ON FÉLIX–TANRÉ RATIONAL MODELS FOR POLYHEDRAL PRODUCTS

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ABSTRACT. The Félix–Tanré rational model for the polyhedral product of a fibre inclusion is considered. In particular, we investigate the rational model for the polyhedral product of a pair of Lie groups corresponding to arbitrary simplicial complex and the rational homotopy group of the polyhedral product. Furthermore, it is proved that for a partial quotient N associated with a toric manifold M, the following conditions are equivalent: (i) $N = M$. (ii) The odd-degree rational cohomology of N is trivial. (iii) The torus bundle map from N to the Davis–Januszkiewicz space is formalizable.

1. INTRODUCTION

Toric varieties are fascinating objects in the study of algebraic geometry, combinatorics, symplectic geometry and topology. For a nonsingular toric variety, so-called a toric manifold, is given by the quotient of a moment-angle manifold by a torus action with Cox's construction. By generalizing the construction of moment-angle manifolds, we obtain a *moment-angle complex* and more general *polyhedral products* [1, 14, 16], which are defined by the colimit of spaces with gluing data obtained from a simplicial complex. Thus we are also interested in the generalized ones.

In [11], Félix and Tanré have given a rational model for a polyhedral products of a tuple of spaces corresponding to arbitrary simplicial complex. One of the aims of this manuscript is to construct a tractable rational model for a polyhedral products by refining the model due to Félix and Tanré. By applying the construction to a polyhedral product for a pair of Lie groups, we have a result on the rational homotopy group of the polyhedral product; see Theorem 1.2 and Proposition 4.1.

Moreover, the formality of a toric manifold and the non-formalizability for a partial quotient, which is not a toric manifold, are discussed with their models induced by the Félix and Tanré rational models for polyhedral products; see Theorems 1.6, 5.1 and 5.5 for more details.

Throughout this article, each space X is assumed to be connected and ($\mathbb{Q}$-)locally finite; that is, the rational cohomology group $H^i(X;\mathbb{Q})$ is of finite dimension for $i \geq 0$. In the rest of this section, we describe our main results more precisely. Following Kishimoto and Levi [16], we define a polyhedral product with the homotopy colimit instead of the colimit; see also [17] for the study of the Davis-Januszkiewicz space with the homotopy colimit functor.

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Definition 1.1. ([16, Definition 1.2]) Let $(\underline{X}, \underline{A}) := ((X_1, A_1), ..., (X_m, A_m))$ be a tuple of spaces with $A_i \subset X_i$ for each i and K a simplicial complex with m. The *polyhedral product* $(\underline{X}, \underline{A})^K$ of the tuple $(\underline{X}, \underline{A})$ corresponding to K is defined by

$$(\underline{X}, \underline{A})^K := \mathrm{hocolim}_{\sigma \in K} (\underline{X}, \underline{A})^\sigma,$$

where $(\underline{X}, \underline{A})^\sigma = Y_1 \times \cdots \times Y_m$ with

$$Y_i = \begin{cases} A_i & i \notin \sigma \\ X_i & i \in \sigma. \end{cases}$$

We write $(X, A)^K$ for $(\underline{X}, \underline{A})^K$ if there are a space X and a subspace A such that $X_i = X$ and $A_i = A$ for each i.

In what follows, we assume that a simplicial complex K has no ghost vertices unless otherwise specified.

Suppose that each (X_i, A_i) is a pair of CW-complexes. Then, the natural map $\mathrm{colim}_{\tau \in \partial(\sigma)} (\underline{X}, \underline{A})^\tau \to (\underline{X}, \underline{A})^\sigma$ is a cofibration. Thus, in veiw of [18, §2 and Proposition 4.8] and also [5, Proposition 8.1.1], we have a weak homotopy equivalence

$$(\underline{X}, \underline{A})^K \xrightarrow{\simeq_w} \mathrm{colim}_{\sigma \in K} (\underline{X}, \underline{A})^\sigma = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma =: \mathcal{Z}_K((\underline{X}, \underline{A}).$$

In particular, by definition, the moment-angle complex $\mathcal{Z}_K(D^2, S^1)$ corresponding to a simplicial complex K is the colimit $\bigcup_{\sigma \in K} (D^2, S^1)^\sigma$ and then it is weak homotopy equivalent to the polyhedral product $(D^2, S^1)^K$.

Our first result concerns with the rational homotopy group of polyhedral product of a pair of Lie groups. We denote by $\pi_*(X)_{\mathbb{Q}}$ the rational homotopy group $\pi_*(X) \otimes \mathbb{Q}$ for a pointed connected space X whose fundamental group is abelian.

Theorem 1.2. *Let G be a connected compact Lie group and $i : H \to G$ the inclusion of a maximal rank subgroup. Suppose that G/H is simply connected and $(Bi)^*(x_k)$ is decomposable in $H^*(BH; \mathbb{Q})$ for each generator x_k of $H^*(BG; \mathbb{Q})$. Then, one has a short exact sequence of rational homotopy groups*

$$0 \longrightarrow \pi_*((G, H)^K)_{\mathbb{Q}} \xrightarrow{q_*} \pi_*((G/H, *)^K)_{\mathbb{Q}} \xrightarrow{\partial_*} \pi_{*-1}(\Pi^m H)_{\mathbb{Q}} \longrightarrow 0$$

for arbitrary simplicial complex K with m verticies, where ∂_* denotes the connecting homomorphism of the homotopy exact sequence of the middle vertical sequence in (1.1) below.

We stress that the exactness in the theorem above does not depend on any property of the given simplicial complex K.

Remark 1.3. While we do not pursue topics on the cohomology $H^*((G, H)^K; \mathbb{K})$ with coefficients in arbitrary field $\mathbb{K}$, in order to compute the cohomology algebra, we may use a commutative diagram

$$\begin{array}{ccccc} (H, H)^K & = & (H, H)^K & = & \Pi^m H \\ \downarrow & & \downarrow & & \downarrow \\ (EG, H)^K & \longleftarrow & (G, H)^K & \longrightarrow & \Pi^m G \\ \downarrow & & \downarrow{\scriptstyle q} & & \downarrow \\ (BH, *)^K & \xleftarrow{j} & (G/H, *)^K & \longrightarrow & \Pi^m G/H. \end{array} \tag{1.1}$$

in which vertical sequences are fibrations; see [8, Lemma 2.3.1]. We can regard the lower squares as pullback diagrams.

Before describing our main result on a partial quotient, we recall some terminology in rational homotopy theory.

A *commutative differential graded algebra* (henceforth, CDGA) (A, d) consists of a nonnagetively graded algebra A and a differential d on A with degree $+1$. Let $A_{PL}(X)$ be the CDGA of polynomial differential forms on a space X; see [9, 10 (c)]. It is worthwhile mentioning that there exists a morphism of cohain complexes from $A_{PL}(X)$ to the singular cochain algebra of X with coefficients in the rational field $\mathbb{Q}$ which induces an isomorphism of algebras between cohomology algebras; see [9, 10(e) Remark].

By definition, a *Sullivan algebra* (A, d) is a CDGA whose underlying algebra A is the free algebra $\wedge W$ generated by a graded vector space W and for which the vector space W admits a filtration $W_0 \subset W_1 \subset \cdots W_n \subset \cdots$ with $W = \bigcup_i W_i$, $d(V_0) = 0$ and $d : W_k \to \wedge W_{k-1}$ for $k \geq 1$. We say that a Sullivan algebra $(\wedge W, d)$ is *minimal* if $d(w)$ is decomposable for each $v \in W$.

A morphism $\varphi : (A, d) \to (B, d')$ of CDGA's is a *quasi-isomorphism* if φ induces an isomorphism on cohomology. A *rational model* (A, d) for a space X is a CDGA which is connected with $A_{PL}(X)$ by using quasi-isomorphisms. We call the rational model (A, d) a *(minimal) Sullivan model* for X if it is a (minimal) Sullivan algebra. Observe that each space has a unique minimal Sullivan model; see [9, 14(b) Corollary]. A space is *formal* if there exists a sequence of quasi-isomorphisms between the cohomology $H^*(X; \mathbb{Q})$ and a Sullivan model for X. We refer the reader to the books [13], [9] and [10] for rational homotopy theory and its applications to topology and geometry.

Definition 1.4. (cf. [19, V]) A map $p : E \to B$ is *formalizable* if there exists a commutative diagram up to homotopy

$$\begin{array}{ccc} A_{PL}(B) & \xrightarrow{A_{PL}(p)} & A_{PL}(E) \\ \simeq\uparrow & & \uparrow\simeq \\ (\wedge W, d) & \xrightarrow{\ l\ } & (\wedge Z, d') \\ \simeq\downarrow & & \downarrow\simeq \\ H^*(B;\mathbb{Q}) & \xrightarrow[p^*]{} & H^*(E;\mathbb{Q}) \end{array}$$

in which $(\wedge W, d)$ and $(\wedge Z, d')$ are minimal Sullivan algebras and vertical arrows are quasi-isomorphisms; see [9, 12 (b)] and [15, Chapter 5] for the homotopy relation.

For a simplicial complex K, define $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)$ by the colimit $\mathrm{colim}_{\tau \in K}(\mathbb{C}, \mathbb{C}^*)^\tau$. Then, we have weak equivalences

$$(D^2, S^1)^K \xrightarrow{\simeq_w} \mathrm{colim}_{\tau\in K}(D^2, S^2)^\tau \underset{\simeq}{\xrightarrow{i}} \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*),$$

where i is the inclusion; see [5, Theorem 4.7.5]. Let X_Σ be a compact toric manifold associated with a complete and smooth fan Σ; see [7, §3.1]. We then have a homeomorphism $X_\Sigma \cong \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H$ via Cox's construction of the manifold, where K is the simplicial complex with m vertices associated with the fan Σ and H is a subgroup of the torus $(\mathbb{C}^*)^m$ which acts on $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)$ canonically and freely; see [7, Theorem 5.1.11] and [5, Theorem 5.4.5, Proposition 5.4.6]. Moreover, the quotient $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H'$ by a subtorus $H' \subset H$ is called a *partial quotient*.

We recall the pullback diagram in [12, The proof of Proposition 3.2]. Let X_Σ be a toric manifold associated with a fan Σ and $\mathcal{Z}_K(\mathbb{C},\mathbb{C}^*)/H$ Cox's construction of X_Σ mentioned above. Then, we have a commutative diagram consisting of two pullbacks

$$\begin{array}{ccccc} EG\times_H \mathcal{Z}_K(\mathbb{C},\mathbb{C}^*) & \xrightarrow{p} & (EG)/H & \longrightarrow & EL \\ {\scriptstyle \pi_H}\downarrow & & \downarrow & & \downarrow \\ EG\times_G \mathcal{Z}_K(\mathbb{C},\mathbb{C}^*) & \xrightarrow[q]{} & BG & \xrightarrow[B\rho]{} & BL, \end{array} \tag{1.2}$$

where $G=(\mathbb{C}^*)^m$ and $L=(\mathbb{C}^*)^m/H$. We observe that right two vertical maps are principal L-bundles and that the maps p and q are fibrations associated with the universal H-bundle and the universal G-bundle, respectively. Since the group H acts on $\mathcal{Z}_K(\mathbb{C},\mathbb{C}^*)$ freely, it follows that the Borel construction $EG\times_H \mathcal{Z}_K(\mathbb{C},\mathbb{C}^*)$ is homotopy equivalent to the toric manifold X_Σ.

Let H' be a subtorus of H. Then, we may replace H and L in the diagram (1.2) with H' and $L':=(\mathbb{C}^*)^m/H$, respectively. With the replacement, the upper left corner in the diagram is regarded as the partial quotient $\mathcal{Z}_K(\mathbb{C},\mathbb{C}^*)/H'\simeq EG\times_{H'}\mathcal{Z}_K(\mathbb{C},\mathbb{C}^*)$.

Remark 1.5. It follows from [5, Theorems 4.3.2 and 4.7.5] that the Borel construction $EG\times_G \mathcal{Z}_K(\mathbb{C},\mathbb{C}^*)$ is homotopy equivalent to the Davis–Januszkiewicz space $DJ(K):=(BS^1,*)^K$. Since the fan that we consider is complete, it follows from the result [7, Theorem 12.1.10] that X_Σ is simply connected. Then, we have an exact sequence $0\longrightarrow \pi_*(X_\Sigma)\xrightarrow{(\pi_H)_*}\pi_*(DJ(K))\xrightarrow{\partial_*}\pi_{*-1}(G/H)\longrightarrow 0$. By considering the center vertical fibration mentioned in (1.1), the exact sequence in Theorem 1.2 is regarded as an analogy of the sequence above.

The following result characterizes a toric manifold among partial quotients associated with the manifold.

Theorem 1.6. *Let $\mathcal{Z}_K(\mathbb{C},\mathbb{C}^*)/H$ be a toric manifold and H' a subtorus of H. For the partial quotient $\mathcal{Z}_K(\mathbb{C},\mathbb{C}^*)/H'$, the following conditions are equivalent.*

(i) $H=H'$.
(ii) $H^{\mathrm{odd}}(\mathcal{Z}_K(\mathbb{C},\mathbb{C}^*)/H';\mathbb{Q})=0$.
(iii) *The map $\pi_{H'}:\mathcal{Z}_K(\mathbb{C},\mathbb{C}^*)/H'\to DJ(K)$ in the diagram (1.2) is formalizable.*

Remark 1.7. As seen in Theorem 5.1, a toric manifold is formal. However, we do not know whether a general partial quotient is formal.

An outline for the article is as follows. Section 2 recalls the construction of the Félix–Tanré rational model for a polyhedral product and discusses the naturality of the models. In Section 3, we give a tractable rational model for a polyhedral product and some examples for the model. Section 4 constructs a rational model for the polyhedral product $(G,H)^K$ of a pair of Lie group and closed subgroup corresponding to arbitrary simplicial complex K. With the model, we prove Theorem 1.2. In Section 5, we show that every compact toric manifold is formal. Section 6 is devoted to proving Theorem 1.6.

2. A recollection of the Félix–Tanré rational models for polyhedral products

While the construction of a rational model in [11] for a polyhedral product is defined by the colimit construction, it is also applicable in constructing a rational model for $(\underline{X}, \underline{A})^K$ obtained by the homotopy colimit as in Definition 1.1. In this section, we summarize the result.

Let $\iota_j : A_j \to X_j$ be the inclusion and $\varphi_j : \mathcal{M}_j \to \mathcal{M}'_j$ a surjective model* for ι_j, namely, an epimorphism of CDGA's which fits in a commutative diagram

$$\begin{array}{ccc} \wedge W_j & \xrightarrow{\varphi_j} & \wedge V_j \\ u\downarrow\simeq & & \simeq\downarrow v \\ A_{PL}(X_j) & \underset{\iota_j^*}{\twoheadrightarrow} & A_{PL}(A_j) \end{array} \tag{2.1}$$

with quasi-isomorphisms u and v. We observe that ι_j^* is surjective; see [9, Proposition 10.4 , Lemma 10.7]. For each $\tau \notin K$, let I_τ denote the ideal of $\bigotimes_{i=1}^m \mathcal{M}_i$ defined by $I_\tau = E_1 \otimes \cdots \otimes E_m$, where

$$E_i = \begin{cases} \mathcal{M}_i & i \notin \tau \\ \text{Ker } \varphi_i & i \in \tau. \end{cases}$$

Theorem 2.1. ([11, Theorem 1]) *There is a sequence of quasi-isomorphisms connecting the CDGA $A_{PL}((\underline{X}, \underline{A})^K)$ and the quotient $(\bigotimes_{i=1}^m \mathcal{M}_i)/J(K)$, where $J(K) := \sum_{\tau \notin K} I_\tau$; that is, the quotient is a rational model for $(\underline{X}, \underline{A})^K$.*

In what follows, we may call the the quotient CDGA in Theorem 2.1 the *Félix–Tanré (rational) model* for the polyhedral product $(\underline{X}, \underline{A})^K$.

Remark 2.2. We observe that the polyhedral product $(\underline{X}, \underline{A})^K$ is defined by the homotopy colimit on the diagram associated to the simplicial complex K. While the nilpotency of each space in the pairs (X_i, A_i) of CW complexes for $1 \le i \le m$ is assumed in [11, Theorem 1], the conditions is not required in Theorem 2.1. In fact, for each inclusion $\iota_j : A_j \to X_j$, we have a commutative diagram

$$\begin{array}{ccc} A_j & \xrightarrow{\iota_j} & X_j \\ \simeq\uparrow & & \uparrow\simeq \\ |S(A_j)| & \underset{|S(\iota)|}{\longrightarrow} & |S(X_j)| \end{array} \tag{2.2}$$

with the singular simplex functor $S(\ )$ and the realization functor $|\ |$. Then, this enables us to obtain a sequence of weak homotopy equivalences

$$(\underline{X}, \underline{A})^K \xleftarrow{\simeq_w} (\underline{X'}, \underline{A'})^K \xrightarrow{\simeq_w} \text{colim}_{\sigma \in K} (\underline{X'}, \underline{A'})^\sigma = \bigcup_{\sigma \in K} (\underline{X'}, \underline{A'})^\sigma, \tag{2.3}$$

where each pair (X'_i, A'_i) denotes the pair $(|S(X_j)|, |S(A_j)|)$; see the paragraph after Definition 1.1. A surjective model for each inclusion $A_j \to X_j$ is regarded as that for the inclusion $A'_j \to X'_j$. Thus, with the models and by applying [9, Proposition

*The existence of the model: We consider a Sullivan representative for ι_j; see [9, page 154]. The proof of [9, Lemma 13.4] enables us to replace the homotopy commutative diagram of the representative with a strictly commutative diagram. By applying the surjective trick ([9, §12 (b)]), we have a surjective model for the inclusion.

13.5] inductively as in the proof of [11, Theorem 1], we can prove Theorem 2.1 without assuming that spaces X_i and A_i are CW-complexes and nilpotent.

In order to confirm the naturality of the model in Theorem 2.1 with respect to an inclusion of simplicial complexes and also given surjective models, the outline of the proof of Theorem 2.1 is recalled below.

By using surjective models $\varphi_j : \mathcal{M}_j \to \mathcal{M}'_j$, for each $\sigma \in K$, we have a CDGA $\widetilde{D}^\sigma := \bigotimes_{i\in\sigma} \mathcal{M}_i \otimes \bigotimes_{i\notin\sigma} \mathcal{M}'_i$ and a map $\xi_\sigma : (\bigotimes_{i=1}^m \mathcal{M}_i)/J(K) \to \widetilde{D}^\sigma$ of CDGA's defined by

$$\xi_\sigma(x_i) = \begin{cases} x_i & i \in \sigma \\ \varphi_i(x_i) & i \notin \sigma. \end{cases}$$

It is readily seen that ξ_σ is well defined. The induction argument in the proof of [11, Theorem 1] yields that the maps ξ_σ of CDGA's give rise to a quasi-isomorphism

$$\alpha : (\bigotimes_{i=1}^m \mathcal{M}_i)/J(K) \overset{\simeq}{\to} \lim_{\sigma\in K} \widetilde{D}^\sigma.$$

We also observe that the fact is proved by using [9, Lemma 13.3] which gives a well-defined quasi-isomorphism between appropriate pullback diagrams in the category of CDGA's. Thus, with the same notation as in Remark 2.2, we have a sequence of quasi-isomorphisms

$$(2.4)\quad A_{PL}((\underline{X},\underline{A})^K) \xrightarrow{\simeq} A^*_{PL}((\underline{X}',\underline{A}')^K) \xleftarrow{\simeq} A_{PL}(\mathrm{colim}_{\sigma\in K}(\underline{X}',\underline{A}')^\sigma) \xrightarrow[\eta]{\simeq} \lim_{\sigma\in K} A_{PL}((\underline{X}',\underline{A}')^\sigma) \xleftarrow[\simeq]{\Phi} \lim_{\sigma\in K}\widetilde{D}^\sigma \xleftarrow[\simeq]{\alpha} (\bigotimes_{i=1}^m \mathcal{M}_i)/J(K)$$

in which the first two quasi-isomorphisms are induced by the weak equivalences in (2.3), Φ is defined by the surjective models φ_i and η is induced by natural maps $(\underline{X},\underline{A})^\sigma \to (\underline{X},\underline{A})^K$. It follows from [18, Proposition 4.8] and [5, Proposition 8.1.4] that Φ and η are quasi-isomorphisms, respectively. This enables us to obtain the rational model for the polyhedral product $(\underline{X},\underline{A})^K$ in Theorem 2.1. Moreover, the construction above of the model yields the following proposition.

Proposition 2.3. *The Félix–Tanré rational models are natural with respect to surjective models which are used in constructing the models of polyhedral products and an inclusion of simplicial complexes.*

The rational cohomology of the moment-angle complex $\mathcal{Z}_K(D^2, S^1)$ is isomorphic to the torsion product $\mathrm{Tor}_{\mathbb{Q}[t_1,...,t_m]}(\mathbb{Q}[t_1,...,t_m]/I(K), \mathbb{Q})$, where $\deg t_i = 2$ and $I(K)$ denotes the ideal generated by monomials $t_{i_1}\cdots t_{i_s}$ for $\{i_1,...,i_s\} \notin K$, which is called the *Stanley–Reisner ideal* associated with K; see [12, 10.1]. Thus, a CDGA of the form

$$(2.5)\qquad (\wedge(x_i,...,x_m)\otimes \mathbb{Q}[t_1,\cdots,t_m]/I(K), d(x_i) = t_i)$$

computes the cohomology algebra $H^*(\mathcal{Z}_K(D^2,S^1);\mathbb{Q})$. Here $SR(K)$ denotes the *Stanley–Reisner algebra* $\mathbb{Q}[t_1,\cdots,t_m]/I(K)$.

Remark 2.4. The inclusion $i : S^1 \to D^2$ admits a surjective model of the form $\pi : (\wedge(x,t), d) \to (\wedge(x), 0)$, where π is the projection, $d(x) = t$ and $\deg x = 1$. By virtue of Theorem 2.1, we see that the CDGA (2.5) above is a rational model for $\mathcal{Z}_K(D^2, S^1)$.

Example 2.5. Let K be a simplicial complex with m vertices and $j : K \to 2^{[m]}$ the inclusion. The map j induces the inclusion $\widetilde{j} : (BS^1, *)^K \to \Pi^m(BS^1)$. We choose the projection $(\wedge(t), 0) \to \mathbb{Q}$ as a surjective model for the inclusion $* \to BS^1$, where $\deg t_i = 2$ for $i = 1, ..., m$. By Theorem 2.1 and Proposition 2.3, we have a model $(\wedge(t_1, ..., t_m), 0) \to (\wedge(t_1, ..., t_m)/I(K), 0) = (SR(K), 0)$ for $\widetilde{j}$ which is the natural projection. As a consequence, we see that the inclusion $\widetilde{j}$ is formalizable in the sense of Definition 1.4.

3. Comparatively tractable rational models for polyhedral products

The Félix–Tanré rational model for a polyhedral product $(\underline{X}, \underline{A})^K$ depends on the choice of surjective models for the inclusions in the given tuple $(\underline{X}, \underline{A})$. While the model is complicated in general, the underlying algebra is adjustable in the sense of Theorem 3.1 below. In fact, we show that the underlying algebra of the model has a particular form which is regarded as a generalization of the rational model for a moment-angle complex; see Remark 2.4.

We recall the CDGA in (2.5). With this mind, we may call a CDGA (A, d) a *Stanley–Reisner (SR) K-type* if the underlying algebra A is of the form

$$\bigotimes_{j=1}^{m}(\wedge V_j \otimes B_j))\big/(b_{j_1} \cdots b_{j_s} \mid b_j \in B_j^+, \{j_1, ..., j_s\} \notin K),$$

where B_j is a free commutative algebra. The term 'K-' may be omitted if it is clear from the context.

Theorem 3.1. *Each polyhedral product $(\underline{X}, \underline{A})^K$ has a SR type CDGA model; that is, there is a sequence of quasi-isomorphisms of CDGA's connecting $A_{PL}((\underline{X}, \underline{A})^K)$ and a Stanley-Reisner K-type CDGA.*

Proof. For each j, let $\varphi_j : \wedge W_j \to \wedge V_j$ be a surjective model for the inclusion $\iota_j : A_j \to X_j$. Since φ_j is surjective, the vector space W_j admits a decomposition $W_j \cong W_j' \oplus W_j''$ which satisfies the condition that $\varphi_j|_{W_j'} : W_j' \xrightarrow{\cong} V_j$ is an isomorphism and $\varphi_j|_{W_j''} \equiv 0$. In fact, we choose indecomposable elements w_λ of $\wedge W_j$ so that $\varphi_j(w_\lambda) = v_\lambda$ for a basis $\{v_\lambda\}_{\lambda \in \Lambda}$ for V_j. Then, we have a decomposition

$$W_j \cong \mathbb{K}\{w_\lambda \mid \lambda \in \Lambda\} \oplus \mathbb{K}\{w_\gamma'' \mid \gamma \in \Gamma\}.$$

Let $P(v_\lambda)$ be the polynomial on v_λ's which represents $\varphi_j(w_\gamma'')$ in $\wedge V_j$. Putting $W_j'' := \mathbb{K}\{w_\gamma'' - P(w_\lambda)\}$, we have the decomposition required above. Theorem 2.1 yields the result. □

We provide a more tractable SR type model for a polyhedral product of a fibre inclusion. For $1 \le j \le m$, let $F_j \xrightarrow{\iota_j} X_j \xrightarrow{p_j} Y_j$ be a fibration with simply-connected base. Assume that $H^*(Y_j; \mathbb{Q})$ is locally finite for each j. Then, a relative Sullivan model $\widetilde{p_j}$ for the map p_j gives a commutative diagram of CDGA's

$$\begin{array}{ccccc}
\wedge W_j & \xrightarrow{\widetilde{p_j}} & (\wedge V_j \otimes \wedge W_j, d_j) & \xrightarrow{\widetilde{\iota_j}} & (\wedge V_j, \overline{d_j}) \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
A_{PL}(Y_j) & \xrightarrow{p_j^*} & A_{PL}(X_j) & \xrightarrow{\iota_j^*} & A_{PL}(F_j)
\end{array}$$

in which vertical maps are quasi-isomorphisms; see [15, 20.3 Theorem]. The upper sequence is called a model for the fibration. It follows from the construction that $\widetilde{\iota_j}$ is a surjective model for ι_j. If $\widetilde{p_j}$ is minimal, by definition, we see that $d(V_j) \subset (\wedge^{\geq 2} V_j) \otimes \wedge W_j + \wedge V_j \otimes \wedge^+ W_j$ in the SR type CDGA. By virtue of Theorem 2.1, we have

Proposition 3.2. *With the same notation as above, the polyhedral product $(\underline{X}, \underline{F})^K$ for the tuple of fibre inclusions ι_j has a SR type CDGA model of the form*

$$\mathcal{M}((\underline{X}, \underline{F})^K) := \Big(\bigotimes_{j=1}^{m} (\wedge V_j \otimes \wedge W_j)) / (b_{j_1} \cdots b_{j_s} \mid b_j \in W_j, \{j_1, ..., j_s\} \notin K), d \Big)$$

for which $d(W_j) \subset \wedge W_j$ and $d(V_j) \subset (\wedge^{\geq 2} V_j) \otimes \wedge W_j + \wedge V_j \otimes \wedge^+ W_j$.

Remark 3.3. The model in Proposition 3.2 is not a Sullivan model in general. However, if we construct a Sullivan model by using the model, then for example, we may obtain information on the rational homotopy group of $(\underline{X}, \underline{F})^K$; see Example 4.3 below.

Example 3.4. (i) Let $S^1 \to ES^1 \to BS^1$ be the universal S^1-bundle and K be a simplicial complex with m vertices. Then, we have a model for the bundle of the form $\wedge(dx) \to \wedge(dx) \otimes \wedge(x) \xrightarrow{\widetilde{\iota}} \wedge(x)$, where $\widetilde{\iota}$ is the canonical projection and $\deg x = 1$. It follows from Proposition 3.2 that $\mathcal{M}((ES^1, S^1)^K) \cong (\wedge(x_1, ..., x_m) \otimes SR(K), d)$ where $d(x_j) = dx_j$; see Remark 2.4. Observe that $\mathcal{Z}_K(D^2, S^1) \simeq \mathcal{Z}_K(ES^1, S^1) \simeq_w (ES^1, S^1)^K$; see [8, page 33] for the first homotopy equivalence.

(ii) Let X be a simply-connected space and LX the free loop space, namely the space of maps from S^1 to X endowed with compact-open topology. The rotation action of S^1 on the domain of maps in LX induces an S^1-action on the free loop space. Thus we have the Borel fibration $LX \xrightarrow{i} ES^1 \times_{S^1} LX \xrightarrow{p} BS^1$. We write $(LX)_h$ for the Borel construction $ES^1 \times_{S^1} LX$. Let $(\wedge V, d)$ be the minimal model for X. Then, the result [21, Theorem A] asserts that the sequence

$$\wedge(t) \xrightarrow{\widetilde{p}} (\wedge(t) \otimes \wedge(V \oplus \overline{V}), \delta) \xrightarrow{\widetilde{i}} (\wedge(V \oplus \overline{V}), \delta')$$

is a model for the Borel fibration, where $\delta'(v) = d(v)$, $\delta'(\overline{v}) = -sd(v)$ and $\delta u = \delta'(u) + ts(u)$ for $u \in V \oplus \overline{V}$. The map $\widetilde{i}$ is the projection and hence a surjective model for i. Thus Proposition 3.2 enables us to obtain a Félix–Tanré model for the polyhedral product $((LX)_h, LX)^K$ of the form $(\otimes_{i=1}^m (\wedge(V_i \oplus \overline{V}_i) \otimes SR(K), \otimes_i \delta_i)$.

(iii) We can apply Theorem 2.1 to an explicit surjective model for an inclusion. Let X be a space as in (ii) and $(\wedge V, d)$ a minimal model for X. Then, the projection $(\wedge(V \oplus \overline{V}, \delta') \to (\wedge V, d)$ is a surjective model for the inclusion $X \to LX$ defined by assigning the constant loop at x to a point x. In fact, the inclusion is a section of the evaluation map $ev_0 : LX \to X$ at zero. The inclusion $(\wedge V, d) \to (\wedge(V \oplus \overline{V}, \delta')$ gives rise to a model for ev_0. By considering the rational homotopy, we have the result. Thus, Theorem 2.1 allows us to construct a model for the polyhedral product $(LX, X)^K$ of the form

$$(\bigotimes_{i=1}^{m} \wedge(V_i \oplus \overline{V_i})) / (\overline{v}_{i_1} \cdots \overline{v}_{i_s} \mid \overline{v}_j \in \overline{V}_j, \{i_1, ..., i_s\} \notin K)$$

for which $d(V_i) \subset \wedge V_i$.

Proposition 3.2 enables us to deduce the following result.

Corollary 3.5. *Let K be a simplicial complex with m vertices. For $1 \le j \le m$, let $F_j \to X_j \to Y_j$ be a fibration with simply-connected base Y_j. Then there is a first quadrant spectral sequence converging to $H^*((\underline{X}, \underline{F})^K; \mathbb{Q})$ as an algebra with*

$$E_2^{*,*} \cong \Big(\bigotimes_{j=1}^{m} H^*(F_j; \mathbb{Q}) \Big) \otimes H^*((\underline{Y}, \underline{*})^K; \mathbb{Q})$$

as a bigraded algebra, where $E_2^{p,q} \cong \big(\big(\bigotimes_{j=1}^{m} H^*(F_j; \mathbb{Q}) \big) \otimes H^p((\underline{Y}, \underline{*})^K; \mathbb{Q}) \big)^{p+q}$.

Proof. With the same notation as in Proposition 3.2, we give the CDGA $\mathcal{M}((\underline{X}, \underline{F})^K)$ a filtration associated with the degrees of elements in $\otimes_j \wedge W_j$. The filtration gives rise to the spectral sequence; see [9, 18(b) Example 2]. □

Let $HH_*(A_{PL}(X))$ denote the Hochschild homology of $A_{PL}(X)$. There exists an isomorphism $HH_*(A_{PL}(X)) \cong H^*(LX; \mathbb{Q})$ of algebras; see [21] and [9, 15(c) Example 1]. Therefore, Example 3.4 (ii) allows us to obtain the following result.

Corollary 3.6. *Let X be a simply-connected space. Then, there exists a first quadrant spectral sequence converging to the cohomology $H^*(((LX)_h, LX)^K; \mathbb{Q})$ as an algebra with*

$$E_2^{*,*} \cong HH_*(A_{PL}(X))^{\otimes m} \otimes SR(K)$$

as a bigraded algebra, where bideg $x = (0, \deg x)$ *for $x \in HH_*(A_{PL}(X))$ and* bideg $t_i = (2, 0)$ *for the generator $t_i \in SR(K)$.*

Remark 3.7. Let $F_j \overset{i_j}{\to} X_j \overset{p_j}{\to} B_j$ be a fibration for each $1 \le j \le m$. Suppose further that each (X_j, F_j) is a pair of CW-complexes. Then, the Félix and Tanré rational model for $(\underline{X}, \underline{F})^K$ in Proposition 3.2 associated with the fibre inclusions is nothing but the relative Sullivan model for the pullback

$$\begin{array}{ccc} (\underline{F}, \underline{F})^K & = & \Pi^m \underline{F} \\ \downarrow & & \downarrow \\ (\underline{X}, \underline{F})^K & \longrightarrow & \Pi^m \underline{X} \\ \downarrow & & \downarrow \\ (\underline{B}, \underline{*})^K & \longrightarrow & \Pi^m \underline{B} \end{array}$$

which is introduced in [8, Lemma 2.3.1]. In fact, This follows from [15, 20.6].

We can apply the replacement in (2.2) to p_j if (X_j, F_j) is not a pair of CW-complexes. As a consequence, we deduce Proposition 3.2 by using a rational model for $(\underline{B}, \underline{*})^K$. Moreover, we have the same spectral sequences with coefficients in *arbitrary* field as in Corollaries 3.5 and 3.6.

4. A rational model for the polyhedral product of a pair of Lie groups

In this section, we consider a more explicit model for the polyhedral product $(G, H)^K$ for a pair of a Lie group and a closed subgroup corresponding to arbitrary simplicial complex K. In particular, we have a manageable SR type model for $(G, H)^K$. Indeed, the rational model is determined by the image of the characteristic classes of BG by the map $(Bi)^* : H^*(BG; \mathbb{Q}) \to H^*(BH; \mathbb{Q})$ for the inclusion $i : H \to G$; see Proposition 4.1 for more details of the model. By using the model, we prove Theorem 1.2.

Let G and H be a connected Lie group G and its closed connected subgroup H, respectively. Let $H \stackrel{i}{\to} G \stackrel{\pi}{\to} G/H$ be the principal H-bundle. In order to obtain a rational model for $(G,H)^K$, we first construct an appropriate surjective model for the fibre inclusion i.

Consider the fibration $EH \to EH \times_H G \stackrel{q}{\to} G/H$ associated with the bundle π. Since EH is contractible, it follows that the map q is a weak homotopy equivalence. Moreover, we have a homotopy pullback diagram

(4.1)
$$\begin{array}{ccc} G & \xrightarrow{=} & G \\ {\scriptstyle \iota}\downarrow & & \downarrow \\ EH \times_H G & \xrightarrow{h} & EG \\ \downarrow & & \downarrow{\scriptstyle p_G} \\ BH & \xrightarrow[Bi]{} & BG, \end{array}$$

where vertical sequences are the associated fibration and the universal G-bundle, respectively, and h is a map defined by $h([x,g]) = Ei(x)g$. There exists a model for the universal bundle of the form

$$\begin{array}{ccccc} (\wedge V_{BG}, 0) & \longrightarrow & (\wedge V_{BG} \otimes \wedge P_G, d) & \longrightarrow & (\wedge P_G, 0) \\ \simeq\downarrow{\scriptstyle m_{BG}} & & \simeq\downarrow{\scriptstyle m_{E_G}} & & \simeq\downarrow \\ A_{PL}(BG) & \xrightarrow{p_G^*} & A_{PL}(E_G) & \longrightarrow & A_{PL}(G), \end{array}$$

such that $d(x_i) = y_i$ and $m_{E_G}(x_i) = \Psi_i$ for $x_i \in \wedge V_G$, where $\Psi_i \in A_{PL}(E_G)$ with $d\Psi_i = p_G^* m_{BG}(y_i)$. Then, by applying the pushout construction [9, Proposition 15.8] to the model of the bundle p_G, we have a model

$$\begin{array}{ccccc} (\wedge V_{BH}, 0) & \longrightarrow & (\wedge V_{BH} \otimes \wedge P_G, d) & \xrightarrow{\bar{\iota}} & (\wedge P_G, 0) \\ \simeq\downarrow & & \simeq\downarrow{\scriptstyle m} & & \simeq\downarrow{\scriptstyle m_G} \\ A_{PL}(BH) & \longrightarrow & A_{PL}(EH \times_H G) & \xrightarrow{A_{PL}(\iota)} & A_{PL}(G) \end{array}$$

of the fibration of the left hand side in the diagram (4.1) in which $d(x_i) = (Bi)^* y_i$. Furthermore, the maps π, ι and q mentioned above fit in the commutative diagram

$$\begin{array}{ccccc} G & \xrightarrow{\simeq} & EH \times G & \xrightarrow{\simeq} & G \\ & {\scriptstyle \iota}\searrow & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & EH \times_H G & \underset{\simeq}{\xrightarrow{q}} & G/H, \end{array}$$

where horizontal arrows are (weak) homotopy equivalences. Thus, the Lifting lemma [9, Proposition 14.6] implies that a Sullivan model ([9, 15(a)]) for ι is regarded as that for π. Consider a commutative diagram

$$\begin{array}{ccccc} (\wedge V_{BH} \otimes \wedge P_G, d) & \xrightarrow{j} & (\wedge V_{BH} \otimes \wedge P_G \otimes \wedge P_H, \partial) & \underset{\simeq}{\xrightarrow{\gamma}} & (\wedge P_G, 0) \\ \simeq\downarrow{\scriptstyle m} & & & \swarrow{\scriptstyle m_G}\,\simeq & \\ A_{PL}(EH \times_H G) & \xrightarrow{A_{PL}(\iota)} & A_{PL}(G) & & \end{array}$$

of CDGA's in which j is an extension, γ is the projection and the differential ∂ is defined by $\partial(u_i) = t_i$ for $u_i \in P_H$, $t_i \in V_{BH}$ and $\partial(x_i) = (Bi)^*(y_i)$ for $x_i \in P_G$. Observe that $H^*(\wedge P_G, 0) \cong H^*(G;\mathbb{Q})$ and γ is a quasi-isomorphism. Then, we see that the projection $\gamma : (\wedge V_{BH} \otimes \wedge P_G \otimes \wedge P_H, \partial) \to (\wedge P_H, 0)$ is a surjective model for the inclusion $H \to G$. Thus, Proposition 3.2 yields the following result.

Proposition 4.1. *One has a rational model of the form*

$$((\wedge P_H)^{\otimes m} \otimes \big((\wedge V_{BH} \otimes \wedge P_G)^{\otimes m}/I(K)\big), \partial) \tag{4.2}$$

for the polyhedral product $(G, H)^K$, where $I(K)$ denotes the Stanley–Reisner ideal generated by elements in $(\wedge V_{BK} \otimes \wedge P_G)^{\otimes m}$.

Example 4.2. With the same notation as above, suppose further that $\operatorname{rank} G = \operatorname{rank} H = N$. Then, the sequence $(Bi)^*(y_j)$ for $j = 1, ..., N$ is regular. This enables us to deduce that G/H is formal. There exists a sequence of quasi-isomorphisms

$$A_{PL}(G/H) \xleftarrow{\simeq} \wedge V_{BH} \otimes \wedge P_G =: \mathcal{M} \xrightarrow[u]{\simeq} H^*(G/H) = (\wedge V_{BH}/((Bi)^*(y_i), d = 0).$$

The naturality (Proposition 2.3) of the rational model in Theorem 2.1 gives rise to a commutative diagram

$$\begin{array}{ccccc} \lim_{\sigma\in K} A_{PL}((G/H,*)^\sigma) & \xleftarrow[\simeq]{\Phi} & \lim_{\sigma\in K} \widetilde{D}^\sigma & \xleftarrow[\simeq]{\alpha} & (\bigotimes^m \mathcal{M})/I(K) =: B_1 \\ \eta\uparrow\simeq & & \simeq\downarrow u_1 & & \downarrow u_2 \\ A_{PL}((G/H,*)^K) & & \lim_{\sigma\in K} H^*(G/H)^\sigma & \xleftarrow[\simeq]{\alpha} & \big(\bigotimes^m H^*(G/H)\big)/I(K) =: B_2, \end{array}$$

where $H^*(G/H)^\sigma := \bigotimes_{i\in\sigma} H^*(G/H)^\sigma \otimes \bigotimes_{i\notin\sigma} \mathbb{Q}$, u_1 and u_2 are maps of CDGA's induced by u; see the sequence (2.4). By virtue of [18, Proposition 4.8], we see that the map u_1 is a quasi-isomorphism. Thus, the commutativity implies that u_2 is also a quasi-isomorphism. Observe that $I(K) = J(K)$ in the case that we deal with; see Section 2. We consider the pushout diagram of ℓ along u_2

$$\begin{array}{ccc} B_1 & \xrightarrow{\ell} & ((\wedge P_H)^{\otimes m} \otimes \big(\bigotimes^m \mathcal{M}\big)/I(K), \partial) \\ u_2\downarrow\simeq & & \downarrow\widetilde{u_2} \\ B_2 & \xrightarrow{\widetilde{\ell}} & ((\wedge P_H)^{\otimes m} \otimes \big(\bigotimes^m H^*(G/H)\big)/I(K), \partial) =: C \end{array} \tag{4.3}$$

where ℓ is the KS-extension induced by the rational model for $(G, H)^K$ in (4.2); see [15, Chapter 1] for a KS-extension. It follows from [9, Lemma 14.2] that $\widetilde{u_2}$ is a quasi-isomorphism and hence C is also a rational model for $(G, H)^K$.

In particular, for the unitary group $U(n)$, the maximal torus T and every simplicial complex K with m vertices, we have a rational model for $(U(n), T)^K$ of the form

$$((\wedge(x_i, ..., x_n))^{\otimes m} \otimes \big(\bigotimes_{i=1}^{m} \mathbb{Q}[t_1, ..., t_n]/(\sigma_1, ..., \sigma_n)\big)/I(K), \partial),$$

where $\partial(x_i) = t_i$ and σ_k denotes the kth elementary symmetric polynomial.

Example 4.3. Let K be arbitrary simplicial complex with m vertices. By virtue of Propositions 2.3 and 4.1, we see that the projection $q : (SU(n), SU(k))^K \to (SU(n)/SU(k), *)^K$ admits a model given by

$$\widetilde{q} : \big(\wedge(x_{k+1}, .., x_n)^{\otimes m}/I(K)), 0\big) \to \big(\wedge(x_2, .., x_k)^{\otimes m} \otimes (\wedge(x_{k+1}, .., x_n)^{\otimes m}/I(K)), 0\big),$$

where $\widetilde{q}(x_i) = x_i$ for $k + 1 \le i \le n$, and $\deg x_i = 2i - 1$. Since the domain of $\widetilde{q}$ admits a Sullivan algebra, we can construct a KS-extesion for $\widetilde{q}$. Then, it follows from Lemma A.1 that the projection q is formalizable in the sense of Definition 1.4.

Suppose further that the 1-skeleton of K does not coincide with that of Δ^m. Then, the minimal model for $(SU(n)/SU(k),*)^K$ has nontrivial differential whose quadratic part is also nontrivial; see [9, pages 144-145] for a way to construct a minimal model for a CDGA. Therefore, the result [9, Theorem 21.6] yields that the rational homotopy groups $\pi_*((SU(n)/SU(k),*)^K)_{\mathbb{Q}}$ and $\pi_*((SU(n),SU(k))^K)_{\mathbb{Q}}$ have nontrivial Whitehead products. Observe that the Whitehead product on $\pi_*((SU(n)/SU(k))_{\mathbb{Q}}$ vanishes.

Let X be a pointed space and $\pi^*(X) := H^*(Q(\wedge V), d_0)$ the homology of the vector space of indecomposable elements of a Sullivan model $(\wedge V, d)$ for X, where $Q(\wedge V)$ is the vector space of indecomposable elements and d_0 denotes the linear part of the differential d. There is a natural map ν_X from $\pi^*(\wedge V)$ to $\mathrm{Hom}(\pi_*(X),\mathbb{Q})$ provided $\pi_*(X)$ is abelian. Moreover, ν_X is an isomorphism if X is a nilpotent space whose fundamental group is abelian; see [4, 11.3].

It follows from the proof of [9, Proposition 15.13] that the natural map $\nu_{(\ )}$ is compatible with the connecting homomorphisms of the dual to the homotopy exact sequence for a fibration and the homology exact sequence for $\pi^*(\ )$ if fundamental groups of spaces of the fibration are abelian. Then, by considering the middle vertical fibration $\mathcal{F}$ in (1.1), we have

Claim 4.4. *For each space X in the fibration $\mathcal{F}$, the map ν_X is an isomorphism.*

Proof of Theorem 1.2. We first observe that $(G/H,*)^K$ is simply-connected. This follows from the Seifert–van Kampen theorem. Then, we see that $\pi_1((G,H)^K)$ is an abelian group.

In what follows, we consider a Sullivan model for $(G,H)^K$ with the same notation as in Example 4.2. Let $\beta_0 : \bigotimes^m \mathcal{M} \to \big(\bigotimes^m H^*(G/H)\big)/I(K) =: B_2$ be the composite of the quasi-isomorphism $u_2 : B_1 \to B_2$ mentioned above and the projection $\bigotimes^m \mathcal{M} \to (\bigotimes^m \mathcal{M})/I(K)$. Observe that $\mathcal{M} = \wedge(V_{BH} \oplus P_G)$. Extending β_0, we define a quasi-isomorphism

$$\beta : A_1 := (\bigotimes^m \mathcal{M}) \otimes \wedge V = \wedge((\oplus^m(V_{BH} \oplus P_G)) \oplus V) \xrightarrow{\simeq} B_2.$$

Let d_0 denote the linear part of the differential of A_1. In order to construct a minimal Sullivan model for A_1, we apply the procedure of the proof of [9, Theorem 14.9]. As a consequence, there exists an isomorphism $(\wedge W, d') \otimes \wedge(U \oplus dU) \cong A_1$ for which $\wedge(U \oplus dU)$ is a contractible CDGA, $(\wedge W, d')$ is minimal, $(\oplus^m(V_{BH} \oplus P_G)) \oplus V = U \oplus \mathrm{Ker}\, d_0 = U \oplus d_0 U \oplus W$ and $d_0(W) = 0$. By the construction, we may assume that $\oplus^m(V_{BH} \oplus P_G) \subset W$. Then, we have a quasi-isomorphism $\beta' : (\wedge W, d') \xrightarrow{\simeq} B_2$ and a pushout diagram

$$\begin{array}{ccc} (\wedge W, d') & \xrightarrow{I} & ((\wedge P_H)^{\otimes m} \otimes \wedge W, \partial) \\ {\scriptstyle \beta'}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \beta''} \\ B_2 & \xrightarrow[\widetilde{\ell}]{} & (\wedge P_H)^{\otimes m} \otimes B_2, \end{array} \tag{4.4}$$

in which $\partial(x_k) = t_k \in V_{BH}$ for $x_k \in P_H$ and I is the canonical inclusion. Therefore, it follows that the map β'' is a quasi-isomorphism. Moreover, we see that the bottom right CDGA in the square above is nothing but the CDGA C in Example 4.2.

By applying Lemma A.1 repeatedly to the diagram obtained by combining the diagrams (4.3) with (4.4) and to a commutative square given by the naturarily of maps in (2.4), we have a commutative diagram

$$\begin{array}{ccc} (\wedge W, d') & \xrightarrow{I} & ((\wedge P_H)^{\otimes m} \otimes \wedge W, \partial) \\ \simeq \downarrow & & \downarrow \simeq \\ A_{PL}((G/H,*)^K) & \xrightarrow[A_{PL}(q)]{} & A_{PL}((G,H)^K). \end{array} \tag{4.5}$$

Recall that $\oplus^m(V_{BG} \oplus P_G)$ is a subspace of W. Then, we may write $(\wedge V_{BH} \otimes \wedge P_G)^{\otimes m} \otimes \wedge W'$ for $\wedge W$. Thus, the upper sequence in the diagram (4.5) gives rise to a short exact sequence of complexes

$$0 \leftarrow (\oplus^m P_H, 0) \leftarrow ((\oplus^m(P_H \oplus V_{BH} \oplus P_G)) \oplus W', \partial_0) \leftarrow (W, 0), \leftarrow 0$$

in which the linear part ∂_0 of ∂ satisfies the condition that $\partial_0 : \wedge P_H \to \wedge V_{BH}$, $\partial_0(t_k) = y_k$ and $\partial_0|_{P_G \oplus W'} = 0$. The last equality follows from the assumption that $(Bi)^*x_k$ is decomposable for each k. The homology long exact sequence is decomposed into a short exact sequence of the form

$$0 \leftarrow H(\oplus^m(P_H \oplus V_{BH} \oplus P_G)) \oplus W', \partial_0) \leftarrow (W, 0) \xleftarrow{d_0'} (\oplus^m P_H, 0) \leftarrow 0,$$

where d_0' denotes the connecting homomorphism. In fact, the linear map d_0' coincides with the composite $\oplus^m P_H \xrightarrow{\partial_0} \oplus^m(P_H \oplus V_{BH} \oplus P_G)) \oplus W' \xrightarrow{pr} W$, where pr is the projection; see the proof of [9, Proposition 15.13]. Thus, Claim 4.4 yields the result. □

Remark 4.5. Let G be a compact Lie group and H be a closed subgroup for which G/H is simply connected and $(Bi)^*(x_k)$ is *indecomposable* in $H^*(BH; \mathbb{Q})$ for some generator x_k of $H^*(BG; \mathbb{Q})$. Then, we see that the connecting homomorphism

$$\partial_* : \pi_*((G/H,*)^K)_{\mathbb{Q}} \to \pi_{*-1}(\Pi^m H)_{\mathbb{Q}}$$

is not surjective. The connecting homomorphism is natural with respect to maps between spaces. Then, in order to prove the fact, it suffices to show that the connecting homomorphism $\partial^* : \pi^*(H) \to \pi^{*+1}(G/H)$ is not injective; see the diagram (1.1). To this end, we show that the map $i^* : \pi^*(G) \to \pi^*(H)$ is non trivial.

Recall the surjective model $\rho : (\wedge V_{BH} \otimes \wedge P_G \otimes \wedge P_H, \partial) \to (\wedge P_H, 0)$ for the inclusion $H \to G$ used in the construction of the model (4.2). Suppose that $(Bi)^*(x_k) = \sum_i \lambda_i t_i +$(decomposable element) for some generator x_k in $H^*(BG; \mathbb{Q})$, where $\lambda_i \neq 0$ for some i. Then, it follows that $x_i + \sum_i \lambda_i u_i$ is a cocycle in the cochain complex $(Q(\wedge V_{BH} \otimes \wedge P_G \otimes \wedge P_H), \partial_0)$ and

$$i^*(x_i + \sum_i \lambda_i u_i) = \sum_i \lambda_i u_i \neq 0$$

for $i^* = H(Q(\rho)) : H(Q(\wedge V_{BH} \otimes \wedge P_G \otimes \wedge P_H), \partial_0) \to H(Q(\wedge P_H), 0) = P_H$.

For example, we see that $\partial_* : \pi_*((U(n)/T,*)^K)_{\mathbb{Q}} \to \pi_{*-1}(\Pi^m T)_{\mathbb{Q}}$ is not surjective for a maximal torus T of $U(n)$.

5. The formality of a compact toric manifold

We prove the following result by using the commutative diagram (1.2).

Theorem 5.1. *Every compact toric manifold X_Σ is formal.*

This result is proved in [18, 3]; see also [5, Theorem 8.1.10]. The proof of [3, Proposition 3.1] indeed uses the algebra structure of the cohomology of the toric manifold. We apply the Félix–Tanré model for $DJ(K)$ in order to prove the fact.

We also use a result due to Baum concerning a characterization of a regular sequence.

Proposition 5.2. ([2, 3.5 Proposition]) *Let A be a connected commutative algebra and $a_1, ..., a_t$ elements of $A^{>0}$. Set $\Lambda = \mathbb{K}[x_1, ..., x_t]$ with $\deg x_i = \deg a_i$ and consider A to be a Λ-module by means of the map $f : \Lambda \to A$ defined by $f(x_i) = a_i$. Then the following are equivalent:*

(i) *$a_1, ..., a_t$ is a regular sequence.*
(ii) $\mathrm{Tor}_\Lambda^{-1,*}(\mathbb{K}, A) = 0$.
(iii) $\mathrm{Tor}_\Lambda^{-j,*}(\mathbb{K}, A) = 0$ *for all* $j \geq 1$.
(iv) *A is a projective Λ-module.*
(v) *As a Λ-module A is isomorphic to $\Lambda \otimes (A/(a_1, ..., a_t))$.*

The following result gives a rational model for the toric manifold X_Σ in the proof of Theorem 5.1.

Lemma 5.3. *The map $(B\rho) \circ q : DJ(K) \to BL'$ in the diagram (1.2) is formalizable; see Definition 1.4 and the paragraph after the diagram (1.2).*

Proof. The result follows from the same argument as in Example 2.5. □

Proof of Theorem 5.1. Let $\{v_j\}_{j=1}^m$ is the set of 1-dimensional cones of the fan Σ of n dimension. Each v_i are in the lattice N of $\mathbb{R}^n$ which defines the fan Σ. Then, it follows from the construction of the diagram (1.2) that $H^*(BL) \cong \mathbb{Q}[t'_1, ..., t'_n]$ as an algebra. Observe that $\dim L = \dim \Sigma = n$. Moreover, we see that for $i = 1, ..., n$,

$$(B\rho)^*(t'_i) = \sum_{j=1}^m \langle m_i, v_j \rangle t_j,$$

where t_j denotes the generator of $H^*(BG) \cong \mathbb{Q}[t_i, ..., t_m]$ and m_i is the dual basis for $M := \mathrm{Hom}(N, \mathbb{Z})$. The Félix–Tanré model for $DJ(K)$ is of the form $(SR(K) = \mathbb{Q}[t_i, ..., t_m]/I(K), 0)$ for which $q^*(t_j) = t_j$ for $j = 1, ..., m$. Consider the pushout construction of models ([15, 9]) for the pullback (1.2). Then, by Lemma 5.3 and [20, Proposition 2.3.4], we have a rational model for X_Σ of the form

$$C := (\wedge(x_1, ..., x_n) \otimes SR(K), d(x_i) = q^*(B\rho)^*(t'_i) = \sum_{j=1}^m \langle m_i, v_j \rangle t_j),$$

where $\deg x_i = 1$. This also computes the torsion functor $\mathrm{Tor}_{H^*(BL)}^{*,*}(H^*(DJ(K)), \mathbb{Q})$ if we assign a bidegree $(-1, 2)$ to each x_i. The result [7, Theorem 12.3.11] asserts that $H^{\mathrm{odd}}(X_\Sigma; \mathbb{Q}) = 0$. This implies that $\mathrm{Tor}_{H^*(BL)}^{-1,*}(H^*(DJ(K)), \mathbb{Q}) = 0$. It follows from Proposition 5.2 that $q^*(B\rho)^*(t'_1), ..., q^*(B\rho)^*(t'_n)$ is a regular sequence in $SR(K)$. Thus, we have a quasi-isomorphism

$$f : C \to SR(K)\big/(d(x_i); i = 1, ..., n) = H^*(X_\Sigma; \mathbb{Q})$$

defined by $f(t_j) = t_j$ and $f(x_i) = 0$. □

Remark 5.4. We can also obtain the rational cohomology of the compact toric manifold by using the Eilenberg-Moore spectral sequence for the pullback (1.2). In fact, it follows from the computation of the spectral sequence that, as algebras,

$$H^*(X_\Sigma) \cong \mathrm{Tor}_{H^*(BL)}(H^*(DJ(K)), \mathbb{Q}) \cong SR(K)\Big/\big(\sum_{j=1}^{m}\langle m_i, v_j\rangle t_j\big).$$

One might be aware that the consideration above for the polyhedral product $(\mathbb{C}, \mathbb{C}^*)^K$ is applicable to more general one, for example, $(EG, G)^K$ for a connected Lie group G. In fact, for a simplicial complex K with m vertices, we have (homotopy) pullback diagrams

$$(5.1)\quad \begin{array}{ccccc} X_{K,(G,H)} := E(\Pi^m G)\times_H (EG,G)^K & \xrightarrow{p} & E(\Pi^m G)/H & \longrightarrow & EL \\ \pi\downarrow & & \downarrow & & \downarrow \\ E(\Pi^m G)\times_{\Pi^m G}(EG,G)^K & \xrightarrow[q]{} & B(\Pi^m G) & \xrightarrow[B\rho]{} & BL, \end{array}$$

where H is a normal (not necessarily connected) closed subgroup of $(\Pi^m G)$ and $L = (\Pi^m G)/H$. The result [8, Lemma 2.3.2] yields that the natural map $E(\Pi^m G)\times_{\Pi^m G} (EG, G)^K \overset{\simeq}{\to} (BG, *)^K$ is a homotopy equivalence. Then, we have

Theorem 5.5. *Suppose that* $H^{\mathrm{odd}}(X_{K,(G,H)}; \mathbb{Q}) = 0$. *Then* $X_{K,(G,H)}$ *is formal.*

Theorem 1.6 asserts that the condition in Theorem 5.5 is satisfied only for the toric manifold M among partial quotients associated with M.

Corollary 5.6. *With the same notation as above, suppose that* $H^{\mathrm{odd}}(X_{K,(G,H)}; \mathbb{Q}) = 0$ *and* $H^*(X_{K,(G,H)}; \mathbb{Q}) \cong H^*(X_{K',(G',H')}; \mathbb{Q})$. *Then* $X_{K,(G,H)} \simeq_{\mathbb{Q}} X_{K',(G',H')}$ *if the spaces are nilpotent.*

Remark 5.7. Suppose that $H^*(BL) \cong \mathbb{Q}[t'_1, ..., t'_n]$. Under the same assumption as in Theorem 5.5, we see that $X_{K,(G,H)}$ admits a rational model of the form

$$(\wedge(x_1, ..., x_n) \otimes (\bigotimes^{m} H^*(G; \mathbb{Q})\big/I(K)), d(x_i) = q^*(B\rho)^*(t'_i))$$

in which $q^*(B\rho)^*(t'_1), ..., q^*(B\rho)^*(t'_n)$ is a regular sequence.

We observe that, for a compact smooth toric manifold X_Σ, there is a homotopy equivalence $X_\Sigma \simeq X_{K,((\mathbb{C}^*)^m,H)}$ for which K is a simplicial complex associated with the fan Σ and $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)/H$ is Cox's construction for X_Σ. Moreover, the toric manifold X_Σ is simply connected and hence nilpotent. Thus, Corollary 5.6 is regarded as an answer of the *rational cohomological rigidity problem* for toric manifolds

6. The (non)formalizability of partial quotients

We begin by considering formalizability for toric manifolds.

Proposition 6.1. *The map* $\pi_H : X_\Sigma \to DJ(K)$ *in (1.2) is formalizable.*

Proof. By considering the sequence (2.4) for $(X, A) = (BS^1, *)$, we have quasi-isomorphisms connecting $A_{PL}(DJ(K))$ with $SR(K)$ the Stanley–Reisner algebra. We can construct a minimal model $\phi : \wedge W \to SR(K)$ for $SR(K)$ so that $W = \mathbb{Q}\{t_1, ..., t_m\} \oplus V$, where $t_1, ..., t_m$ give the generators of $SR(K)$, $\phi(V) = 0$ and $V = V^{\geq 2}$. The Lifting lemma ([9, Proposition 12.9]) yields a quasi-isomorphism

$\phi' : \wedge W \to A_{PL}(DJ(K))$. Thus, in particular, the Davis–Januszkiewicz space $DJ(K)$ is formal. The pushout construction in the proof of Theorem 5.1 gives rise to a commutative diagram

$$\begin{array}{ccc} A_{PL}((DJ(K)) & \xrightarrow{A_{PL}(\pi_H)} & A_{PL}(X_\Sigma) \\ \phi' \uparrow \simeq & & \simeq \uparrow \\ \wedge W & \xrightarrow{\quad i \quad} & \wedge(x_1, ..., x_n) \otimes \wedge W, \end{array}$$

where i is a KS-extension. Moreover, we have a commutative diagram of CDGA's

$$\begin{array}{ccc} \wedge W & \xrightarrow{\quad i \quad} & \wedge(x_1, ..., x_n) \otimes \wedge W \\ \phi \downarrow \simeq & & \simeq \downarrow f \\ H^*(DJ(K); \mathbb{Q}) & \xrightarrow[(\pi_H)^*]{} & H^*(X_\Sigma; \mathbb{Q}) \end{array}$$

in which ϕ is a quasi-isomorphism defined by $\phi(t_j) = t_j$ and $\phi|_V \equiv 0$, the map i is an extension and f is the quasi-isomorphism given in the end of the proof of Theorem 5.1. □

Proof of Theorem 1.6. Let $\mathcal{Z}_K$ denote the space $\mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*)$. Suppose that $H = H'$. Then the partial quotient $\mathcal{Z}_K/H'$ is nothing but the toric manifold X_Σ. Then, the result [7, Theorem 12.3.11] implies the assertion (ii).

We recall the proof of Theorem 5.1. Then, we have a rational model

$$C' := (\wedge(x_1, ..., x_l) \otimes SR(K), d(x_i) = q^*(B\rho)^*(t_i')) \tag{6.1}$$

for $\mathcal{Z}_K/H'$. Under the assumption (ii), by Proposition 5.2, we see that the sequence $d(x_1), ..., d(x_l)$ is regular. Thus the same argument as in the proof of Proposition 6.1 yields (iii).

Suppose that H' is a connected proper subgroup of H. We show that $\pi_{H'}$ is not formalizable. We may replace three spaces $\mathcal{Z}_K$, the tori H and H' acting the moment-angle manifold with the polyhedral product $(D^2, S^1)^K$, a compact Lie group T^k and its subtorus with an appropriate integer k, respectively. Assume that the fan Σ has m rays and hence K is a simplicial complex with m vertices. If the fan is of dimension n, then we have an exact sequence $1 \to T^k \to T^m \xrightarrow{\rho} T^n \to 1$ via Cox's construction of the toric manifold X_Σ. With the same notation as in the proof of Theorem 5.1, since $d(x_1), ..., d(x_n)$ is a regular sequence, it follows from Proposition 5.2 that $H^*(DJ(K)) \cong \mathbb{Q}[t_1, ..., t_n] \otimes H^*(X_\Sigma)$ as a $\mathbb{Q}[t_1, ..., t_n]$-module.

For a proper subgroup H' of T^k, the cokernel L' of the inclusion $H' \to T^m$ is the torus of dimension l greater than n. Consider a rational model (6.1) for $\mathcal{Z}_K/H'$. We assume that $d(x_1), ..., d(x_l)$ is a regular sequence. Then, by the same argument as in the proof of Theorem 5.1 with the diagram (1.2), we have $H^*(DJ(K)) \cong \mathbb{Q}[t_1, ..., t_l] \otimes H^*(\mathcal{Z}_K/H')$ as a $\mathbb{Q}[t_1, ..., t_l]$-module. By considering the Poincaré series of $H^*(DJ(K))$ with two ways, we have an equality

$$\frac{1}{\Pi^{l-n}(1-t^2)} P_1(t) = P_2(t),$$

where $P_1(t)$ and $P_2(t)$ are the Poincaré series of $H^*(\mathcal{Z}_K/H')$ and $H^*(\mathcal{Z}_K/H) \cong H^*(X_\Sigma)$, respectively. Since the partial quotients are manifolds of finite dimensions, it follows that $P_1(t)$ and $P_2(t)$ are polynomials. This contradicts the equality above and hence $d(x_1), ..., d(x_l)$ is not a regular sequence.

Suppose that the map $\pi_{H'} : \mathcal{Z}_K/H' \to DJ(K)$ is formalizable. By virtue of [20, Proposition 2.3.4], we have a commutative diagram

$$\begin{array}{ccc} A_{PL}((DJ(K)) & \xrightarrow{A_{PL}(\pi_{H'})} & A_{PL}(\mathcal{Z}_K/H') \\ \phi' \uparrow \simeq & & \uparrow \simeq \\ \wedge W & \xrightarrow{\quad i \quad} & \wedge(x_1, ..., x_l) \otimes \wedge W \\ \phi \downarrow \simeq & & \eta \downarrow \simeq \\ H^*(DJ(K); \mathbb{Q}) & \xrightarrow[(\pi_{H'})^*]{} & H^*(\mathcal{Z}_K/H'; \mathbb{Q}). \end{array}$$

Consider the fibration $\mathcal{Z}_K \to EG \times_{H'} \mathcal{Z}_K \xrightarrow{p} (EG)/H'$ which fits in the diagram (1.2). The argument in [8, 4.1] enables us to conclude that $\mathcal{Z}_K$ is 2-connected; see also [5, Proposition 4.3.5]. Thus, the homotopy exact sequence of the fibration above yields that $\mathcal{Z}_K/H' \simeq EG \times_{H'} \mathcal{Z}_K$ is simply connected and hence $H^1(\mathcal{Z}_K/H') = 0$.

By Lemma 5.3 and [20, Proposition 2.3.4], we see that $\mathrm{Tor}^*_{\mathbb{Q}[t_1,...,t_l]}(\mathbb{Q}, SR(K)) \cong \mathrm{Tor}^*_{\mathbb{Q}[t_1,...,t_l]}(\mathbb{Q}, \wedge W)$. This implies that the spectral sequence converging to the torsion group $\mathrm{Tor}^*_{\mathbb{Q}[t_1,...,t_l]}(\mathbb{Q}, \wedge W)$ with $E_2^{*,*} \cong \mathrm{Tor}^{*,*}_{\mathbb{Q}[t_1,...,t_l]}(\mathbb{Q}, SR(K))$ collapses at the E_2-term. Then, the fact allows us to obtain a sequence

$$\mathrm{Tor}_P^{-1,*}(\mathbb{Q}, SR(K)) \cong E_0^{-1,*} \xleftarrow{\ p\ } F^{-1}\mathrm{Tor}_P^{*-1}(\mathbb{Q}, \wedge W) \xrightarrow{\ \iota\ } \mathrm{Tor}_P^{*-1}(\mathbb{Q}, \wedge W).$$

Here $\{F^j\}$ denotes the filtration associated to the spectral sequence, P is the polynomial algebra $\mathbb{Q}[t_1, ..., t_l]$, p and ι are the canonical projection and the inclusion, respectively. Since $d(x_1), ..., d(x_l)$ is not a regular sequence, it follows form Proposition 5.2 that there is a non-exact cocycle

$$w = \sum_{j=1}^{l} u_j x_j - z$$

in $F^{-1}\mathrm{Tor}^*_{\mathbb{Q}[t_1,...,t_l]}(\mathbb{Q}, \wedge W)$, where u_i and z are in $\wedge W$. Observe that the torsion algebra $\mathrm{Tor}^*_{\mathbb{Q}[t_1,...,t_l]}(\mathbb{Q}, \wedge W)$ is isomorphic to the cohomology $H^*(\wedge(x_1, ..., x_l) \otimes \wedge W, d)$ as an algebra. We see that $\deg x_j = 1$ and then $\eta(x_j) = 0$ for each j. The element z is of odd degree and in the image of the map i. Thus, since $H^{\mathrm{odd}}(DJ(K); \mathbb{Q}) = 0$, it follows that

$$H^*(\eta)([w]) = [\eta(w)] = \eta(w) = \sum_{j=1}^{l} \eta(u_j)\eta(x_j) - (\pi')^*\phi(z) = 0,$$

which is a contradiction. □

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Appendix A. A lifting lemma

In this section, we describe an algebraic result obtained by the Lifting lemmas [9, Lemma 12.4 and Proposition 14.6].

Lemma A.1. *For a commutative diagram (A.1) below with a Sullivan algebra A_1 and a KS-extension I, one has a commutative diagram (A.2) in which $\widetilde{u_i}$ is a quasi-isomorphism if so is u_i for $i = 1$ or 2.*

$$\text{(A.1)}\qquad \begin{array}{ccc} A_1 & \xrightarrow{I} & A_1 \otimes \wedge W_1 \\ {\scriptstyle u_1}\downarrow & & \downarrow{\scriptstyle u_2} \\ B_2 & \xrightarrow{\ell_2} & C_2 \\ {\scriptstyle v}\uparrow\simeq & & \simeq\uparrow{\scriptstyle v'} \\ B_1 & \xrightarrow{\ell_1} & C_1 \end{array} \qquad\qquad \text{(A.2)}\qquad \begin{array}{ccc} A_1 & \xrightarrow{I} & A_1 \otimes \wedge W_1 \\ {\scriptstyle \widetilde{u_1}}\downarrow & & \downarrow{\scriptstyle \widetilde{u_2}} \\ B_1 & \xrightarrow{\ell_1} & C_1 \end{array}$$

Proof. By applying the surjective toric ([9, page 148]) to v, we have a diagram

$$\begin{array}{ccccc} & B_1 & \xrightarrow{\ell_1} & C_1 & \\ & {\scriptstyle v}\downarrow\simeq & & \simeq\downarrow{\scriptstyle v'} & \\ {\scriptstyle\lambda}\ \simeq & B_2 & \xrightarrow{\ell_2} & C_2 & \simeq\ {\scriptstyle\lambda'} \\ {\scriptstyle u_1}\nearrow & {\scriptstyle \widetilde{v}}\uparrow\simeq & & \simeq\uparrow{\scriptstyle \widetilde{v'}} & \\ A_1 \overset{\xi}{\dashrightarrow} & B_1 \otimes \wedge S & \xrightarrow{\ell_1\otimes 1} & C_1 \otimes \wedge S & \end{array}$$

of solid arrows in which three squares are commutative. We observe that $S \cong B_2 \oplus dB_2$ and that $\widetilde{v'}$ is defined by $\widetilde{v'}(c_1) = v'(c_1)$ for $c_1 \in C_1$ and $\widetilde{v'}(s) = \ell_2(s)$ for $s \in S$. Since A_1 is a Sullivan algebra, the Lifting lemma [9, Lemma 12.4] enables us to obtain the map ξ which fits in the commutative triangle. Thus we have a commutative diagram of solid arrows

$$\begin{array}{ccccc} A_1 & \xrightarrow{\xi} & B_1 \otimes \wedge S & \xrightarrow{\ell_1\otimes 1} & C_1 \otimes \wedge S \\ {\scriptstyle I}\downarrow & & \overset{\overline{u_2}}{\nearrow} & & \simeq\downarrow{\scriptstyle \widetilde{v'}} \\ A_1 \otimes \wedge W_1 & & \xrightarrow{\quad u_2 \quad} & & C_2. \end{array}$$

Since the map I is a KS-extension, by using [9, Proposition 14.6], we have a dots arrow $\overline{u_2}$ which makes the upper triangle commutative and the lower triangle commutative up to homotopy relative to A_1. Define $\widetilde{u_1} := \lambda \circ \xi$ and $\widetilde{u_2} := \lambda' \circ \overline{u_2}$. Then, we have the commutative diagram (A.2). By the construction of the map $\widetilde{u_i}$, we see that $\widetilde{u_i}$ is a quasi-isomorphism if so is u_i for $i = 1$ or 2. □

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