prod^m (G/H) \end{array} \tag{1.1}$$

in which vertical sequences are fibrations; see [7, Lemma 2.3.1]. We can regard the lower squares as pullback diagrams.

Before describing our main result on partial quotients, we recall some terminology of rational homotopy theory.

A *commutative differential graded algebra* (henceforth CDGA) (A,d) consists of a nonnegatively graded algebra A and a differential d on A with degree $+1$. Let $A_{\mathrm{PL}}(X)$ be the CDGA of polynomial differential forms on a space X; see [8, 10(c)]. It is worth mentioning that there exists a morphism of cochain complexes from $A_{\mathrm{PL}}(X)$ to the singular cochain algebra of X with coefficients in $\mathbb{Q}$ which induces an isomorphism of cohomology algebras; see [8, 10(e), Remark].

By definition, a *Sullivan algebra* (A,d) is a CDGA whose underlying algebra A is the free algebra $\bigwedge W$ generated by a graded vector space W and for which the vector space W admits a filtration $W_0 \subset W_1 \subset \cdots$ with $W = \bigcup_i W_i$, $d(W_0) = 0$ and $d : W_k \to \bigwedge W_{k-1}$ for $k \geq 1$. We say that a Sullivan algebra $(\bigwedge W, d)$ is *minimal* if $d(w)$ is decomposable for each $v \in W$.

A morphism $\varphi : (A,d) \to (B,d')$ of CDGAs is a *quasi-isomorphism* if φ induces an isomorphism on cohomology. A *rational model* (A,d) for a space X is a CDGA which is connected to $A_{\mathrm{PL}}(X)$ by a sequence of quasi-isomorphisms. We call the rational model (A,d) a (*minimal*) *Sullivan model* for X if it is a (minimal) Sullivan algebra. Observe that each space has a unique minimal Sullivan model; see [8, 14(b), Corollary]. A space X is *formal* if there exists a sequence of quasi-isomorphisms between a Sullivan model for X and the cohomology $H^*(X;\mathbb{Q})$ which is regarded as a CDGA with zero differential. We refer the reader to the books [13, 8, 9] for rational homotopy theory and its applications to topology and geometry.

DEFINITION 1.4 (cf. [18, V]). A map $p : E \to B$ is *formalizable* if there exists a diagram commutative up to homotopy,

$$\begin{array}{ccc} A_{\mathrm{PL}}(B) & \xrightarrow{A_{\mathrm{PL}}(p)} & A_{\mathrm{PL}}(E) \\ \simeq\uparrow & & \uparrow\simeq \\ (\bigwedge W, d) & \xrightarrow{l} & (\bigwedge Z, d') \\ \simeq\downarrow & & \downarrow\simeq \\ H^*(B;\mathbb{Q}) & \xrightarrow{p^*} & H^*(E;\mathbb{Q}) \end{array}$$

in which $(\bigwedge W, d)$ and $(\bigwedge Z, d')$ are minimal Sullivan algebras and the vertical arrows are quasi-isomorphisms; see [8, 12(b)] and [14, Chapter 5] for the homotopy relation.

For a simplicial complex K, define $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)$ as the colimit

$$\mathrm{colim}_{\tau\in K}(\mathbb{C}, \mathbb{C}^*)^\tau.$$

Then we have weak equivalences

$$(D^2, S^1)^K \xrightarrow{\simeq_w} \mathrm{colim}_{\tau\in K}(D^2, S^2)^\tau \underset{\simeq}{\xrightarrow{i}} \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*),$$

where i is the inclusion; see [5, Theorem 4.7.5]. Let X_Σ be a compact toric manifold associated with a complete and smooth fan Σ; see [6, §3.1]. We then have a homeomorphism $X_\Sigma \cong \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H$ via Cox's construction of the manifold X_Σ, where K is the simplicial complex with m vertices associated with the fan Σ and H is a subgroup of the torus $(\mathbb{C}^*)^m$ which acts on $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)$ canonically and freely; see [6, Theorem 5.1.11] and [5, Theorem 5.4.5, Proposition 5.4.6]. The quotient $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H'$ by a subtorus $H' \subset H$ is called a *partial quotient*.

We recall the pullback diagram in [11, proof of Proposition 3.2]. Let X_Σ be a toric manifold associated with a fan Σ, and let $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H$ be Cox's construction of X_Σ mentioned above. Then we have a commutative diagram consisting of two pullbacks

$$\begin{array}{ccccc} EG \times_H \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*) & \xrightarrow{p} & (EG)/H & \longrightarrow & EL \\ \pi_H\downarrow & & \downarrow & & \downarrow \\ EG \times_G \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*) & \xrightarrow{q} & BG & \xrightarrow{B\rho} & BL \end{array} \tag{1.2}$$

where $G = (\mathbb{C}^*)^m$ and $L = (\mathbb{C}^*)^m/H$. We observe that the two nonlabelled vertical maps are principal L-bundles and that the maps p and q are fibrations associated with the universal H-bundle and the universal G-bundle, respectively. Since the group H acts on $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)$ freely, it follows that the Borel construction $EG \times_H \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)$ is homotopy equivalent to the toric manifold X_Σ.

Let H' be a subtorus of H. Then we may replace H and L in the diagram (1.2) with H' and $L' := (\mathbb{C}^*)^m/H$, respectively. With this replacement, the upper left corner in the diagram can be regarded as the partial quotient $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H' \simeq EG \times_{H'} \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)$.

REMARK 1.5. It follows from [5, Theorems 4.3.2, 4.7.5] that the Borel construction $EG \times_G \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)$ is homotopy equivalent to the Davis–Januszkiewicz space $DJ(K) := (BS^1, *)^K$. Since the fan we consider is complete, it follows from [6, Theorem 12.1.10] that X_Σ is simply-connected. Then we have an exact sequence

$$0 \to \pi_*(X_\Sigma) \xrightarrow{(\pi_H)_*} \pi_*(DJ(K)) \xrightarrow{\partial_*} \pi_{*-1}(G/H) \to 0.$$

By considering the center vertical fibration mentioned in (1.1), the exact sequence in Theorem 1.2 can be regarded as an analogy of the sequence above.

The following result characterizes the toric manifolds among the associated partial quotients.

THEOREM 1.6. *Let $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H$ be a toric manifold and H' a subtorus of H. For the partial quotient $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H'$, the following conditions are equivalent.*

(i) $H = H'$.
(ii) $H^{\mathrm{odd}}(\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H'; \mathbb{Q}) = 0$.
(iii) *The map $\pi_{H'} : \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H' \to DJ(K)$ in the diagram* (1.2) *is formalizable.*

REMARK 1.7. As seen in Theorem 5.1, a toric manifold is formal. However, we do not know whether every partial quotient is formal.

An outline of the article is as follows. Section 2 recalls the construction of the Félix–Tanré rational model for a polyhedral product and discusses the naturality of the models. In Section 3, we give a tractable rational model for a polyhedral product and some examples for the model. Section 4 constructs a rational model for the polyhedral product $(G, H)^K$ of a pair of a Lie group and a closed subgroup corresponding to an arbitrary simplicial complex K. With this model, we prove Theorem 1.2. In Section 5, we show that every compact toric manifold is formal. Section 6 is devoted to proving Theorem 1.6.

2. A recollection of the Félix–Tanré rational models for polyhedral products. While the construction of a rational model in [10] for a polyhedral product is defined by the colimit construction, it is also applicable to constructing a rational model for $(\underline{X},\underline{A})^K$ obtained by the homotopy colimit as in Definition 1.1. In this section, we summarize this construction.

Let $\iota_j : A_j \to X_j$ be the inclusion and $\varphi_j : \mathcal{M}_j \to \mathcal{M}'_j$ a surjective model ([1]) for ι_j, namely, an epimorphism of CDGAs which fits in a commutative diagram

$$\begin{array}{ccc} \bigwedge W_j & \xrightarrow{\varphi_j} & \bigwedge V_j \\ u\downarrow\simeq & & \simeq\downarrow v \\ A_{\mathrm{PL}}(X_j) & \xrightarrow{\iota_j^*} & A_{\mathrm{PL}}(A_j) \end{array} \tag{2.1}$$

with quasi-isomorphisms u and v. We observe that ι_j^* is surjective; see [8, Proposition 10.4, Lemma 10.7]. For each $\tau \notin K$, let I_τ denote the ideal of $\bigotimes_{i=1}^m \mathcal{M}_i$ defined by $I_\tau = E_1 \otimes \cdots \otimes E_m$, where

$$E_i = \begin{cases} \mathcal{M}_i, & i \notin \tau, \\ \operatorname{Ker}\varphi_i, & i \in \tau. \end{cases}$$

Theorem 2.1 ([10, Theorem 1]). *There is a sequence of quasi-isomorphisms connecting the CDGA $A_{\mathrm{PL}}((\underline{X},\underline{A})^K)$ and $(\bigotimes_{i=1}^m \mathcal{M}_i)/J(K)$, where $J(K) := \sum_{\tau\notin K} I_\tau$; that is, the quotient is a rational model for $(\underline{X},\underline{A})^K$.*

In what follows, we may call the quotient CDGA in Theorem 2.1 the *Félix–Tanré* (*rational*) *model* for the polyhedral product $(\underline{X},\underline{A})^K$.

Remark 2.2. We observe that the polyhedral product $(\underline{X},\underline{A})^K$ is defined by the homotopy colimit of the diagram associated to the simplicial complex K. While the nilpotency of each space in the pairs (X_i, A_i) of CW complexes for $1 \le i \le m$ is assumed in [10, Theorem 1], this condition is not required in Theorem 2.1. In fact, for each inclusion $\iota_j : A_j \to X_j$, we have a commutative diagram

$$\begin{array}{ccc} A_j & \xrightarrow{\iota_j} & X_j \\ \simeq\uparrow & & \uparrow\simeq \\ |S(A_j)| & \xrightarrow{|S(\iota)|} & |S(X_j)| \end{array} \tag{2.2}$$

with the singular simplex functor $S(\ )$ and the realization functor $|\ |$. This enables us to obtain a sequence of weak homotopy equivalences

$$(\underline{X},\underline{A})^K \xleftarrow{\simeq_w} (\underline{X'},\underline{A'})^K \xrightarrow{\simeq_w} \mathrm{colim}_{\sigma\in K}(\underline{X'},\underline{A'})^\sigma = \bigcup_{\sigma\in K}(\underline{X'},\underline{A'})^\sigma, \tag{2.3}$$

([1]) The existence of the model: We consider a Sullivan representative for ι_j; see [8, p. 154]. The proof of [8, Lemma 13.4] enables us to replace the homotopy commutative diagram of the representative with a strictly commutative diagram. By applying the surjective trick [8, §12(b)], we have a surjective model for the inclusion.

where (X_i', A_i') denotes the pair $(|S(X_i)|, |S(A_i)|)$; see the paragraph after Definition 1.1. A surjective model for each inclusion $A_j \to X_j$ is regarded as that for the inclusion $A_j' \to X_j'$. Thus, with these models and by applying [8, Proposition 13.5] inductively as in [10, proof of Theorem 1], we can prove Theorem 2.1 without assuming that the spaces X_i and A_i are CW-complexes and nilpotent.

In order to prove the naturality of the model in Theorem 2.1 with respect to inclusions of simplicial complexes and also given surjective models, the outline of the proof of Theorem 2.1 is presented below.

By using surjective models $\varphi_j : \mathcal{M}_j \to \mathcal{M}_j'$, for each $\sigma \in K$, we have a CDGA $\widetilde{D}^\sigma := \bigotimes_{i\in\sigma} \mathcal{M}_i \otimes \bigotimes_{i\notin\sigma} \mathcal{M}_i'$ and a map $\xi_\sigma : (\bigotimes_{i=1}^m \mathcal{M}_i)/J(K) \to \widetilde{D}^\sigma$ of CDGAs defined by

$$\xi_\sigma(x_i) = \begin{cases} x_i, & i \in \sigma, \\ \varphi_i(x_i), & i \notin \sigma. \end{cases}$$

It is readily seen that ξ_σ is well-defined. The induction argument in [10, proof of Theorem 1] shows that the maps ξ_σ of CDGAs give rise to a quasi-isomorphism

$$\alpha : \Big(\bigotimes_{i=1}^m \mathcal{M}_i\Big)/J(K) \xrightarrow{\simeq} \lim_{\sigma\in K} \widetilde{D}^\sigma.$$

We also observe that this fact is proved by using [8, Lemma 13.3] which gives a well-defined quasi-isomorphism between appropriate pullback diagrams in the category of CDGAs. Thus, with the same notation as in Remark 2.2, we have a sequence of quasi-isomorphisms

(2.4)

$$\begin{array}{c} A_{\mathrm{PL}}((\underline{X},\underline{A})^K) \xrightarrow{\simeq} A^*_{\mathrm{PL}}((\underline{X'},\underline{A'})^K) \xleftarrow{\simeq} A_{\mathrm{PL}}(\mathrm{colim}_{\sigma\in K}(\underline{X'},\underline{A'})^\sigma) \\ \swarrow_{\eta}^{\simeq} \\ \lim_{\sigma\in K} A_{\mathrm{PL}}((\underline{X'},\underline{A'})^\sigma) \xleftarrow[\simeq]{\Phi} \lim_{\sigma\in K} \widetilde{D}^\sigma \xleftarrow[\simeq]{\alpha} (\bigotimes_{i=1}^m \mathcal{M}_i)/J(K) \end{array}$$

in which the first two quasi-isomorphisms are induced by the weak equivalences in (2.3), Φ is defined by the surjective models φ_i, and η is induced by the natural maps $(\underline{X},\underline{A})^\sigma \to (\underline{X},\underline{A})^K$. It follows from [17, Proposition 4.8] and [5, Proposition 8.1.4] respectively that Φ and η are quasi-isomorphisms. This enables us to obtain the rational model for the polyhedral product $(\underline{X},\underline{A})^K$ in Theorem 2.1. Moreover, the above construction of the model yields the following proposition.

PROPOSITION 2.3. *The Félix–Tanré rational models are natural with respect to surjective models which are used in constructing the models of polyhedral products and inclusions of simplicial complexes.*

The rational cohomology of the moment-angle complex $\mathcal{Z}_K(D^2, S^1)$ is isomorphic to the torsion product $\mathrm{Tor}_{\mathbb{Q}[t_1,\dots,t_m]}(\mathbb{Q}[t_1,\dots,t_m]/I(K), \mathbb{Q})$, where $\deg t_i = 2$ and $I(K)$ denotes the ideal generated by monomials $t_{i_1}\cdots t_{i_s}$ for $\{i_1,\dots,i_s\} \notin K$, which is called the *Stanley–Reisner ideal* associated with K; see [11, 10.1]. Thus, a CDGA of the form

$$(2.5)\qquad (\bigwedge(x_i,\dots,x_m) \otimes \mathbb{Q}[t_1,\dots,t_m]/I(K), d(x_i) = t_i)$$

computes the cohomology algebra $H^*(\mathcal{Z}_K(D^2, S^1); \mathbb{Q})$. Here $SR(K)$ denotes the *Stanley–Reisner algebra* $\mathbb{Q}[t_1,\dots,t_m]/I(K)$.

Remark 2.4. The inclusion $i : S^1 \to D^2$ admits a surjective model of the form $\pi : (\bigwedge(x,t), d) \to (\bigwedge(x), 0)$, where π is the projection, $d(x) = t$ and $\deg x = 1$. By virtue of Theorem 2.1, we see that the CDGA (2.5) is a rational model for $\mathcal{Z}_K(D^2, S^1)$.

Example 2.5. Let K be a simplicial complex with m vertices and $j : K \to 2^{[m]}$ be the inclusion. The map j induces the inclusion $\widetilde{j} : (BS^1, *)^K \to \prod^m (BS^1)$. We choose the projection $(\bigwedge(t), 0) \to \mathbb{Q}$ as a surjective model for the inclusion $* \to BS^1$, where $\deg t_i = 2$ for $i = 1,\dots,m$. It follows from Theorem 2.1 and Proposition 2.3 that we have a model $(\bigwedge(t_1,\dots,t_m), 0) \to (\bigwedge(t_1,\dots,t_m)/I(K), 0) = (SR(K), 0)$ for which $\widetilde{j}$ is the natural projection. As a consequence, we see that the inclusion $\widetilde{j}$ is formalizable in the sense of Definition 1.4.

3. Comparatively tractable rational models for polyhedral products. The Félix–Tanré rational model for a polyhedral product $(\underline{X}, \underline{A})^K$ depends on the choice of surjective models for the inclusions in the given tuple $(\underline{X}, \underline{A})$. While the model is complicated in general, the underlying algebra is adjustable in the sense of Theorem 3.1 below. In fact, we show that the underlying algebra of the model has a particular form which can be regarded as a generalization of the rational model for a moment-angle complex; see Remark 2.4.

We recall the CDGA in (2.5). With this mind, we may call a CDGA (A, d) a *Stanley–Reisner* (*SR*) *K-type* if the underlying algebra A is of the form

$$\Big(\bigotimes_{j=1}^{m} (\bigwedge V_j \otimes B_j)\Big)/(b_{j_1}\cdots b_{j_s} \mid b_j \in B_j^+, \{j_1,\dots,j_s\} \notin K),$$

where B_j is a free commutative algebra. The 'K-' may be omitted if it is clear from the context.

Theorem 3.1. *Each polyhedral product $(\underline{X}, \underline{A})^K$ has an SR type CDGA model; that is, there is a sequence of quasi-isomorphisms of CDGAs connecting $A_{\mathrm{PL}}((\underline{X}, \underline{A})^K)$ and a Stanley–Reisner K-type CDGA.*

Proof. For each j, let $\varphi_j : \bigwedge W_j \to \bigwedge V_j$ be a surjective model for the inclusion $\iota_j : A_j \to X_j$. Since φ_j is surjective, the vector space W_j admits a decomposition $W_j \cong W_j' \oplus W_j''$ which satisfies the condition that $\varphi_j|_{W_j'} : W_j' \overset{\cong}{\to} V_j$ is an isomorphism and $\varphi_j|_{W_j''} \equiv 0$. In fact, we choose indecomposable elements w_λ of $\bigwedge W_j$ so that $\varphi_j(w_\lambda) = v_\lambda$ for a basis $\{v_\lambda\}_{\lambda\in\Lambda}$ for V_j. Then we have a decomposition

$$W_j \cong \mathbb{Q}\{w_\lambda \mid \lambda \in \Lambda\} \oplus \mathbb{Q}\{w_\gamma'' \mid \gamma \in \Gamma\}$$

with some index set Γ. Let $P(v_\lambda)$ be the polynomial in v_λ's which represents $\varphi_j(w_\gamma'')$ in $\bigwedge V_j$. Putting $W_j'' := \mathbb{Q}\{w_\gamma'' - P(w_\lambda)\}$, we have the decomposition required above. Theorem 2.1 yields the result. ■

We now provide a more tractable SR type model for a polyhedral product of a fibre inclusion. For $1 \le j \le m$, let $F_j \xrightarrow{\iota_j} X_j \xrightarrow{p_j} Y_j$ be a fibration with a simply-connected base. Assume that $H^*(Y_j;\mathbb{Q})$ is locally finite for each j. Then a relative Sullivan model $\widetilde{p_j}$ for the map p_j gives a commutative diagram of CDGAs

$$\begin{array}{ccccc}
\bigwedge W_j & \xrightarrow{\widetilde{p_j}} & (\bigwedge V_j \otimes \bigwedge W_j, d_j) & \xrightarrow{\widetilde{\iota_j}} & (\bigwedge V_j, \overline{d_j}) \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
A_{\mathrm{PL}}(Y_j) & \xrightarrow{p_j^*} & A_{\mathrm{PL}}(X_j) & \xrightarrow{\iota_j^*} & A_{\mathrm{PL}}(F_j)
\end{array}$$

in which the vertical maps are quasi-isomorphisms; see [14, 20.3, Theorem]. The upper sequence is called a model for the fibration. It follows from the construction that $\widetilde{\iota_j}$ is a surjective model for ι_j. If $\widetilde{p_j}$ is minimal, by definition, we see that $d(V_j) \subset (\bigwedge^{\ge 2} V_j)\otimes \bigwedge W_j + \bigwedge V_j \otimes \bigwedge^+ W_j$ in the SR type CDGA. By virtue of Theorem 2.1, we have

Proposition 3.2. *With the same notation as above, the polyhedral product $(\underline{X}, \underline{F})^K$ for the tuple of fibre inclusions ι_j has an SR type CDGA model of the form*

$$\mathcal{M}((\underline{X}, \underline{F})^K) := \Big(\Big(\bigotimes_{j=1}^m (\bigwedge V_j \otimes \bigwedge W_j)\Big)/(b_{j_1}\cdots b_{j_s} \mid b_j \in W_j, \{j_1, \dots, j_s\} \notin K), d\Big)$$

for which $d(W_j) \subset \bigwedge W_j$ and $d(V_j) \subset (\bigwedge^{\ge 2} V_j) \otimes \bigwedge W_j + \bigwedge V_j \otimes \bigwedge^+ W_j$.

Remark 3.3. The model in Proposition 3.2 is not a Sullivan model in general. However, if we construct a Sullivan model by using this model, then for example, we may obtain information on the rational homotopy group of $(\underline{X}, \underline{F})^K$; see Example 4.3 below.

Example 3.4. (i) Let $S^1 \to ES^1 \to BS^1$ be the universal S^1-bundle and K be a simplicial complex with m vertices. Then we have a model for the bundle of the form $\bigwedge(dx) \to \bigwedge(dx) \otimes \bigwedge(x) \xrightarrow{\widetilde{\iota}} \bigwedge(x)$, where $\widetilde{\iota}$ is the canonical projection and $\deg x = 1$. It follows from Proposition 3.2 that $\mathcal{M}((ES^1, S^1)^K) \cong (\bigwedge(x_1, \dots, x_m) \otimes SR(K), d)$ where $d(x_j) = dx_j$; see Remark 2.4. Observe that $\mathcal{Z}_K(D^2, S^1) \simeq \mathcal{Z}_K(ES^1, S^1) \simeq_w (ES^1, S^1)^K$; see [7, p. 33] for the first homotopy equivalence.

(ii) Let X be a simply-connected space and LX the free loop space, that is, the space of maps from S^1 to X endowed with the compact-open topology. The rotation action of S^1 on the domain of maps in LX induces an S^1-action on the free loop space. Thus we have the Borel fibration $LX \xrightarrow{i} ES^1 \times_{S^1} LX \xrightarrow{p} BS^1$. We write $(LX)_h$ for the Borel construction $ES^1 \times_{S^1} LX$. Let $(\bigwedge V, d)$ be the minimal model for X. Then [20, Theorem A] asserts that the sequence

$$\bigwedge(t) \xrightarrow{\widetilde{p}} (\bigwedge(t) \otimes \bigwedge(V \oplus \overline{V}), \delta) \xrightarrow{\widetilde{i}} (\bigwedge(V \oplus \overline{V}), \delta')$$

is a model for the Borel fibration, where $\delta'(v) = d(v)$, $\delta'(\overline{v}) = -sd(v)$ and $\delta u = \delta'(u) + ts(u)$ for $u \in V \oplus \overline{V}$. The map $\widetilde{i}$ is the projection and hence a surjective model for i. Thus Proposition 3.2 enables us to obtain a Félix–Tanré model for the polyhedral product $((LX)_h, LX)^K$ of the form $\bigotimes_{i=1}^m (\bigwedge(V_i \oplus \overline{V}_i) \otimes SR(K), \bigotimes_i \delta_i)$.

(iii) We can apply Theorem 2.1 to an explicit surjective model for an inclusion. Let X be a space as in (ii) and $(\bigwedge V, d)$ a minimal model for X. Then the projection $(\bigwedge(V \oplus \overline{V}), \delta') \to (\bigwedge V, d)$ is a surjective model for the inclusion $X \to LX$ defined by assigning the constant loop at x to a point x. In fact, this inclusion is a section of the evaluation map $\mathrm{ev}_0 : LX \to X$ at zero. The inclusion $(\bigwedge V, d) \to (\bigwedge(V \oplus \overline{V}), \delta')$ gives rise to a model for ev_0. By considering rational homotopy groups, the result follows. Thus, Theorem 2.1 allows us to construct a model for the polyhedral product $(LX, X)^K$ of the form

$$\Big(\bigotimes_{i=1}^m \bigwedge(V_i \oplus \overline{V_i})\Big)/(\overline{v}_{i_1} \cdots \overline{v}_{i_s} \mid \overline{v}_j \in \overline{V}_j,\ \{i_1, \dots, i_s\} \notin K)$$

for which $d(V_i) \subset \bigwedge V_i$.

Proposition 3.2 enables us to deduce the following result.

Corollary 3.5. *Let K be a simplicial complex with m vertices. For $1 \le j \le m$, let $F_j \to X_j \to Y_j$ be a fibration with a simply-connected base Y_j. Then there is a first quadrant spectral sequence converging to $H^*((\underline{X}, \underline{F})^K; \mathbb{Q})$*

as an algebra with

$$E_2^{*,*} \cong \Big(\bigotimes_{j=1}^{m} H^*(F_j;\mathbb{Q})\Big) \otimes H^*((\underline{Y},\underline{*})^K;\mathbb{Q})$$

as a bigraded algebra, where $E_2^{p,q} \cong \big((\bigotimes_{j=1}^m H^*(F_j;\mathbb{Q}))\otimes H^p((\underline{Y},\underline{*})^K;\mathbb{Q})\big)^{p+q}$.

Proof. With the same notation as in Proposition 3.2, we give the CDGA $\mathcal{M}((\underline{X},\underline{F})^K)$ a filtration associated with the degrees of elements in $\bigotimes_j \bigwedge W_j$. This filtration gives rise to the spectral sequence; see [8, 18(b), Example 2]. ∎

Let $HH_*(A_{\mathrm{PL}}(X))$ denote the Hochschild homology of $A_{\mathrm{PL}}(X)$. There exists an isomorphism $HH_*(A_{\mathrm{PL}}(X)) \cong H^*(LX;\mathbb{Q})$ of algebras; see [20] and [8, 15(c), Example 1]. Therefore, Example 3.4(ii) allows us to obtain the following result.

Corollary 3.6. *Let X be a simply-connected space. Then there exists a first quadrant spectral sequence converging to $H^*(((LX)_h, LX)^K;\mathbb{Q})$ as an algebra with*

$$E_2^{*,*} \cong HH_*(A_{\mathrm{PL}}(X))^{\otimes m} \otimes SR(K)$$

as a bigraded algebra, where bideg $x = (0,\deg x)$ *for* $x \in HH_*(A_{\mathrm{PL}}(X))$ *and* bideg $t_i = (2,0)$ *for the generator* $t_i \in SR(K)$.

Remark 3.7. Let $F_j \xrightarrow{i_j} X_j \xrightarrow{p_j} B_j$ be a fibration for each $1 \le j \le m$. Suppose further that each (X_j, F_j) is a pair of CW-complexes. Then the Félix and Tanré rational model for $(\underline{X},\underline{F})^K$ in Proposition 3.2 associated with the fibre inclusions is nothing other than the relative Sullivan model for the pullback

$$\begin{array}{ccc}
(\underline{F},\underline{F})^K & = & \prod^m \underline{F} \\
\downarrow & & \downarrow \\
(\underline{X},\underline{F})^K & \longrightarrow & \prod^m \underline{X} \\
\downarrow & & \downarrow \\
(\underline{B},\underline{*})^K & \longrightarrow & \prod^m \underline{B}
\end{array}$$

which is introduced in [7, Lemma 2.3.1]. In fact, this follows from [14, 20.6].

4. A rational model for the polyhedral product of a pair of Lie groups. In this section, we consider a more explicit model for the polyhedral product $(G,H)^K$ for a pair of a Lie group and a closed subgroup corresponding to an arbitrary simplicial complex K. In particular, we have a manageable SR type model for $(G,H)^K$. Indeed, the rational model is determined by the image of the characteristic classes of BG under the map $(Bi)^* : H^*(BG;\mathbb{Q}) \to H^*(BH;\mathbb{Q})$ for the inclusion $i : H \to G$; see Proposition 4.1 for more details on the model. By using this model, we prove Theorem 1.2.

Let G be a connected Lie group and H a closed connected subgroup of G. Let $H \overset{i}{\to} G \overset{\pi}{\to} G/H$ be the principal H-bundle. In order to obtain a rational model for $(G,H)^K$, we first construct an appropriate surjective model for the fibre inclusion i.

Consider the fibration $EH \to EH \times_H G \overset{q}{\to} G/H$ associated with the bundle π. Since EH is contractible, it follows that the map q is a weak homotopy equivalence. Moreover, we have a homotopy pullback diagram

$$
\begin{CD}
G @>{=}>> G \\
@V{\iota}VV @VVV \\
EH \times_H G @>{h}>> EG \\
@VVV @VV{p_G}V \\
BH @>{Bi}>> BG
\end{CD}
\tag{4.1}
$$

where vertical sequences are the associated fibration and the universal G-bundle, respectively, and h is defined by $h([x,g]) = Ei(x)g$. There exists a model for the universal bundle of the form

$$
\begin{CD}
(\bigwedge V_{BG}, 0) @>>> (\bigwedge V_{BG} \otimes \bigwedge P_G, d) @>>> (\bigwedge P_G, 0) \\
@V{\simeq}V{m_{BG}}V @V{\simeq}V{m_{E_G}}V @V{\simeq}VV \\
A_{\mathrm{PL}}(BG) @>{p_G^*}>> A_{\mathrm{PL}}(E_G) @>>> A_{\mathrm{PL}}(G)
\end{CD}
$$

such that $d(x_i) = y_i$ and $m_{E_G}(x_i) = \Psi_i$ for $x_i \in \bigwedge V_G$, where $\Psi_i \in A_{\mathrm{PL}}(E_G)$ with $d\Psi_i = p_G^* m_{BG}(y_i)$. Then, by applying the pushout construction [8, Proposition 15.8] to the model of the bundle p_G, we obtain a model

$$
\begin{CD}
(\bigwedge V_{BH}, 0) @>>> (\bigwedge V_{BH} \otimes \bigwedge P_G, d) @>{\bar{\iota}}>> (\bigwedge P_G, 0) \\
@V{\simeq}VV @V{\simeq}V{m}V @V{\simeq}V{m_G}V \\
A_{\mathrm{PL}}(BH) @>>> A_{\mathrm{PL}}(EH \times_H G) @>{A_{\mathrm{PL}}(\iota)}>> A_{\mathrm{PL}}(G)
\end{CD}
$$

of the fibration of the left hand side in the diagram (4.1) in which $d(x_i) = (Bi)^* y_i$. Furthermore, the maps π, ι and q mentioned above fit in the commutative diagram

$$
\begin{array}{ccccc}
G & \xrightarrow{\simeq} & EH \times G & \xrightarrow{\simeq} & G \\
 & {\scriptstyle\iota}\searrow & \downarrow & & \downarrow{\scriptstyle\pi} \\
 & & EH \times_H G & \underset{\simeq}{\xrightarrow{q}} & G/H
\end{array}
$$

where the horizontal arrows are (weak) homotopy equivalences. Thus, the lifting lemma [8, Proposition 14.6] implies that a Sullivan model [8, 15(a)] for ι can be regarded as that for π. Consider a commutative diagram

$$\begin{array}{ccccc} (\bigwedge V_{BH} \otimes \bigwedge P_G, d) & \xrightarrow{j} & (\bigwedge V_{BH} \otimes \bigwedge P_G \otimes \bigwedge P_H, \partial) & \underset{\simeq}{\xrightarrow{\gamma}} & (\bigwedge P_G, 0) \\ \simeq \big\downarrow m & & & \swarrow^{\simeq}_{m_G} & \\ A_{\mathrm{PL}}(EH \times_H G) & \xrightarrow{A_{\mathrm{PL}}(\iota)} & A_{\mathrm{PL}}(G) & & \end{array}$$

of CDGAs in which j is an extension, γ is the projection and the differential ∂ is defined by $\partial(u_i) = t_i$ for $u_i \in P_H$, $t_i \in V_{BH}$ and $\partial(x_i) = (Bi)^*(y_i)$ for $x_i \in P_G$. Observe that $H^*(\bigwedge P_G, 0) \cong H^*(G; \mathbb{Q})$ and γ is a quasi-isomorphism. Then, we see that the projection $\gamma : (\bigwedge V_{BH} \otimes \bigwedge P_G \otimes \bigwedge P_H, \partial) \to (\bigwedge P_H, 0)$ is a surjective model for the inclusion $H \to G$. Thus, Proposition 3.2 yields the following result.

Proposition 4.1. *One has a rational model of the form*

$$\big((\bigwedge P_H)^{\otimes m} \otimes ((\bigwedge V_{BH} \otimes \bigwedge P_G)^{\otimes m}/I(K)), \partial\big) \tag{4.2}$$

for the polyhedral product $(G, H)^K$, *where* $I(K)$ *denotes the Stanley–Reisner ideal generated by elements in* $(\bigwedge V_{BK} \otimes \bigwedge P_G)^{\otimes m}$.

Example 4.2. With the same notation as above, suppose further that $\operatorname{rank} G = \operatorname{rank} H = N$. Then the sequence $(Bi)^*(y_j)$ for $j = 1, \dots, N$ is regular. This enables us to deduce that G/H is formal. There exists a sequence of quasi-isomorphisms

$$\begin{aligned} A_{\mathrm{PL}}(G/H) \xleftarrow{\simeq} \bigwedge V_{BH} \otimes \bigwedge P_G =: \mathcal{M} \underset{u}{\xrightarrow{\simeq}} H^*(G/H) \\ = (\bigwedge V_{BH}/(Bi)^*(y_i), d = 0). \end{aligned}$$

The naturality (Proposition 2.3) of the rational model in Theorem 2.1 gives rise to a commutative diagram

$$\begin{array}{ccccc} \lim_{\sigma \in K} A_{\mathrm{PL}}((G/H, *)^\sigma) & \underset{\simeq}{\xleftarrow{\Phi}} & \lim_{\sigma \in K} \widetilde{D}^\sigma & \underset{\simeq}{\xleftarrow{\alpha}} & (\bigotimes^m \mathcal{M})/I(K) =: B_1 \\ \eta \big\uparrow \simeq & & \simeq \big\downarrow u_1 & & \big\downarrow u_2 \\ A_{\mathrm{PL}}((G/H, *)^K) & & \lim_{\sigma \in K} H^*(G/H)^\sigma & \underset{\simeq}{\xleftarrow{\alpha}} & (\bigotimes^m H^*(G/H))/I(K) =: B_2 \end{array}$$

where $H^*(G/H)^\sigma := \bigotimes_{i \in \sigma} H^*(G/H)^\sigma \otimes \bigotimes_{i \notin \sigma} \mathbb{Q}$ and u_1 and u_2 are maps of CDGAs induced by u; see (2.4). By virtue of [17, Proposition 4.8], we see that the map u_1 is a quasi-isomorphism. Thus, commutativity implies that u_2 is also a quasi-isomorphism. Observe that $I(K) = J(K)$ in the case that we deal with; see Section 2. We consider the pushout diagram of ℓ along u_2:

$$\begin{array}{ccc} B_1 & \xrightarrow{\ell} & ((\bigwedge P_H)^{\otimes m} \otimes (\bigotimes^m \mathcal{M})/I(K), \partial) \\ u_2 \big\downarrow \simeq & & \big\downarrow \widetilde{u_2} \\ B_2 & \xrightarrow{\widetilde{\ell}} & ((\bigwedge P_H)^{\otimes m} \otimes (\bigotimes^m H^*(G/H))/I(K), \partial) =: C \end{array} \tag{4.3}$$

where ℓ is the KS-extension induced by the rational model for $(G,H)^K$ in (4.2); see [14, Chapter 1] for the definition of a KS-extension. It follows from [8, Lemma 14.2] that $\widetilde{u_2}$ is a quasi-isomorphism and hence C is also a rational model for $(G,H)^K$.

In particular, for the unitary group $U(n)$, the maximal torus T and every simplicial complex K with m vertices, we have a rational model for $(U(n),T)^K$ of the form

$$\Big((\bigwedge(x_i,\dots,x_n))^{\otimes m}\otimes\Big(\bigotimes_{i=1}^{m}\mathbb{Q}[t_1,\dots,t_n]/(\sigma_1,\dots,\sigma_n)\Big)/I(K),\partial\Big),$$

where $\partial(x_i)=t_i$ and σ_k denotes the kth elementary symmetric polynomial.

Example 4.3. Let K be an arbitrary simplicial complex with m vertices. By virtue of Propositions 2.3 and 4.1, we see that the projection

$$q:(SU(n),SU(k))^K\to(SU(n)/SU(k),*)^K$$

admits a model given by

$$\begin{aligned}\widetilde{q}:(\textstyle\bigwedge(x_{k+1},\dots,x_n)^{\otimes m}/I(K)),0)\\ \to\big(\textstyle\bigwedge(x_2,\dots,x_k)^{\otimes m}\otimes(\bigwedge(x_{k+1},\dots,x_n)^{\otimes m}/I(K)),0\big),\end{aligned}$$

where $\widetilde{q}(x_i)=x_i$ for $k+1\le i\le n$, and $\deg x_i=2i-1$. Since the domain of $\widetilde{q}$ admits the structure of a Sullivan algebra, we can construct a KS-extension for $\widetilde{q}$. Then it follows from Lemma A.1 that the projection q is formalizable in the sense of Definition 1.4.

Suppose further that the 1-skeleton of K does not coincide with that of Δ^m. Then the minimal model for $(SU(n)/SU(k),*)^K$ has a nontrivial differential whose quadratic part is also nontrivial; see [8, pp. 144–145] for a way to construct a minimal model for a CDGA. Therefore, [8, Theorem 21.6] implies that the rational homotopy groups $\pi_*((SU(n)/SU(k),*)^K)_{\mathbb{Q}}$ and $\pi_*((SU(n),SU(k))^K)_{\mathbb{Q}}$ have nontrivial Whitehead products. Observe that the Whitehead product on $\pi_*(SU(n)/SU(k))_{\mathbb{Q}}$ vanishes.

Let X be a pointed space and $\pi^*(X):=H^*(Q(\bigwedge V),d_0)$ the homology of the vector space of indecomposable elements of a Sullivan model $(\bigwedge V,d)$ for X, where $Q(\bigwedge V)$ is the vector space of indecomposable elements and d_0 denotes the linear part of the differential d. There is a natural map ν_X from $\pi^*(\bigwedge V)$ to $\mathrm{Hom}(\pi_*(X),\mathbb{Q})$ provided $\pi_*(X)$ is abelian. Moreover, ν_X is an isomorphism if X is a nilpotent space whose fundamental group is abelian; see [4, 11.3].

It follows from [8, proof of Proposition 15.13] that the natural map $\nu_{(\ )}$ is compatible with the connecting homomorphisms of the dual to the homotopy exact sequence for a fibration and the homology exact sequence for $\pi^*(\ )$ if the fundamental groups of the spaces of the fibration are abelian. Then, by considering the middle vertical fibration $\mathcal{F}$ in (1.1), we arrive at

LEMMA 4.4. *For each space X in the fibration $\mathcal{F}$, the map ν_X is an isomorphism.*

Proof of Theorem 1.2. We first observe that $(G/H,*)^K$ is simply-connected. This follows from the Seifert–van Kampen theorem. Then we see that $\pi_1((G,H)^K)$ is an abelian group.

In what follows, we consider a Sullivan model for $(G,H)^K$ with the same notation as in Example 4.2. Let $\beta_0 : \bigotimes^m \mathcal{M} \to (\bigotimes^m H^*(G/H))/I(K) =: B_2$ be the composition of the quasi-isomorphism $u_2 : B_1 \to B_2$ mentioned above and the projection $\bigotimes^m \mathcal{M} \to (\bigotimes^m \mathcal{M})/I(K)$. Observe that $\mathcal{M} = \bigwedge(V_{BH} \oplus P_G)$. Extending β_0, we define a quasi-isomorphism

$$\beta : A_1 := (\textstyle\bigotimes^m \mathcal{M}) \otimes \bigwedge V = \bigwedge((\bigoplus^m (V_{BH} \oplus P_G)) \oplus V) \xrightarrow{\simeq} B_2.$$

Let d_0 denote the linear part of the differential of A_1. In order to construct a minimal Sullivan model for A_1, we apply the procedure from [8, proof of Theorem 14.9]. As a consequence, there exists an isomorphism $(\bigwedge W, d') \otimes \bigwedge(U \oplus dU) \cong A_1$ for which $\bigwedge(U \oplus dU)$ is a contractible CDGA, $(\bigwedge W, d')$ is minimal, $(\bigoplus^m (V_{BH} \oplus P_G)) \oplus V = U \oplus \operatorname{Ker} d_0 = U \oplus d_0 U \oplus W$ and $d_0(W) = 0$. By the construction, we may assume that $\bigoplus^m (V_{BH} \oplus P_G) \subset W$. Then we have a quasi-isomorphism $\beta' : (\bigwedge W, d') \xrightarrow{\simeq} B_2$ and a pushout diagram

$$\begin{array}{ccc} (\bigwedge W, d') & \xrightarrow{I} & ((\bigwedge P_H)^{\otimes m} \otimes \bigwedge W, \partial) \\ {\scriptstyle\beta'}\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\beta''} \\ B_2 & \xrightarrow{\widetilde{\ell}} & (\bigwedge P_H)^{\otimes m} \otimes B_2 \end{array} \tag{4.4}$$

in which $\partial(x_k) = t_k \in V_{BH}$ for $x_k \in P_H$ and I is the canonical inclusion. Therefore, the map β'' is a quasi-isomorphism. Moreover, we see that the bottom right CDGA in the square above is nothing other than the CDGA C from Example 4.2.

By applying Lemma A.1 repeatedly to the diagram obtained by combining the diagrams (4.3) and (4.4) and to a commutative square given by the naturality of maps in (2.4), we obtain a commutative diagram

$$\begin{array}{ccc} (\bigwedge W, d') & \xrightarrow{I} & ((\bigwedge P_H)^{\otimes m} \otimes \bigwedge W, \partial) \\ \simeq\downarrow & & \downarrow\simeq \\ A_{\mathrm{PL}}((G/H,*)^K) & \xrightarrow{A_{\mathrm{PL}}(q)} & A_{\mathrm{PL}}((G,H)^K) \end{array} \tag{4.5}$$

Recall that $\bigoplus^m (V_{BG} \oplus P_G)$ is a subspace of W. Then we may write $(\bigwedge V_{BH} \otimes \bigwedge P_G)^{\otimes m} \otimes \bigwedge W'$ for $\bigwedge W$. Thus, the upper sequence in the diagram (4.5) gives rise to a short exact sequence of complexes

$$0 \longleftarrow (\textstyle\bigoplus^m P_H, 0) \longleftarrow ((\bigoplus^m (P_H \oplus V_{BH} \oplus P_G)) \oplus W', \partial_0) \longleftarrow (W, 0) \longleftarrow 0$$

in which the linear part ∂_0 of ∂ satisfies $\partial_0 : \bigwedge P_H \to \bigwedge V_{BH}$, $\partial_0(t_k) = y_k$ and $\partial_0|_{P_G \oplus W'} = 0$. The last equality follows from the assumption that $(Bi)^* x_k$ is decomposable for each k. The homology long exact sequence is decomposed into a short exact sequence of the form

$$0 \leftarrow H(\textstyle\bigoplus^m (P_H \oplus V_{BH} \oplus P_G)) \oplus W', \partial_0) \leftarrow (W, 0) \stackrel{d_0'}{\leftarrow} (\textstyle\bigoplus^m P_H, 0) \leftarrow 0,$$

where d_0' denotes the connecting homomorphism. In fact, the linear map d_0' coincides with the composition $\bigoplus^m P_H \xrightarrow{\partial_0} \bigoplus^m (P_H \oplus V_{BH} \oplus P_G) \oplus W' \xrightarrow{pr} W$, where pr is the projection; see [8, proof of Proposition 15.13]. Thus, Lemma 4.4 yields the desired result. ∎

Remark 4.5. Let G be a compact Lie group and H be a closed subgroup for which G/H is simply-connected and $(Bi)^*(x_k)$ is *indecomposable* in $H^*(BH; \mathbb{Q})$ for some generator x_k of $H^*(BG; \mathbb{Q})$. Then we see that the connecting homomorphism

$$\partial_* : \pi_*((G/H, *)^K)_{\mathbb{Q}} \to \pi_{*-1}(\textstyle\prod^m H)_{\mathbb{Q}}$$

is not surjective. Indeed, the connecting homomorphism is natural with respect to maps between spaces. Thus it suffices to show that the connecting homomorphism $\partial^* : \pi^*(H) \to \pi^{*+1}(G/H)$ is not injective; see the diagram (1.1). To this end, we show that the map $i^* : \pi^*(G) \to \pi^*(H)$ is nontrivial.

Recall the surjective model $\rho : (\bigwedge V_{BH} \otimes \bigwedge P_G \otimes \bigwedge P_H, \partial) \to (\bigwedge P_H, 0)$ for the inclusion $H \to G$ used in the construction of the model (4.2). Suppose that $(Bi)^*(x_k) = \sum_i \lambda_i t_i +$(decomposable element) for some generator x_k in $H^*(BG; \mathbb{Q})$, where $\lambda_i \neq 0$ for some i. Then it follows that $x_i + \sum_i \lambda_i u_i$ is a cocycle in the cochain complex $(Q(\bigwedge V_{BH} \otimes \bigwedge P_G \otimes \bigwedge P_H), \partial_0)$ and

$$i^*\Big(x_i + \sum_i \lambda_i u_i\Big) = \sum_i \lambda_i u_i \neq 0$$

for $i^* = H(Q(\rho)) : H(Q(\bigwedge V_{BH} \otimes \bigwedge P_G \otimes \bigwedge P_H), \partial_0) \to H(Q(\bigwedge P_H), 0) = P_H$.

For example, we see that $\partial_* : \pi_*((U(n)/T, *)^K)_{\mathbb{Q}} \to \pi_{*-1}(\prod^m T)_{\mathbb{Q}}$ is not surjective for a maximal torus T of $U(n)$.

5. The formality of a compact toric manifold. We prove the following result by using the commutative diagram (1.2).

Theorem 5.1. *Every compact toric manifold X_Σ is formal.*

This result is proved in [17, 3]; see also [5, Theorem 8.1.10]. The proof of [3, Proposition 3.1] indeed uses the algebra structure of the cohomology of the toric manifold. We will use the Félix–Tanré model for $DJ(K)$ in order to give another proof.

We will also use a result due to Baum characterizing regular sequences.

Proposition 5.2 ([2, 3.5 Proposition]). *Let A be a connected commutative algebra and $a_1, \dots, a_t$ be elements of $A^{>0}$. Set $\Lambda = \mathbb{K}[x_1, \dots, x_t]$ with $\deg x_i = \deg a_i$ and consider A to be a Λ-module through the map $f : \Lambda \to A$ defined by $f(x_i) = a_i$. Then the following are equivalent:*

(i) *$a_1, \dots, a_t$ is a regular sequence.*
(ii) $\mathrm{Tor}_{\Lambda}^{-1,*}(\mathbb{K}, A) = 0$.
(iii) $\mathrm{Tor}_{\Lambda}^{-j,*}(\mathbb{K}, A) = 0$ *for all* $j \geq 1$.
(iv) *A is a projective Λ-module.*
(v) *As a Λ-module, A is isomorphic to $\Lambda \otimes (A/(a_1, \dots, a_t))$.*

The following result gives a rational model for the toric manifold X_Σ in the proof of Theorem 5.1.

Lemma 5.3. *The map $(B\rho) \circ q : DJ(K) \to BL'$ in the diagram* (1.2) *is formalizable* (see Definition 1.4 and the paragraph after the diagram (1.2)).

Proof. This follows from the same argument as in Example 2.5. ∎

Proof of Theorem 5.1. Let $\{v_j\}_{j=1}^m$ be the set of 1-dimensional cones of the fan Σ of dimension n. Each v_i is in the lattice N of $\mathbb{R}^n$ which defines the fan Σ. Then it follows from the construction of the diagram (1.2) that $H^*(BL) \cong \mathbb{Q}[t'_1, \dots, t'_n]$ as algebras. Observe that $\dim L = \dim \Sigma = n$. Moreover, we see that for $i = 1, \dots, n$,

$$(B\rho)^*(t'_i) = \sum_{j=1}^{m} \langle m_i, v_j \rangle t_j,$$

where t_j denotes the generator of $H^*(BG) \cong \mathbb{Q}[t_i, \dots, t_m]$ and m_i is the dual basis for $M := \mathrm{Hom}(N, \mathbb{Z})$. The Félix–Tanré model for $DJ(K)$ is of the form $(SR(K) = \mathbb{Q}[t_i, \dots, t_m]/I(K), 0)$ for which $q^*(t_j) = t_j$ for $j = 1, \dots, m$. Consider the pushout construction of models [14, 8] for the pullback (1.2). Then, by Lemma 5.3 and [19, Proposition 2.3.4], we have a rational model for X_Σ of the form

$$C := \Big(\bigwedge(x_1, \dots, x_n) \otimes SR(K), d(x_i) = q^*(B\rho)^*(t'_i) = \sum_{j=1}^{m} \langle m_i, v_j \rangle t_j\Big),$$

where $\deg x_i = 1$. This also computes the torsion functor

$$\mathrm{Tor}_{H^*(BL)}^{*,*}(H^*(DJ(K)), \mathbb{Q})$$

if we assign bidegree $(-1, 2)$ to each x_i. Then [6, Theorem 12.3.11] asserts that $H^{\mathrm{odd}}(X_\Sigma; \mathbb{Q}) = 0$. This implies that $\mathrm{Tor}_{H^*(BL)}^{-1,*}(H^*(DJ(K)), \mathbb{Q}) = 0$. It follows from Proposition 5.2 that $q^*(B\rho)^*(t'_1), \dots, q^*(B\rho)^*(t'_n)$ is a regular sequence in $SR(K)$. Thus, we have a quasi-isomorphism

$$f : C \to SR(K)/(d(x_i); i = 1, \dots, n) = H^*(X_\Sigma; \mathbb{Q})$$

defined by $f(t_j) = t_j$ and $f(x_i) = 0$. ∎

Remark 5.4. We can also obtain the rational cohomology of the compact toric manifold by using the Eilenberg–Moore spectral sequence for the pullback (1.2). In fact, it follows from the computation of the spectral sequence that, as algebras,

$$\begin{aligned} H^*(X_\Sigma) &\cong \mathrm{Tor}_{H^*(BL)}(H^*(DJ(K)), \mathbb{Q}) \\ &\cong SR(K)/\Big(\sum_{j=1}^{m} \langle m_i, v_j\rangle t_j\Big). \end{aligned}$$

The consideration above for the polyhedral product $(\mathbb{C}, \mathbb{C}^*)^K$ is applicable to more general ones, for example, $(EG, G)^K$ for a connected Lie group G. In fact, for a simplicial complex K with m vertices, we have (homotopy) pullback diagrams

$$\begin{array}{ccccc} X_{K,(G,H)} := E(\prod^m G) \times_H (EG, G)^K & \xrightarrow{p} & E(\prod^m G)/H & \longrightarrow & EL \\ \pi\downarrow & & \downarrow & & \downarrow \\ E(\prod^m G) \times_{\prod^m G} (EG, G)^K & \xrightarrow{q} & B(\prod^m G) & \xrightarrow{B\rho} & BL \end{array} \tag{5.1}$$

where H is a normal (not necessarily connected) closed subgroup of $\prod^m G$ and $L = (\prod^m G)/H$. Then [7, Lemma 2.3.2] implies that the natural map $E(\prod^m G) \times_{\prod^m G} (EG, G)^K \xrightarrow{\simeq} (BG, *)^K$ is a homotopy equivalence. Thus, we arrive at

Theorem 5.5. *Suppose that* $H^{\mathrm{odd}}(X_{K,(G,H)}; \mathbb{Q}) = 0$. *Then* $X_{K,(G,H)}$ *is formal.*

Theorem 1.6 asserts that among partial quotients associated with M the condition in Theorem 5.5 is satisfied only by the toric manifold M.

Corollary 5.6. *With the notation as above, suppose further that* $H^{\mathrm{odd}}(X_{K,(G,H)}; \mathbb{Q}) = 0$ *and* $H^*(X_{K,(G,H)}; \mathbb{Q}) \cong H^*(X_{K',(G',H')}; \mathbb{Q})$. *Then* $X_{K,(G,H)} \simeq_{\mathbb{Q}} X_{K',(G',H')}$ *if the spaces are nilpotent.*

Remark 5.7. Suppose that $H^*(BL) \cong \mathbb{Q}[t'_1, \dots, t'_n]$. Under the same assumption as in Theorem 5.5, we see that $X_{K,(G,H)}$ admits a rational model of the form

$$\big(\bigwedge(x_1, \dots, x_n) \otimes (\textstyle\bigotimes^m H^*(G; \mathbb{Q})/I(K)), d(x_i) = q^*(B\rho)^*(t'_i)\big)$$

in which $q^*(B\rho)^*(t'_1), \dots, q^*(B\rho)^*(t'_n)$ is a regular sequence.

We observe that, for a compact smooth toric manifold X_Σ, there is a homotopy equivalence $X_\Sigma \simeq X_{K,((\mathbb{C}^*)^m,H)}$ for which K is a simplicial complex associated with the fan Σ and $\mathcal{Z}_K(\mathbb{C},\mathbb{C}^*)/H$ is Cox's construction for X_Σ. Moreover, the toric manifold X_Σ is simply-connected and hence nilpotent. Thus, Corollary 5.6 is regarded as an answer to the *rational cohomological rigidity problem* for toric manifolds.

6. The (non)formalizability of partial quotients. We begin by considering formalizability for toric manifolds.

PROPOSITION 6.1. *The map $\pi_H : X_\Sigma \to DJ(K)$ in* (1.2) *is formalizable.*

Proof. By considering the sequence (2.4) for $(X, A) = (BS^1, *)$, we find quasi-isomorphisms connecting $A_{\mathrm{PL}}(DJ(K))$ to the Stanley–Reisner algebra $SR(K)$. We can construct a minimal model $\phi : \bigwedge W \to SR(K)$ for $SR(K)$ so that $W = \mathbb{Q}\{t_1,\dots,t_m\} \oplus V$, where $t_1,\dots,t_m$ give the generators of $SR(K)$, $\phi(V) = 0$ and $V = V^{\geq 2}$. The lifting lemma [8, Proposition 12.9] produces a quasi-isomorphism $\phi' : \bigwedge W \to A_{\mathrm{PL}}(DJ(K))$. Thus, in particular, the Davis–Januszkiewicz space $DJ(K)$ is formal. The pushout construction in the proof of Theorem 5.1 gives rise to a commutative diagram

$$\begin{array}{ccc} A_{\mathrm{PL}}(DJ(K)) & \xrightarrow{A_{\mathrm{PL}}(\pi_H)} & A_{\mathrm{PL}}(X_\Sigma) \\ \phi'\uparrow\simeq & & \simeq\uparrow \\ \bigwedge W & \xrightarrow{\quad i \quad} & \bigwedge(x_1,\dots,x_n)\otimes\bigwedge W \end{array}$$

where i is a KS-extension. Moreover, we have a commutative diagram of CDGAs

$$\begin{array}{ccc} \bigwedge W & \xrightarrow{\quad i \quad} & \bigwedge(x_1,\dots,x_n)\otimes\bigwedge W \\ \phi\downarrow\simeq & & \simeq\downarrow f \\ H^*(DJ(K);\mathbb{Q}) & \xrightarrow{(\pi_H)^*} & H^*(X_\Sigma;\mathbb{Q}) \end{array}$$

in which ϕ is a quasi-isomorphism defined by $\phi(t_j) = t_j$ and $\phi|_V \equiv 0$, the map i is an extension and f is the quasi-isomorphism given at the end of the proof of Theorem 5.1. ∎

Proof of Theorem 1.6. Let $\mathcal{Z}_K$ denote the space $\mathcal{Z}_K(\mathbb{C},\mathbb{C}^*)$. Suppose that $H = H'$. Then the partial quotient $\mathcal{Z}_K/H'$ is nothing other than the toric manifold X_Σ. Then [6, Theorem 12.3.11] implies (ii).

We recall the proof of Theorem 5.1. We have a rational model

(6.1) $$C' := \left(\bigwedge(x_1,\dots,x_l)\otimes SR(K), d(x_i) = q^*(B\rho)^*(t'_i)\right)$$

for $\mathcal{Z}_K/H'$. Under the assumption (ii), by Proposition 5.2, we see that the sequence $d(x_1),\dots,d(x_l)$ is regular. Thus the same argument as in the proof of Proposition 6.1 yields (iii).

Suppose that H' is a connected proper subgroup of H. We show that $\pi_{H'}$ is not formalizable. We may replace $\mathcal{Z}_K$ and the tori H and H' acting on the moment-angle manifold with the polyhedral product $(D^2, S^1)^K$, a compact Lie group T^k and its subtorus with an appropriate integer k, respectively. Assume that the fan Σ has m rays and hence K is a simplicial complex with m vertices. If the fan is of dimension n, then we have an exact sequence $1 \to T^k \to T^m \xrightarrow{\rho} T^n \to 1$ via Cox's construction of the toric manifold X_Σ. With the same notation as in the proof of Theorem 5.1, since $d(x_1), \dots, d(x_n)$ is a regular sequence, it follows from Proposition 5.2 that $H^*(DJ(K)) \cong \mathbb{Q}[t_1, \dots, t_n] \otimes H^*(X_\Sigma)$ as a $\mathbb{Q}[t_1, \dots, t_n]$-module.

For a proper subgroup H' of T^k, the quotient L' of the inclusion $H' \to T^m$ is a torus of dimension l greater than n. Consider the rational model (6.1) for $\mathcal{Z}_K/H'$. We assume that $d(x_1), \dots, d(x_l)$ is a regular sequence. Then, by the same argument as in the proof of Theorem 5.1 with the diagram (1.2), we find that $H^*(DJ(K)) \cong \mathbb{Q}[t_1, \dots, t_l] \otimes H^*(\mathcal{Z}_K/H')$ as a $\mathbb{Q}[t_1, \dots, t_l]$-module. By considering the Poincaré series of $H^*(DJ(K))$ in two ways, we see that

$$\frac{1}{\prod^{l-n}(1-t^2)} P_1(t) = P_2(t),$$

where $P_1(t)$ and $P_2(t)$ are the Poincaré series of $H^*(\mathcal{Z}_K/H')$ and $H^*(X_\Sigma) \cong H^*(\mathcal{Z}_K/H)$, respectively. Since the partial quotients are manifolds of finite dimensions it follows that $P_1(t)$ and $P_2(t)$ are polynomials. This contradicts the equality above and hence $d(x_1), \dots, d(x_l)$ is not a regular sequence.

Suppose that the map $\pi_{H'} : \mathcal{Z}_K/H' \to DJ(K)$ is formalizable. By virtue of [19, Proposition 2.3.4], we have a commutative diagram

$$\begin{array}{ccc}
A_{\mathrm{PL}}((DJ(K)) & \xrightarrow{A_{\mathrm{PL}}(\pi_{H'})} & A_{\mathrm{PL}}(\mathcal{Z}_K/H') \\
{\scriptstyle \phi'}\uparrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \simeq} \\
\bigwedge W & \xrightarrow{\quad i \quad} & \bigwedge(x_1, \dots, x_l) \otimes \bigwedge W \\
{\scriptstyle \phi}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \eta}\downarrow{\scriptstyle \simeq} \\
H^*(DJ(K);\mathbb{Q}) & \xrightarrow{(\pi_{H'})^*} & H^*(\mathcal{Z}_K/H';\mathbb{Q})
\end{array}$$

Consider the fibration $\mathcal{Z}_K \to EG \times_{H'} \mathcal{Z}_K \xrightarrow{p} (EG)/H'$ which fits in the diagram (1.2). The argument in [7, 4.1] enables us to conclude that $\mathcal{Z}_K$ is 2-connected; see also [5, Proposition 4.3.5]. Thus, the homotopy exact sequence of the fibration above implies that $\mathcal{Z}_K/H' \simeq EG \times_{H'} \mathcal{Z}_K$ is simply-connected and hence $H^1(\mathcal{Z}_K/H') = 0$.

By Lemma 5.3 and [19, Proposition 2.3.4], we see that

$$\mathrm{Tor}^*_{\mathbb{Q}[t_1,\dots,t_l]}(\mathbb{Q}, SR(K)) \cong \mathrm{Tor}^*_{\mathbb{Q}[t_1,\dots,t_l]}(\mathbb{Q}, \bigwedge W).$$

This implies that the spectral sequence converging to the torsion group $\mathrm{Tor}^*_{\mathbb{Q}[t_1,\dots,t_l]}(\mathbb{Q}, \bigwedge W)$ with $E_2^{*,*} \cong \mathrm{Tor}^{*,*}_{\mathbb{Q}[t_1,\dots,t_l]}(\mathbb{Q}, SR(K))$ collapses at the E_2-page. This fact allows us to obtain a sequence

$$\mathrm{Tor}_P^{-1,*}(\mathbb{Q}, SR(K)) \cong E_0^{-1,*} \xleftarrow{p} F^{-1}\mathrm{Tor}_P^{*-1}(\mathbb{Q}, \bigwedge W) \xrightarrow{\iota} \mathrm{Tor}_P^{*-1}(\mathbb{Q}, \bigwedge W).$$

Here $\{F^j\}$ denotes the filtration associated to the spectral sequence, P is the polynomial algebra $\mathbb{Q}[t_1, \dots, t_l]$, p and ι are the canonical projection and the inclusion, respectively. Since $d(x_1), \dots, d(x_l)$ is not a regular sequence, it follows from Proposition 5.2 that there is a nonexact cocycle

$$w = \sum_{j=1}^{l} u_j x_j - z$$

in $F^{-1}\mathrm{Tor}^*_{\mathbb{Q}[t_1,\dots,t_l]}(\mathbb{Q}, \bigwedge W)$, where u_i and z are in $\bigwedge W$. Observe that the torsion algebra $\mathrm{Tor}^*_{\mathbb{Q}[t_1,\dots,t_l]}(\mathbb{Q}, \bigwedge W)$ is isomorphic to the cohomology

$$H^*(\textstyle\bigwedge(x_1, \dots, x_l) \otimes \bigwedge W, d)$$

as an algebra. We see that $\deg x_j = 1$ and then $\eta(x_j) = 0$ for each j. The element z is of odd degree and is in the image of the map i. Thus, since $H^{\mathrm{odd}}(DJ(K); \mathbb{Q}) = 0$, it follows that

$$H^*(\eta)([w]) = [\eta(w)] = \eta(w) = \sum_{j=1}^{l} \eta(u_j)\eta(x_j) - (\pi')^*\phi(z) = 0,$$

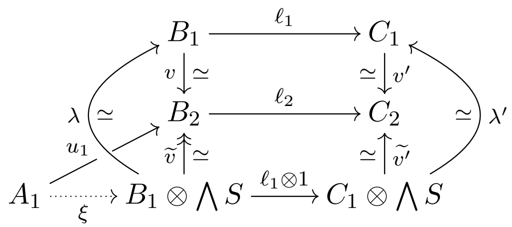
a contradiction. ∎

Appendix A. A lifting lemma. In this section, we describe an algebraic result obtained by the lifting lemmas [8, Lemma 12.4 and Proposition 14.6].

LEMMA A.1. *For the commutative diagram* (A.1) *below with a Sullivan algebra A_1 and a KS-extension I, one has a commutative diagram* (A.2) *in which $\widetilde{u_i}$ is a quasi-isomorphism if u_i is, for $i = 1$ or 2.*

$$\text{(A.1)}\quad \begin{array}{ccc} A_1 & \xrightarrow{I} & A_1 \otimes \bigwedge W_1 \\ {\scriptstyle u_1}\downarrow & & \downarrow{\scriptstyle u_2} \\ B_2 & \xrightarrow{\ell_2} & C_2 \\ {\scriptstyle v}\uparrow\simeq & & \simeq\uparrow{\scriptstyle v'} \\ B_1 & \xrightarrow{\ell_1} & C_1 \end{array} \qquad\qquad \text{(A.2)}\quad \begin{array}{ccc} A_1 & \xrightarrow{I} & A_1 \otimes \bigwedge W_1 \\ {\scriptstyle \widetilde{u_1}}\downarrow & & \downarrow{\scriptstyle \widetilde{u_2}} \\ B_1 & \xrightarrow{\ell_1} & C_1 \end{array}$$

Proof. By applying the surjective trick [8, p. 148] to v, we obtain a diagram



of solid arrows in which the three squares are commutative. We observe that

$S \cong B_2 \oplus dB_2$ and that $\widetilde{v'}$ is defined by $\widetilde{v'}(c_1) = v'(c_1)$ for $c_1 \in C_1$ and $\widetilde{v'}(s) = \ell_2(s)$ for $s \in S$. Since A_1 is a Sullivan algebra, the lifting lemma [8, Lemma 12.4] enables us to obtain the map ξ which fits in the commutative triangle. Thus we have a commutative diagram of solid arrows

$$\begin{array}{ccccc} A_1 & \xrightarrow{\ \xi\ } & B_1 \otimes \bigwedge S & \xrightarrow{\ell_1 \otimes 1} & C_1 \otimes \bigwedge S \\ {\scriptstyle I}\downarrow & & {\scriptstyle \overline{u_2}}\ (\text{dotted, } A_1 \otimes \bigwedge W_1 \dashrightarrow C_1 \otimes \bigwedge S) & & \simeq \downarrow {\scriptstyle \widetilde{v'}} \\ A_1 \otimes \bigwedge W_1 & & \xrightarrow{\qquad u_2 \qquad} & & C_2 \end{array}$$

Since the map I is a KS-extension, by using [8, Proposition 14.6] we find a dotted arrow $\overline{u_2}$ which makes the upper triangle commutative and the lower triangle commutative up to homotopy relative to A_1. Define $\widetilde{u_1} := \lambda \circ \xi$ and $\widetilde{u_2} := \lambda' \circ \overline{u_2}$. Then we obtain the commutative diagram (A.2). By the construction of the map $\widetilde{u_i}$, we see that $\widetilde{u_i}$ is a quasi-isomorphism if u_i is for $i = 1$ or 2. ∎

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