TOWARD RIEMANNIAN DIFFEOLOGY

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Abstract. We introduce a framework for Riemannian diffeology. To this end, we use the tangent functor in the sense of Blohmann and one of the options of a metric on a diffeological space in the sense of Iglesias-Zemmour. With a technical condition for a definite Riemannian metric, we show that the psudodistance induced by the metric is indeed a distance. As examples of Riemannian diffeological spaces, an adjunction space of manifolds, a space of smooth maps and the mixed one are considered.

1. Introduction

Diffeology [35, 18] provides a natural generalization of differential topology and geometry. The de Rham theory [13, 14, 16, 17, 25, 27, 12], sheaf theory [30, 23], infinite dimensional geometry for partial differential equations [10] and (abstract) homotopy theory [9, 21, 22, 26, 34] have also been developed in the diffeological setting. Moreover, categorical comparisons of diffeology with other smooth and topological structures are made in [1, 8, 36]. However, it is hard to say that the development of Riemannian notions of diffeological spaces is sufficient.

In [19], Iglesias-Zemmour has introduced a notion of Riemannian metrics in diffeology. The definition of the definiteness of the metric on a diffeological space is described with differential 1-forms on the given space; see [19, Page 3]. Making a quote from [20, Page 227], "It is not clear what definition is the best, for many examples built with manifolds and spaces of smooth maps they do coincide. But they may differ in general and, depending on the problem, one must choose one or the other."

This article introduces a weak Riemannian metric of a diffeological space and its definiteness using the tangent functor due to Blohmann [2, 3]. In particular, the colimit construction of the tangent functor works well in considering the metrics. As a consequence, we may deal with Riemannian metrics for a diffeological adjunction space and the space of smooth maps in our framework simultaneously. A comparison between a weak Riemannian metric and the metric due to Iglesias-Zemmour is made in Propositions 3.5 and 3.10. It is worthwhile mentioning that Goldammer and Welker introduce another definition of a Riemannian diffeological space in [11] by using the tangent space in the sense of Vincent [37]. While it is important to consider the relationship between two frameworks of Riemannian diffeolgy, we do not pursue the issue in this article.

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Here is a summary of our main results. Let X be a diffeological space. The pseudodistance d defined by a weak Riemannian metric on X gives the topology $\mathcal{O}_d$ of X. Then, Theorem 3.13 allows us to conclude that the D-topology of X is finer than $\mathcal{O}_d$. Theorem 3.19 asserts that the pseudodistance defined by a weak Riemannian metric on X is indeed a distance if the metric is definite and the diffeology of X is generated by a family of plots which *separates points*; see Definition 3.17 for the separation condition. The necessity of the separateness for a diffeology is clarified in Example 4.4.

Theorem 4.2 yields that an adjunction diffeological space obtained by attaching two definite weak Riemannian diffeological spaces admits again a definite weak Riemannian metric. We also see that the space $C^\infty(M, N)$ of smooth maps endowed with appropriate diffeology (which may be coarser than the functional diffeology) admits a weak Riemannian metric, here M is a closed manifold and N is a weak Riemannian diffeological space; see Section 5.2. Moreover, it turns out that the pseudodistance on $C^\infty(M, N)$ is a distance; see Theorem 5.6.

These constructions are mixed. In fact, we obtain a fascinating example of a definite weak Riemannian diffeological space.

Example 1.1. (See Example 5.9 and Proposition 5.8 for a more general setting.) Let M be a closed orientable manifold and (N, g_N) a Remannian manifold. The diffeological adjunction space

$$C^\infty(M, N) \coprod_N C^\infty(M, N)$$

obtained by the section $N \to C^\infty(M, N)$ of the evaluation map admits a definite weak Riemannian metric $\widetilde{g}$ for which

$$\iota^*(\widetilde{g}) = (\int_M \mathrm{vol}_M) \times g_N,$$

where the left-hand side is the pullback of the metric $\widetilde{g}$ by the canonical injection $\iota : N \to C^\infty(M, N) \coprod_N C^\infty(M, N)$ and vol_M denotes the volume form of M.

By applying Proposition 3.11, which is a result on the pullback of a metric along an induction, and a general result on a weak Riemannian metric on a mapping space, we investigate a special example of mapping spaces, a *free loop space*. Proposition 5.11 shows that the *concatenation map*, that is important in the study of *string topology* [4], preserves the metrics on the loop spaces. This result suggests that our weak Riemannian metric might be well-suited to string topology; see Section 5.3.

The rest of this article is organized as follows. Section 2 recalls the definition of a diffeological space and gives its examples. In Section 3, we introduce a weak Riemannian metric and the (puseudo-)distance on a diffeological space. As mentioned above, the metric is related to that in the sense of Iglesias-Zemmour. Sections 4 and 5 address metrics on a diffeological adjunction space and a diffeological mapping space, respectively. Section 6 is devoted to considering the warped product in diffeology. In particular, we investigate a pullback of definite weak Riemannian diffeological spaces.

1.1. **Future work and perspective.** The category of diffeological spaces contains manifolds and other spaces with smooth structures; see Example 2.2 below.

Therefore, diffeology together with the notion of Riemannian metrics (Riemannian diffeology) produces many issues on smooth spaces. Here are some of them.

i) With the framework of Rimannian metrics, we may consider geodesic calculus in diffeology as seen in [20].

ii) There exists a fully faithful embedding from the category of manifolds modeled by locally convex spaces in the sense in [24, Section 27] to the category Diff; see [22, Lemma 2.5]. Then, we are interested in descriptions of weak and strong Riemannian metrics on manifolds containing mapping spaces with our metrics in diffeology; see, for example, [33, Section 4.1] for the metrics on infinite-dimensional manifolds.

iii) We expect a new development of convergence theory of smooth spaces with concepts such as connections, curvatures and the Gromov-Hausdorff distance if the notion of metrics of diffeological spaces combines with *elastic diffeology**.

iv) Topological data analysis (TDA) begins usually by putting a data set into an appropriate metric space, in particular an Euclidean space, and investigate shapes of the data. Thus, future work includes moreover studying TDA by using a Riemannian diffeological space as the stage putting a data set.

2. Preliminaries

In this section, we recall examples of diffeological spaces and important fundamental constructions in diffeology which are used throughout this article. We begin with the definition of a diffeological space. A comprehensive reference for diffeology is the book by Iglesias-Zemmour [18].

Definition 2.1. We call an open subset U of a Euclidean space (of arbitrary dimension) a *domain*. A map from a domain to a set X is called a *parametrization* of X. The domain of a parametrization P of X is denoted by U_P or $\mathrm{dom}(P)$.

Let X be a set. A set $\mathcal{D}$ of parametrizations is a *diffeology* of X if the following three conditions hold:

(1) (Covering) Every constant map $U \to X$ for all domains U is in $\mathcal{D}$;

(2) (Compatibility) If $U \to X$ is in $\mathcal{D}$, then for any smooth map $V \to U$ from another domain V, the composite $V \to U \to X$ is also in $\mathcal{D}$;

(3) (Locality) If $U = \bigcup_i U_i$ is an open cover of a domain U and $U \to X$ is a map such that each restriction $U_i \to X$ is in $\mathcal{D}$, then $U \to X$ is in $\mathcal{D}$.

An element of a diffeology $\mathcal{D}$ of X is called a *plot* of X. A *diffeological space* $(X, \mathcal{D})$ consists of a set X and a diffeology $\mathcal{D}$ of X.

For diffeological spaces $(X, \mathcal{D}^X)$ and $(Y, \mathcal{D}^Y)$, a map $f\colon X \to Y$ is *smooth* if the composite $f \circ P$ is in $\mathcal{D}^Y$ for each plot $P \in \mathcal{D}^X$. Then, we have the category Diff consisting of all diffeological spaces and smooth maps.

A typical example of a diffeological space is a manifold, as in the following Example 2.2 (1). We also have many other examples each of which is not a manifold in general.

Example 2.2. (1) Let M be a manifold. Then, the underlying set M and the *standard diffeology* $\mathcal{D}^M_{\mathrm{std}}$ give rise to a diffeological space $(M, \mathcal{D}^M_{\mathrm{std}})$, where $\mathcal{D}^M_{\mathrm{std}}$ is defined to be the set of all smooth maps $U \to M$ from domains to M in the usual sense. We have a functor $m\colon \mathsf{Mfd} \to \mathsf{Diff}$ from the category of manifolds defined by $m(M) = (M, \mathcal{D}^M_{\mathrm{std}})$. This functor is fully faithful; see, for example, [1, 2.1 Example].

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*See [2, 5, 6] for Lie brackets of vector fields which are defined in a category with a tangent structure in the sense of Rosicý [32]. We refer the reader to [7] for connections in tangent categories.

(2) Let $\mathcal{G}$ be a set of parametrizations of X. Then define $\langle\mathcal{G}\rangle$ to be the set of all parametrizations $P\colon U_P \to X$ satisfying the following condition:

For any $r \in U_P$, there exists an open neighborhood V_r of r in U_P such that $P|_{V_r}$ is a constant plot, or there exist $Q \in \mathcal{G}$ and a smooth map $f\colon V_r \to U_Q$ such that $P|_{V_r} = Q \circ f$.

Then $\langle\mathcal{G}\rangle$ is a diffeology of X, called the *diffeology generated by* $\mathcal{G}$.

Let $(X, \mathcal{D})$ be a diffeological space. A subset $\mathcal{G} \subset \mathcal{D}$ is a *generating family* of $\mathcal{D}$ if $\langle\mathcal{G}\rangle = \mathcal{D}$. Examples include generating families $\mathcal{G}_{\mathrm{atlas}}$ and $\mathcal{G}_{\mathrm{imm}}$ of the standard diffeology $\mathcal{D}^M_{\mathrm{std}}$ of a manifold M with an atlas $\{(V_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ given respectively by $\mathcal{G}_{\mathrm{atlas}} = \{i_\lambda \circ \varphi_\lambda^{-1}\colon \varphi_\lambda(V_\lambda) \to M \mid i_\lambda \text{ is the inclusion}, \lambda \in \Lambda\}$ and the set of immersions $\mathcal{G}_{\mathrm{imm}} = \{\text{immersions } f\colon U \looparrowright M\}$.

(3) For a diffeological space $(X, \mathcal{D}^X)$ and a subset A of X, we define $\mathcal{D}^A_{\mathrm{sub}}$ by $\mathcal{D}^A_{\mathrm{sub}} \coloneqq \{P\colon U_P \to A \mid U_P \text{ is a domain and } i \circ P \in \mathcal{D}^X\}$, where $i\colon A \to X$ is the inclusion. Then, the set $\mathcal{D}^A_{\mathrm{sub}}$ is a diffeology of A, which is called the *sub-diffeology*. We call $(A, \mathcal{D}^A_{\mathrm{sub}})$ a *diffeological subspace* of X.

(4) Let $\{(X_i, \mathcal{D}_i)\}_{i\in I}$ be a family of diffeological spaces. Then, the product $\Pi_{i\in I}X_i$ has a diffeology $\mathcal{D}$, called the *product diffeology*, defined to be the set of all parameterizations $P\colon U_P \to \Pi_{i\in I}X_i$ such that $\pi_i \circ p$ are plots of X_i for each $i \in I$, where $\pi_i\colon \Pi_{i\in I}X_i \to X_i$ denotes the canonical projection.

(5) More general, the *initial diffeology* $\mathcal{D}^Y$ for maps $h_i\colon Y \to (X_i, \mathcal{D}_i)$ for $i \in I$ is defined by $\mathcal{D}^Y \coloneqq \{P\colon U_P \to Y \mid h_i \circ P \in \mathcal{D}_i \text{ for } i \in I\}$. This is the largest diffeology on Y making all h_i smooth. We call an injective map $j\colon X \to Y$ between diffeological spaces an *induction* if $\mathcal{D}^X$ coincides with the initial diffeology for j. It is immediate that if A is a diffeological subspace of X, then the inclusion $A \to X$ is an induction.

(6) Let $(X, \mathcal{D}^X)$ and $(Y, \mathcal{D}^Y)$ be diffeological spaces. Let $C^\infty(X, Y)$ denote the set of all smooth maps from X to Y. The *functional diffeology* $\mathcal{D}_{\mathrm{func}}$ is the set of parametrizations $P\colon U_P \to C^\infty(X, Y)$ whose adjoints $\mathrm{ad}(P)\colon U_P \times X \to Y$ are smooth.

(7) Let $\mathcal{F} \coloneqq \{f_i\colon Y_i \to X\}_{i\in I}$ be a set of maps from diffeological spaces $(Y_i, \mathcal{D}^{Y_i})$ $(i \in I)$ to a set X. Then, a diffeology $\mathcal{D}^X$ of X is defined to be the set of parametrizations $P\colon U \to X$ satisfying the following condition; for $r \in U$, (i) there exists an open neighborhood V_r of r in U such that $P|_{V_r}$ is constant, or (ii) for $i \in I$, there exists an open neighborhood $V_{r,i}$ of r in U and a plot $(P_i\colon V_{r,i} \to Y_i) \in \mathcal{D}^{Y_i}$ with $P|_{V_{r,i}} = f_i \circ P_i$. We call $\mathcal{D}^X$ the *final diffeology* of X with respect to $\mathcal{F}$. This is the smallest diffeollogy on X making all f_i smooth.

Moreover, by definition, a surjective map $\pi\colon X \to Y$ between diffeological spaces is a *subduction* if the diffeology of Y is the final diffeology with respect to π.

We shall say that a smooth surjection $\pi\colon X \to Y$ is a *local subduction* if for a point $x \in X$ and each plot $P\colon U_P \to Y$ with $P(0) = \pi(x)$, there exist an open neighborhood W of 0 in U_P and a plot $Q\colon W \to X$ such that $Q(0) = x$ and $\pi \circ Q = P|_W$.

(8) For a family of diffeological spaces $\{(X_i, \mathcal{D}_i)\}_{i\in I}$, the coproduct $\coprod_{i\in I} X_i$ has the final diffeology with respect to the set of canonical inclusions. The diffeology is called the *sum diffeology*.

(9) Let $(X, \mathcal{D})$ be a diffeological space with an equivalence relation $\sim$. Then, the final diffeology of $X/{\sim}$ with respect to the quotient map $q\colon X \to X/{\sim}$ is called the *quotient diffeology*. In particular q is a subduction.

The constructions (4) and (6) above enable us to obtain an adjoint pair

$$\text{-} \times X : \mathsf{Diff} \leftrightarrows \mathsf{Diff} : C^\infty(X, \text{-}).$$

Moreover, the limit and colimit in Diff are constructed explicitly by (4) with (3) and (8) with (9), respectively, via the forgetful functor from Diff to the category of sets; see also [8, Section 2]. Thus, we have

Theorem 2.3. ([1, 16]) *The category* Diff *is complete, cocomplete and cartesian closed.*

Another adjoint pair is given by the *D-topology functor* and the *continuous diffeology functor*

$$D\colon \mathsf{Diff} \leftrightarrows \mathsf{Top} : C,$$

where Top is the category of topological spaces and continuous maps, and

- for $(X, \mathcal{D}^X) \in \mathsf{Diff}$, the topological space $D(X)$ consists of the underlying set X and the set of open sets $\mathcal{O}$ defined by
$$\mathcal{O} := \{O \subset X \mid P^{-1}(O) \text{ is an open set of } U_P \text{ for each } P \in \mathcal{D}^X\},$$
- for $Y \in \mathsf{Top}$, the diffeological space $C(Y)$ consists of the underlying set Y and the diffeology $\mathcal{D}^Y := \{\text{continuous maps } P\colon U_P \to Y\}$.

Example 2.4. The composite of the functor $\mathsf{Mfd} \xrightarrow{m} \mathsf{Diff} \xrightarrow{D} \mathsf{Top}$ coincides with the forgetful functor; see [8, Section 3]. We refer the reader to [8, Proposition 3.3], [34, Proposition 2.1] and [22, Theorem 1.5] for other properties of the adjoint pair.

3. A Riemannian diffeological space

3.1. **Diffeological Riemannian metrics.** Let Euc denote the category consisting of domains and smooth maps. Let $\widehat{T}\colon \mathsf{Euc} \to \mathsf{Euc}$ be the functor defined by

$$\widehat{T}(U) := U \times \mathbb{R}^{\dim U}$$

and $\mathcal{Y}\colon \mathsf{Euc} \to \mathsf{Diff}$ the Yoneda functor. We recall the tangent functor $T\colon \mathsf{Diff} \to \mathsf{Diff}$ from the category of diffeological spaces to itself in the sense of Blohmann [2, 3], which is the left Kan extension $\mathbb{L}\mathcal{Y}\widehat{T} := \mathrm{Lan}_{\mathcal{Y}}\mathcal{Y}\widehat{T}$ of the functor $\mathcal{Y}\widehat{T}\colon \mathsf{Euc} \to \mathsf{Diff}$ along the Yoneda functor $\mathcal{Y}$; see also [28]. For a diffeological space $(X, \mathcal{D})$, it turns out that

$$T(X) = \operatorname*{colim}_{P \in \mathcal{D}} (U_P \times \mathbb{R}^{\dim U_P}).$$

Here, the diffeology $\mathcal{D}$ is regarded as a category whose objects are plots of X and whose morphisms are smooth maps $h\colon U_P \to U_Q$ with $Q \circ h = P$.

By using the functor $\widehat{T}_2\colon \mathsf{Euc} \to \mathsf{Diff}$ defined by $\widehat{T}_2(U) := \widehat{T}(U) \times_U \widehat{T}(U) = U \times \mathbb{R}^{\dim U} \times \mathbb{R}^{\dim U}$, we have the functor $T_2 := \mathbb{L}\mathcal{Y}\widehat{T}_2\colon \mathsf{Diff} \to \mathsf{Diff}$ via the left Kan extension along the Yoneda functor $\mathcal{Y}$. Observe that as a diffeological space,

$$T_2(X) = \operatorname*{colim}_{P \in \mathcal{D}} (U_P \times \mathbb{R}^{\dim U_P} \times \mathbb{R}^{\dim U_P}).$$

We may write $[x, v_1, v_2]_P$ for an element in $T_2(X)$ which has an element (x, v_1, v_2) in $U_P \times \mathbb{R}^{\dim U_P} \times \mathbb{R}^{\dim U_P}$ for some plot P as a representative.

Definition 3.1. A map $g\colon T_2(X) \to \mathbb{R}$ is a *weak Riemannian metric* on X if the composite $g \circ \pi_{\widehat{T}_2(U_P)}$ is a symmetric and positive covariant 2-tensor on U_P for each plot P of X, where $\pi_{\widehat{T}_2(U_P)}$ denotes the canonical map $\widehat{T}_2(U_P) \to T_2(X)$.

Remark 3.2. In the definition above, each map $g \circ \pi_{\widehat{T}_2(U_P)}$ is smooth in the usual sense and then g is smooth in the diffeological sense. In fact, $T_2(X)$ is endowed with the quotient diffeology; see Example 2.2 (9).

We introduce the Riemannian metric on a diffeological space described in [19, Page 3]. To this end, we recall the definition of covariant tensors of a diffeological space.

Let $(X, \mathcal{D}^X)$ be a diffeological space and $\mathcal{D}^X(U) \subset \mathcal{D}^X$ the set of plots whose domains are the common domain U. Let $T^k(U)$ denote the set of covariant k-tensors on a domain U. A *covariant k-tensor* ν in the sense in Iglesias-Zemmour is a natural transformation which fits in the diagram

$$\begin{tikzcd} \mathsf{Euc}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{D}^X"] \arrow[r, bend right, "T^k"'] \arrow[r, phantom, "\Downarrow \nu"] & \mathsf{Sets}\,. \end{tikzcd} \tag{3.1}$$

Here, we regard $\mathcal{D}^X$ as a functor in which $\mathcal{D}^X(f) \colon D^X(V) \to \mathcal{D}^X(U)$ is defined by $\mathcal{D}^X(f)(P_V) = P_V \circ f$ for a smooth map $f \colon U \to V$.

We may write $\nu(P)$ for $\nu_{U_P}(P)$. A covariant k-tensor ν is *(anti-) symmetric* if $\nu(P)$ is (anti-) symmetric in the usual sense for each P. A *(differential) k-form* on a diffeological space X is an anti-symmetric k-tensor on X. The set of all k-forms on X is denoted by $\Omega^k(X)$.

Definition 3.3 ([19, p. 3])**.** Let X be a diffeological space. A 2-tensor g on X is a *Riemannian metric* on X if it satisfies the following three conditions.

(1) (Symmetric) The tensor g is symmetric.
(2) (Positivity) For all path $\gamma \in \mathrm{Path}(X) := C^\infty(\mathbb{R}, X)$, we have $g(\gamma) \geq 0$; that is, $g(\gamma)_t(1,1) \geq 0$ for $t \in \mathrm{dom}(\gamma) = \mathbb{R}$, where $1 = (\frac{d}{dt})_t$ is the canonical base of $T_t(\mathrm{dom}(\gamma)) = \mathbb{R}$.
(3) (Definiteness) If $g(\gamma)_t(1,1) = 0$ for $\gamma \in \mathrm{Path}(X)$, then we have $\alpha(\gamma)_t(1) = 0$ for any $\alpha \in \Omega^1(X)$.

Remark 3.4. Another definition of the definiteness is proposed in [19, p. 3] with "pointed differential forms".

Proposition 3.5. *Let $(X, \mathcal{D})$ be a diffeological space. There exists a one-to-one correspondence between the set of weak Riemannian metrics $g \colon T_2(X) \to \mathbb{R}$ and that of symmetric and positive covariant 2-tensors in the sense of Definition 3.3.*

Proof. A natural transformation $\{g(P)\}_{P\in\mathcal{D}}$ consisting of covariant 2-tensors gives rise to the smooth map g (see Remark 3.2). Given a map $g \colon T_2(X) \to \mathbb{R}$ such that $g(P) := g \circ \pi_{\widehat{T}_2(U_P)}$ is a 2-tensor on U_P for each $P \in \mathcal{D}$, we obtain a natural transformation $\{g(P)\}_{P\in\mathcal{D}}$. Then g and $g(P)$ fit into the commutative diagram

$$\begin{tikzcd} U_P \times \mathbb{R}^{\dim U_P} \times \mathbb{R}^{\dim U_P} \arrow[d, "\pi_{\widehat{T}_2(U_P)}"'] \arrow[dr, "g(P)"] & \\ T_2(X) \arrow[r, "g"'] & \mathbb{R}. \end{tikzcd} \tag{3.2}$$

It is easy to see that g is symmetric if and only if $g(P)$ is symmetric for each P.

We consider the equivalence between the positivity of $g(P)$ and the condition (2) in Definition 3.3. Suppose that $g(P)$ is positive for each plot P. Since a path γ is a plot, it is immediate that the condition (2) holds.

Conversely, suppose that $g(P)_r(v,v)=0$, where $P\in\mathcal{D}$, $r\in U_P$ and $v\in T_rU_P$. We apply the same argument as in the proof of [20, 243 Exercise 2]. Let $\gamma_v(t)=r+tv$ and $\gamma^v=P\circ\gamma_v$. Then, we see that $g(P)_r(v,v)=g(\gamma^v)_0(1,1)$. Thus, the condition (2) implies that $g(P)$ is positive. As a consequence, the universality of the colimit yields the result. $\square$

In what follows, we call a diffeological space X endowed with a weak Riemannian metric g a *weak Riemannian diffeological space* and denote by (X,g).

Remark 3.6. In order to define a weak Riemannian metric on a diffeological space X, the elastic conditions on X as in [2, Section 1.2] and [3, Section 2.3.3] are not assumed. In particular, we do not require the condition that the tangent functor T is compatible with limits; see [2, Axiom (E1)].

We observe that $T_2(X)$ is not diffeomorphic to the pullback $T(X)\times_X T(X)$ along the natural map $T(X)\to X$ in general; see [3, Examples 2.3.12 and 2.3.13] and [28, Example 3.9]

Definition 3.7. A weak Riemannian metric $g\colon T_2(X)\to\mathbb{R}$ is *definite* if there exists a generating family $\mathcal{G}$ of the diffeology $\mathcal{D}$ of X such that the symmetric positive covariant 2-tensor $g(P)$, which corresponds to g via the bijection in Proposition 3.5, is definite in the usual sense for every $P\in\mathcal{G}$.

Remark 3.8. Let $(X,\mathcal{D},g)$ be a weak Riemannian diffeological space. In general, if the diffeology $\mathcal{D}$ is generated by the empty set, then the metric g is definite. In this case, the diffeology $\mathcal{D}$ is indeed discrete; see [18, 1.67].

Example 3.9. Let N be a Riemannian manifold endowed with a metric g_N. We see that the metric g_N is a definite weak Riemannian metric in the sense of Definition 3.7 with respect to both generating families $\mathcal{G}_{\text{atlas}}$ and $\mathcal{G}_{\text{imm}}$ in Example 2.2 (2).

Proposition 3.10. *The definiteness of a weak Riemannian metric in Definition 3.7 induces that in Definition 3.3.*

Proof. Let g be a definite weak Riemannina metric. To show that g is definite in the sense of Definition 3.3, suppose that $g(\gamma)_t(1,1)=0$ for $\gamma\in\mathrm{Path}(X)$. By assumption, there exist an open neighborhood V_t of t in $\mathbb{R}$, a plot $Q\in\mathcal{G}$ and a smooth map $f\colon V_t\to U_Q$ such that $\gamma|_{V_t}=Q\circ f$. Then, we see that

$$0=g(\gamma|_{V_t})_t(1,1)=g(Q\circ f)_t(1,1)=g(Q)_{f(t)}\left(\frac{df}{dt}(t),\frac{df}{dt}(t)\right).$$

The definiteness of $g(Q)$ yields that $\frac{df}{dt}(t)=0$. Thus, for any $\alpha\in\Omega^1(X)$, we have

$$\alpha(\gamma)_t(1)=\alpha(\gamma|_{V_t})_t(1)=\alpha(Q)_{f(t)}\left(\frac{df}{dt}(t)\right)=\alpha(Q)_{f(t)}(0)=0.$$

This completes the proof. $\square$

Proposition 3.11. *Let (X,g) be a weak Riemannian diffeological space whose metric g is definite with respect to a generating family $\mathcal{G}$ and $i\colon A\to X$ an induction. Then, the map g_A defined by the composite*

$$T_2(A)\xrightarrow{T_2(i)}T_2(X)\xrightarrow{g}\mathbb{R}$$

is a weak Riemannian metric on A which is definite with respect to the generating family

$$i^*\mathcal{G} := \{P \in \mathcal{D}^A \mid i \circ P \in \mathcal{G}\}. \tag{3.3}$$

Proof. We show that $\langle i^*\mathcal{G}\rangle$ coincides with $\mathcal{D}^A$. Since $\langle i^*\mathcal{G}\rangle \subset \mathcal{D}^A$ is obvious, it suffices to show that $\langle i^*\mathcal{G}\rangle \supset \mathcal{D}^A$. For any $P \in \mathcal{D}^A$, since i is an induction, we have $i \circ P \in \mathcal{D}^X$. Since $\mathcal{G}$ generates $\mathcal{D}^X$, it follows that for any $r \in U_P$, there exist an open neighborhood $V \subset U_P$ of r, an element $Q\colon U_Q \to X$ of $\mathcal{G}$, and a smooth map $f\colon V \to U_Q$ with $i \circ P|_V = Q \circ f$. Since i is an induction, we have $(A, \mathcal{D}^A) \cong (\mathrm{Im}(i), \mathcal{D}^{\mathrm{Im}(i)}_{\mathrm{sub}})$ ([18, 1.36]). Let $\bar{i}\colon \mathrm{Im}(i) \to A$ be the inverse to i. The fact that $i \circ \bar{i} \circ Q = Q \in \mathcal{G}$ enables us to deduce that $\bar{i} \circ Q \in i^*\mathcal{G}$. Since the image of $Q \circ f = i \circ P|_V$ lies in $\mathrm{Im}(i)$, we have

$$P|_V = \bar{i} \circ i \circ P|_V = \bar{i} \circ Q \circ f.$$

This implies $P \in \langle i^*\mathcal{G}\rangle$.

The map $T_2(i)$ is smooth. Therefore, we see that $g_A = g \circ T_2(i)$ is smooth. Thus, the symmetry and positivity of g imply that g_A is a weak Riemannian metric.

To show that g_A is definite with respect to $i^*\mathcal{G}$, suppose $g_A(P)_r(v, v) = 0$ for $P \in i^*\mathcal{G}$, $r \in U_P$ and $v \in T_rU_P$. By definition we have

$$0 = g_A(P)_r(v, v) = g(i \circ P)_r(v, v).$$

Since $i \circ P$ is in $\mathcal{G}$, it follows from the definiteness of g that $v = 0$. $\square$

Remark 3.12. We observe that the pullback (3.3) of $\mathcal{G}$ is smaller than that in the sense in [18, 1.75]. But (3.3) is enough to generate $\mathcal{D}^A$ when i is an induction.

3.2. **Riemannian (pseudo-) distance.** A weak Riemannian metric $g\colon T_2(X) \to \mathbb{R}$ on a diffeological space X gives a pseudodistance d of X by applying the usual procedure, as in the case of Riemannian manifolds [29]; that is, the pseudodistance $d\colon X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is defined by

$$d(x, y) = \inf_{\gamma \in \mathrm{Path}(X;x,y)} \ell(\gamma), \quad \text{where} \quad \ell(\gamma) = \int_0^1 (g(\gamma)_t(1,1))^{\frac{1}{2}}\,dt \tag{3.4}$$

and $d(x, y) = \infty$ if there is no smooth path connecting x and y. Here $\mathrm{Path}(X; x, y)$ denotes the subset of $\mathrm{Path}(X)$ consisting of γ with $\gamma(0) = x$ and $\gamma(1) = y$.

Theorem 3.13. *Let $d\colon X \times X \to \mathbb{R}_{\geq 0}$ be the pseudodistance on a diffeological space X defined by a weak Riemannian metric $g\colon T_2(X) \to \mathbb{R}$. Then the D-topology of X is finer than the topology $\mathcal{O}_d$ defined by d; that is, the D-topology contains $\mathcal{O}_d$. In particular, the function d on $D(X) \times D(X)$ is continuous.*

In order to prove the theorem, we need a lemma.

Lemma 3.14. *Let d_E be the Euclidian metric on U_P and $x \in U_P \cap V$ for some domain $V \subset \mathbb{R}^{\dim U_P}$. Then there exists a real number $k > 0$ and an open ball B of $U_P \cap V$ cenetered at x such that $\overline{B} \subset U_P \cap V$ and $d_P \leq k d_E$ on $B \times B$, where d_P is the distance defined by the symmetric, positive covariant 2-tensor $g(P)$. Moreover, if the 2-tensor $g(P)$ is definite, then, one has $\frac{1}{k} d_E \leq d_P \leq k d_E$ on $B \times B$ for some open ball B of U_P with $x \in B \subset \overline{B} \subset U_P \cap V$ and $k > 0$.*

Proof. This follows from the proof of [29, Theorem 4.1.8]. The upper bound is given by the continuity of $g(P)$. The definiteness of $g(P)$ yields the lower bound. $\square$

Proof of Theorem 3.13. We observe that $X \cong \mathrm{colim}_{P\in\mathcal{D}} U_P$ as a diffeological space. Since the D-topology functor $D\colon \mathsf{Diff} \to \mathsf{Top}$ is the left adjoint, it follows that D preserves colimits. Therefore, natural maps give rise to homeomorphisms

$$D(X) \cong D(\operatorname*{colim}_{P\in\mathcal{D}} U_P) \cong \operatorname*{colim}_{P\in\mathcal{D}} D(U_P) \cong \operatorname*{colim}_{P\in\mathcal{D}} U_P,$$

where the last space is endowed with the quotient topology; see also Example 2.4 for the fact that $D(U_P) = U_P$. Let $\pi\colon \coprod_{P\in\mathcal{D}} U_P \to X$ be the subduction. Then $\pi\colon \coprod_{P\in\mathcal{D}} U_P \to D(X)$ is the quotient map. We show that, for $V \in \mathcal{O}_d$, the set $\pi^{-1}(V) \cap U_P$ is open in U_P with the usual topology for each $P \in \mathcal{D}$.

Let x be in $\pi^{-1}(V)\cap U_P$. Since $V \in \mathcal{O}_d$, we have a δ-neighborhood $U_d(P(x);\delta)$ of $P(x)$ which is contained in V. Lemma 3.14 yields that $d_P \le kd_E$ on $B\times B$ for some open ball B of U_P centered at x with $\overline{B} \subset U_P$ and $k > 0$. Let ε be a positive number less than $\min\{\delta, \text{the radius of } B\}$. Then, we see that $U_{d_E}(x; \frac{\varepsilon}{k}) \subset U_{d_P}(x;\varepsilon) \subset B$. The following Claim 3.15 allows one to conclude that

$$\pi(U_{d_E}(x; \frac{\varepsilon}{k})) \subset \pi(U_{d_P}(x;\varepsilon)) \subset U_d(P(x);\varepsilon) \subset U_d(P(x);\delta) \subset V.$$

This completes the proof. $\square$

Claim 3.15. $\pi(U_{d_P}(x;\varepsilon)) \subset U_d(P(x);\varepsilon)$.

Proof. Let q be in $U_{d_P}(x;\varepsilon)$. By definition, we see that on U_P,

$$d_P(x,q) = \inf_{\gamma\in\mathrm{Path}(U_P;x,q)} \ell_P(\gamma), \quad \text{where} \quad \ell_P(\gamma) = \int_0^1 (g(P)_{\gamma(t)}(\gamma',\gamma'))^{\frac{1}{2}}\, dt.$$

Since $d_P(x,q) < \varepsilon$, it follows that there exists a path $\gamma \in \mathrm{Path}(U_P;x,q)$ such that $d_P(x,q) \le \ell_P(\gamma) =: \widetilde{\delta} < \varepsilon$. Define a path $\widetilde{\gamma}$ by $\widetilde{\gamma} := P\circ\gamma = \pi\circ\gamma$. Then, we see that $g(\widetilde{\gamma})_t(1,1) = g(P\circ\gamma)_t(1,1) = \gamma^*(g(P))_t(1,1) = g(P)_{\gamma(t)}(\gamma',\gamma')$. This implies that $\ell(\widetilde{\gamma}) = \widetilde{\delta}$ and then $d(P(x),P(q)) \le \ell(\widetilde{\gamma}) = \widetilde{\delta} < \varepsilon$; see (3.4). It turns out that $\pi(q)$ is in $U_d(P(x);\varepsilon)$. $\square$

Remark 3.16. In the proof of Theorem 3.13, we do *not* need the definiteness of the weak Riemannian metric g. We do not clarify when the topology $\mathcal{O}_d$ induced by a weak Riemannian metric g on a diffeological space X contains the D-topology on X. Suppose that the metric g is definite and a generating family $\mathcal{G}$ gives the definiteness. Let $\mathcal{O}_{d_P}$ be the topology of U_P defined by the metric $g(P)$. Then, the latter half of Lemma 3.14 yields that $P^{-1}(O)\cap U_P \in \mathcal{O}_{d_P}$ for every D-open subset $O \subset X$ and every plot $P \in \mathcal{G}$.

Definition 3.17. Let X be a diffeological space. A generating family $\mathcal{G}$ of a diffeology of X *separates points* if for distinct points p and q in X, there exist a plot $P \in \mathcal{G}$ and an open ball B with center x such that $P(x) = p$, $B \subset \overline{B} \subset U_P$ and each smooth path γ from p to q admits a local lift $\widetilde{\gamma}$ in U_P with $\widetilde{\gamma}(0) = x$ and $\mathrm{Im}\,\widetilde{\gamma} \not\subset B$.

Remark 3.18. Both generating families $\mathcal{G}_{\mathrm{atlas}}$ and $\mathcal{G}_{\mathrm{imm}}$ of a manifold M separate points.

With the technical condition above, we have the following result.

Theorem 3.19. *The pseudodistance d defined by a weak Riemannian metric g is indeed a distance provided g is definite with respect to a generating family $\mathcal{G}$ which separates points.*

Proof. The proof is verbatim the same as that of [29, Theorem 4.1.6] by replacing the use of a chart with that of a plot. Suppose that $d(p,q)=0$ for distinct points p and q. We choose a plot $P\colon U_P \to X$ in $\mathcal{G}$ and an open ball B with center x which satisfy the condition in Definition 3.17. The proof of [29, Theorem 4.1.6] allows us to deduce that there exists a positive number k such that

$$\frac{1}{k}||v|| \leq (g(P)_a(v,v))^{\frac{1}{2}} \leq k||v|| \tag{3.5}$$

for $(a,v)\in \overline{B}\times \mathbb{R}^{\dim U_P}$. Let γ be a smooth path from p to q and $\widetilde{\gamma}$ a local lift in U_P stating from a point x with $P(x)=p$. Then there exists the smallest number $s\in (0,1]$ satisfying $u := \gamma(s) \in \gamma([0,1])\cap \partial B$. Let $\gamma_1 := \gamma|_{[0,s]}$. Observe that $s>0$ and $\int_0^s (g(P)_{\widetilde{\gamma}_1(t)}(\widetilde{\gamma}_1', \widetilde{\gamma}_1'))^{\frac{1}{2}}dt \neq 0$. In fact, if the integration is equal to zero, then the definiteness condition allows us to conclude that $\widetilde{\gamma}_1$ is constant. However, $\widetilde{\gamma}_1(0) = x \neq u = \widetilde{\gamma}_1(s)$. With the radius r of B, we have

$$\ell(\gamma) = \int_0^1 (g(\gamma)_t(1,1))^{\frac{1}{2}}dt \geq \int_0^s (g(P)_{\widetilde{\gamma}_1(t)}(\widetilde{\gamma}_1', \widetilde{\gamma}_1'))^{\frac{1}{2}}dt \geq \frac{1}{k}r.$$

The last inequality follows from (3.5) and a change of variables in the integration. Then, we see that $d(p,q) = \inf \ell(\gamma) \geq \frac{1}{k}r$, which is a contradiction. □

Remark 3.20. If $\pi\colon \coprod_{P\in\mathcal{D}} U_P \to X$ is a *local subduction* (Example 2.2 (7), see also [18, 2.16]), then we may replace the condition "a generating family $\mathcal{G}$ separates points" with a simpler one; "for any distinct $p,q\in X$, there exists a plot $P\in\mathcal{G}$ and an open ball $B\subset U_P$ with center x such that $P(x)=p$, $B\subset \overline{B}\subset U_P$ and $q\notin P(B)$." Indeed, for any distinct p and q, the following two conditions are equivalent;

(1) for any $P\in\mathcal{G}$ with $p\in P(U_P)$ and any open ball B centered at x ($P(x)=p$), we have $q\in P(B)$

(2) $d(p,q)=0$

It is easy to see that (1) implies (2). Suppose (1) does not hold and let $P\in\mathcal{G}$ be a plot that does not satisfy (1). For any smooth path γ from p to q, we can find a local lift $\widetilde{\gamma}$ of γ on U_P defined near 0. Then, we can prove that $\ell(\gamma) > r/k$ (r is the radius of B, $q\notin P(B)$) as in the proof of Theorem 3.19.

4. A diffeological adjunction space

We introduce an appropriate setting to construct a weak Riemannian diffeological space by attaching two such spaces. As seen in Example 1.1, the construction of the metric is applicable to the spaces of smooth maps.

We consider the diffeological adjunction space obtained by two inductions i and j into weak Riemannian diffeological spaces

$$X \xleftarrow{\ i\ } A \xrightarrow{\ j\ } Y. \tag{4.1}$$

We assume further that

(I) weak Riemannian metrics $g_X\colon T_2(X)\to\mathbb{R}$ and $g_Y\colon T_2(Y)\to\mathbb{R}$ on X and Y satisfy the condition that $g_X\circ T_2(i) = g_Y\circ T_2(j)\colon T_2(A)\to\mathbb{R}$.

Then, we will construct a weak Riemannian metric $g\colon T_2(X\coprod_A Y)\to\mathbb{R}$, where $Z := X\coprod_A Y$ is endowed with the quotient diffeology $\mathcal{D}^Z$.

Let $f\colon P \to Q$ be a morphism in $\mathcal{D}^Z$. Then, for $s \in U_P$, we have a commutative diagram

$$\begin{tikzcd}
W' \arrow[rr, "f|_{W'}"] \arrow[d, hook] \arrow[ddd, bend right=50, "P_{XY}"'] & & W \arrow[d, hook] \arrow[ddd, bend left=50, "Q_{XY}"] \\
U_P \arrow[rr, "f"] \arrow[dr, "P"'] & & U_Q \arrow[dl, "Q"] \\
& Z & \\
& X \coprod Y \arrow[u, "\pi"'] &
\end{tikzcd}$$

here open neighborhoods W' and W of s and $f(s) =: r$ respectively are taken so that P_{XY} and Q_{XY} are plots on X *or* Y. In fact, plots P and Q are in the quotient diffeology $\mathcal{D}^Z$ and then the local lifting condition of the plots gives the diagram. We define $g_{U_P}\colon T_2(U_P) \to \mathbb{R}$ by $g_{U_P}((s, v_1, v_2)) = g(P_{XY})((s, v_1, v_2))$. We observe that $g_{U_P}((s, v_1, v_2)) = g_X([s, v_1, v_2])$; see the diagram (3.2).

Suppose that P_{XY} and Q_{XY} are plots P_X on X and Q_Y on Y, respectively. Then, we see that $\operatorname{Im} P_X \subset i(A)$, $\operatorname{Im}(Q_Y \circ f) \subset j(A)$ and

$$P_A := i^{-1}P_X = j^{-1} \circ Q_Y \circ f =: \rho. \tag{4.2}$$

Observe that an induction gives a diffeomorphism between the domain and its image; see [18, 1.36]. We write $P\colon T_2(U) \to T_2(Y)$ for the canonical inclusion

$$\{P\} \times T_2(U_P) \to T_2(Y) = \operatorname*{colim}_{P \in \mathcal{D}_Y} (U_P \times \mathbb{R}^{\dim U_P} \times \mathbb{R}^{\dim U_P}) \tag{4.3}$$

for a weak Riemannian diffeological space Y.

Lemma 4.1. *With the notation above, suppose that $f_*\colon T_2(U_P) \to T_2(U_Q)$ assigns (r, u_1, u_2) to (s, v_1, v_2). Then, one has $g_{U_P}((s, v_1, v_2)) = g_{U_Q}((r, u_1, u_2))$.*

Proof. By the equalities in (4.2), we have the commutative diagram of diffeological spaces and smooth maps

$$\begin{tikzcd}
T_2(W') \arrow[r, "(\rho)_* = (P_A)_*"'] \arrow[rrr, bend left=15, "(f|_{W'})_*"] \arrow[dr, bend right, "(P_{U_P})_*"'] & T_2(A) \arrow[r, "j_*"] \arrow[d, "i_*"'] & T_2(Y) \arrow[d, "g_Y"] & T_2(W) \arrow[l, "(Q_{U_Q})_*"] \\
& T_2(X) \arrow[r, "g_X"'] & \mathbb{R}. &
\end{tikzcd} \tag{4.4}$$

Thus, it follows that

$$\begin{aligned}
&g_{U_P}((s, v_1, v_2)) = g_X([s, v_1, v_2]) = g_X(i_*((P_A)_*([s, v_1, v_2]))) \\
&= g_X(i_*(\rho_*([s, v_1, v_2]))) = g_Y(j_*(\rho_*([s, v_1, v_2]))) \\
&= g_Y((Q_{U_Q})_*((f|_{W'})_*([s, u_1, u_2]))) = g_{U_Q}((r, u_1, u_2)).
\end{aligned}$$

We have the result. $\square$

Theorem 4.2. *Under the assumption* (I) *for inductions in the diagram (4.1), the map $g\colon T_2(Z) \to \mathbb{R}$ defined by $g((s, v_1, v_2)) := g_{U_P}((s, v_1, v_2)) = g_X((s, v_1, v_2))$ for $(s, v_1, v_2) \in T_2(U_{P_X})$ is a well-defined weak Riemannian metric on the diffeological adjunction space $Z = X \coprod_A Y$. Moreover, if g_X and g_Y are definite, then so is g.*

Proof. The proof of Lemma 4.1 is valid for the case where P_{XY} and Q_{XY} are both plots on X or Y. Then, the definition g_{U_P} does not depend on the choice of a neighborhood of s if f is an inclusion. Therefore, the map g is well defined. We show the smoothness of g. A plot $R\colon U_R \to T_2(Z)$ locally factors through $\{P\} \times T_2(U_P)$. Moreover, R locally factor through an open subset $T_2(W')$ of $\{P\} \times T_2(U_P)$ which is used when defining the metric g. We see that g is indeed g_X or g_Y on the open subset $T_2(W')$. This implies that g is smooth.

The latter half of the assertion follows from the definition of g. In fact, let $\mathcal{G}_X$ and $\mathcal{G}_Y$ be generating families which give the definiteness of g_X and g_Y, respectively. Then, the definiteness of g is given with the set $\mathcal{G}_X \coprod \mathcal{G}_Y$. □

Remark 4.3. For the inductions in (4.1), the universality of the pushout gives the commutative diagram

$$\begin{tikzcd}
T_2(A) \arrow[r, "j_*"] \arrow[d, "i_*"'] & T_2(Y) \arrow[d, "\widehat{i}"] \arrow[dr, bend left, "\bar{i}_*"] & \\
T_2(X) \arrow[r, "\widehat{j}"] \arrow[rr, bend right, "\bar{j}_*"'] & T_2(X) \coprod_{T_2(A)} T_2(Y) \arrow[r, dotted, "\alpha"] & T_2(X \coprod_A Y)
\end{tikzcd}$$

The proof of Theorem 4.2 enables us to obtain the inverse of α. In fact, by replacing g_X and g_Y in the diagram (4.4) with $\widehat{j}$ and $\widehat{i}$, respectively, we have the result.

A smooth embedding between manifolds is an induction. Thus, Theorem 4.2 provides a crucial example. An adjunction diffeological space of M and N does not necessarily have a generating family which separates points even if M and N do.

Example 4.4. We consider the pushout diffeological space $\mathsf{Y} := \mathbb{R}_1 \coprod_{(1,\infty)} \mathbb{R}_2$ of the induction $i\colon (1,\infty) \to \mathbb{R}$ along itself, where $\mathbb{R}_i$ denotes the copy of $\mathbb{R}$. By virtue of Theorem 4.2, we see that the usual metric on $\mathbb{R}$ gives rise to a weak Riemannian metric on Y. A point $x \in \mathbb{R}_i$ is denoted by x_i for $i = 1, 2$. We observe that $D(\mathsf{Y})$ is non–Hausdorff. In fact, the distinct points $[1_1]$ and $[1_2]$ is not separated by any neighborhoods of the points.

For each $\varepsilon > 0$, we have a smooth path γ from $[1_1]$ to $[1_2]$ with $\ell(\gamma) < \varepsilon$. This yields that $d([1_1], [1_2]) = 0$. This example implies that the inverse of the inclusion relation of the topologies in Theorem 3.13 does not hold in general even if the metric g is definite.

We will see that the diffeological space Y does not satisfy the condition in Definition 3.17. Let $\pi\colon \mathbb{R}_1 \coprod \mathbb{R}_2 \to \mathsf{Y}$ be the subduction. We write $(a, b)_i$ for the open interval (a, b) in $\mathbb{R}_i$. Then, the set $D(\pi)((1-\varepsilon, 1+\varepsilon)_1)$ is open in $D(\mathsf{Y})$ for a positive number $\varepsilon > 0$. However, the set $\pi((1-\varepsilon, 1+\varepsilon)_1)$ is not in $\mathcal{O}_d$. In fact, $\pi((1-\varepsilon, 1+\varepsilon)_1)$ does not contain the δ-open ball $U_d([1_1], \delta)$ with center $[1_1]$ for every δ with $0 < \delta < \varepsilon$. The element $[(1-\frac{\delta}{2})_2]$ is in $U_d([1_1], \delta)$ but not in $\pi((1-\varepsilon, 1+\varepsilon)_1)$. This follows from the fact that

$$d([(1-\frac{\delta}{2})_2], [1_1]) \leq d([(1-\frac{\delta}{2})_2], [1_2]) + d([1]_2, [1]_1) = d([(1-\frac{\delta}{2})_2], [1_2]) \leq \frac{\delta}{2}.$$

We refer the reader to [15, Section 6] for non-Hausdorff Riemannian manifolds; see also [31] for many examples of non-Hausdorff manifolds. Moreover, Theorem 4.2 allows us to conclude that the adjunction space $M := \mathbb{R} \coprod_{(1,\infty)} \mathbb{R}^2$ is a definite weak Riemannian diffeological space which is not a manifold. Observe that $D(M)$ is non-Hausdorff.

Example 4.5. We consider the pushout diffeological space $+ := \mathbb{R}_1 \coprod_{\{0\}} \mathbb{R}_2$. We see that the diffeology of $+$ has a generating family which separates points; see Definition 3.17. Moreover, the usual metrics on $\mathbb{R} = \mathbb{R}_1$ and $\mathbb{R} = \mathbb{R}_2$ satisfy the assumption (I).

5. A diffeological mapping space

5.1. **A subdiffeology of the functional diffeology.** Let X and Y be diffeological spaces and $\mathcal{G}$ a generating family of $\mathcal{D}^Y$. Then, the set $C^\infty(X,Y)$ of smooth maps is endowed with the functional diffeology $\mathcal{D}_{\text{func}}$; see Example 2.2 (6). We introduce a subdiffeology of $\mathcal{D}_{\text{func}}$. To this end, we consider the following condition (E) for a plot $P \in \mathcal{D}_{\text{func}}$.

(E) For $r \in U_P$ and $m \in X$, there exists an open neighborhood $W_{r,m}$ of r in U_P such that the composite $W_{r,m} \hookrightarrow U_P \xrightarrow{P} C^\infty(X,Y) \xrightarrow{\mathrm{ev}_m} Y$ is in $\mathcal{G}$, where ev_m denotes the evaluation map at m.

Let $\mathcal{F}_{\mathcal{G}}^{XY}$ be the subset of plots in $\mathcal{D}_{\text{func}}$ each of which satisfies the condition (E). We may write $\mathcal{F}_{\mathcal{G}}$ for $\mathcal{F}_{\mathcal{G}}^{XY}$ when the domain X and the codomain Y are clear in the context. Let $\mathcal{D}' := \langle \mathcal{F}_{\mathcal{G}}^{XY} \rangle$ be the diffeology generated by $\mathcal{F}_{\mathcal{G}}^{XY}$. It is immediate that $\mathcal{D}'$ is contained in $\mathcal{D}_{\text{func}}$ and that the identity map $(C^\infty(X,Y),\mathcal{D}') \to (C^\infty(X,Y),\mathcal{D}_{\text{func}})$ is smooth.

Example 5.1. The diffeology $\mathcal{D}'$ does not coincide with $\mathcal{D}_{\text{func}}$ in general. In fact, we consider the case where $Y = U$ is an open subset of $\mathbb{R}^n$ with $n > 1$ and $\mathcal{G} := \{id \colon U \to U\}$. Then, the standard diffeology of U is generated by $\mathcal{G}$.

Let Q be a plot of the mapping space $C^\infty(U,U)$ which satisfies the condition (E). Then, for each $r \in U_Q$, there exists an open neighborhood W_r of r in U_Q such that $Q|_{W_r} = P \circ h$ for some smooth map h and $P \in \mathcal{F}_{\mathcal{G}}$. Thus, we see that the image of $Q : h^{-1}(U) \to C^\infty(U,U)$ consists of constant maps. Therefore, the diffeology $\mathcal{D}'$ is strictly finer than the functional diffeology.

The following lemma describes fundamental properties of the diffeology $\mathcal{D}'$.

Lemma 5.2. *Let $f \colon X \to Y$ be a smooh map.*

(1) Let Z be a diffeological space with a generating family $\mathcal{G}_Z$ of the diffeology. Then, the map

$$f^* \colon (C^\infty(Y,Z), \langle \mathcal{F}_{\mathcal{G}_Z}^{YZ} \rangle) \to (C^\infty(X,Z), \langle \mathcal{F}_{\mathcal{G}_Z}^{XZ} \rangle)$$

defined by $f^*(\varphi) = \varphi \circ f$ is smooth.

(2) Let $\mathcal{G}_Y$ be a generating family of the diffeology of Y and $\mathcal{G}_X$ be the pullback $f^*\mathcal{G}_Y$ of $\mathcal{G}_Y$ by f; see (3.3). Then, for any Z, the map

$$f_* \colon (C^\infty(Z,(X,\langle \mathcal{G}_X \rangle)), \langle \mathcal{F}_{\mathcal{G}_X}^{ZX} \rangle) \to (C^\infty(Z,Y), \langle \mathcal{F}_{\mathcal{G}_Y}^{ZY} \rangle)$$

defined by $f_*(\psi) = f \circ \psi$ is smooth.

Proof. Since both $f \colon X \to Y$ and $f \colon (X,\langle \mathcal{G}_X \rangle) \to Y$ are smooth, it follows that the maps f^* and f_* are smooth with respect to the functional diffeology. Moreover, we see that $\mathrm{ev}_x \circ (f^* \circ Q) = \mathrm{ev}_{f(x)} \circ Q$ for every $Q \colon U_Q \to C^\infty(Y,Z)$ and $x \in X$. Thus, if $Q \in \mathcal{F}_{\mathcal{G}_Z}^{YZ}$, then the map $\mathrm{ev}_x \circ (f^* \circ Q)$ restricted to an appropriate domain is in

$\mathcal{G}_Z$. This implies that $f^* \circ Q \in \mathcal{F}^{XZ}_{\mathcal{G}_Z}$, proving (1). The fact that $\mathrm{ev}_z \circ f_* = f \circ \mathrm{ev}_z$ and the definition of $\mathcal{G}_X$ yield that $f_* \circ P$ is in $\mathcal{F}^{ZY}_{\mathcal{G}_Y}$ for each $P \in \mathcal{F}^{ZX}_{\mathcal{G}_X}$. We have the results. $\square$

Lemma 5.3. *Let $\mathcal{G}$ be a generating family of the diffeology of a diffeological space Y. The functional diffeology $\mathcal{D}_{\mathrm{func}}$ of $C^\infty(*, Y)$ coincides with $\langle \mathcal{F}^{*Y}_{\mathcal{G}} \rangle$.*

Proof. It suffices to show that $\mathcal{D}_{\mathrm{func}} \subset \langle \mathcal{F}^{*Y}_{\mathcal{G}} \rangle$. For a plot $P \in \mathcal{D}_{\mathrm{func}}$, the adjoint $\mathrm{ad}(P)\colon U_P \times * \to Y$ is regarded as a plot of Y. Thus, for each $r \in U_P$, there exist a neighborhood W_r of r in U_P, $Q \in \mathcal{G}$ and a smooth map $h\colon W_r \to U_Q$ such that $\mathrm{ad}(P)|_{W_r \times *} = \widetilde{Q} \circ (h \times *)$, where $\widetilde{Q}$ is defined by $\widetilde{Q} := Q \circ \mathrm{pr} : U_Q \times * \to Y$ with the projection pr to the first factor. We see that $P|_{W_r} = \mathrm{ad}(\widetilde{Q}) \circ h$ and $\mathrm{ev}_* \circ \mathrm{ad}(\widetilde{Q}) = Q$. This implies that P is in $\langle \mathcal{F}^{*Y}_{\mathcal{G}} \rangle$. $\square$

Let $s\colon Y \to C^\infty(X, Y)$ be the section of ev_x at any x; that is, $s(y)(x) = y$ for $y \in Y$ and $x \in X$.

Lemma 5.4. *The section $s\colon (Y, \langle \mathcal{G} \rangle) \to (C^\infty(X, Y), \mathcal{D}')$ is an induction.*

Proof. The map $q\colon Y \to (C^\infty(*, Y), \mathcal{D}_{\mathrm{func}})$ defined by $q(y)(*) = y$ is smooth. By Lemma 5.2, we see that the trivial map $u\colon X \to *$ induces the smooth map $u^*\colon (C^\infty(*, Y), \langle \mathcal{F}^{*Y}_{\mathcal{G}} \rangle) \to (C^\infty(X, Y), \langle \mathcal{F}_{\mathcal{G}} \rangle)$. Since $s = u^* \circ q$, it follows from Lemma 5.3 that the section s is smooth.

For a parametrization $Q\colon U \to Y$, suppose that $s \circ Q \in \mathcal{D}' = \langle \mathcal{F}_{\mathcal{G}} \rangle$. Since ev_x is smooth for each x, it follows that $Q = \mathrm{ev}_x \circ s \circ Q \in \langle \mathcal{G} \rangle$. We see that s is an induction. $\square$

5.2. **Weak Riemannian metrics on mapping spaces.** In this section, the Riemannian metric on the space of smooth maps due to Iglesias-Zemmour [19] is interpreted in our framework.

Let M be a closed orientable manifold and N a definite weak Riemannian diffeological space with a generating family $\mathcal{G}$ which gives the definiteness; see Definition 3.7 and Example 3.9. Let $\{(V_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ be an atlas of M. Then, for a plot $P \in \mathcal{D}_{\mathrm{func}}$, $r \in U_P$ and tangent vectors $v, w \in T_rU_P$, we define a smooth map $\Theta_{g_N}(P)(v, w)(r, \text{–})\colon M \to \mathbb{R}$ by

$$\Theta_{g_N}(P)(v, w)(r, m) = g_N(\mathrm{ad}(P) \circ (1 \times \varphi_\lambda^{-1}))_{(r, \varphi_\lambda(m))}(\underline{v}, \underline{w})$$

for $m \in V_\lambda$, where $\underline{u} = (u, 0) \in T_rU_P \times T_{\varphi_\lambda(m)}\varphi_\lambda(V_\lambda) = T_rU_P \times \mathbb{R}^{\dim M}$. For the functional diffeology $\mathcal{D}_{\mathrm{func}}$, we define a metric g on $C^\infty(M, N)$ by

$$g(P)_r(v, w) = \int_M \Theta_{g_N}(P)(v, w)(r, m)\mathrm{vol}_M \tag{5.1}$$

for $r \in U_P$, where vol_M is a fixed unit $(\dim M)$-form on M.

The following Lemma 5.5 shows the well-definedness of Θ_{g_N}.

Lemma 5.5. $\Theta_{g_N}(P)(v, w)(r, m) = g_N(\mathrm{ev}_m \circ P)_r(v, w)$.

Proof. Define $j_{m,\lambda}\colon U_P \to U_P \times \varphi_\lambda(V_\lambda)$ by $j_{m,\lambda}(x) := (x, \varphi_\lambda(m))$. Then $\underline{v} = (j_{m,\lambda})_*(v)$ and

$$\begin{aligned}\Theta_{g_N}(P)(v, w)(r, m) &= g_N(\mathrm{ad}(P) \circ (\mathrm{id}_{U_P} \times \varphi_\lambda^{-1}))_{(r, \varphi(m))}(\underline{v}, \underline{w}) \\ &= g_N(\mathrm{ad}(P) \circ (\mathrm{id}_{U_P} \times \varphi_\lambda^{-1}) \circ j_{m,\lambda})_r(v, w).\end{aligned}$$

It is easy to see that $\mathrm{ad}(P) \circ (\mathrm{id}_{U_P} \times \varphi_\lambda^{-1}) \circ j_{m,\lambda} = \mathrm{ev}_m \circ P$. $\square$

Since g_N is a 2-tensor, it follows that $g(P)$ is an ordinary symmetric covariant 2-tensor field on U_P for a plot $P\colon U_P \to C^\infty(M,N)$.

To see the smoothness of g, let V and W be vector fields on U_P. Since g_N is a weak Riemannian metric on N, $g_N(\mathrm{ad}(P)\circ(\mathrm{id}_{U_P}\times\varphi_\lambda^{-1}))$ is a 2-tensor field on $U_P \times U_\lambda$, where $(V_\lambda, \varphi_\lambda)$ is a coordinate neighborhood of $m \in M$, and the function on $U_P \times V_\lambda$ defined by

$$(r, \varphi^{-1}(m)) \mapsto g_N(\mathrm{ev}_m \circ P)_r(V(r), W(r))$$

is smooth. Thus the map $r \mapsto g(P)_r(V(r), W(r))$ on U_P is smooth and hence $g(P)$ is a smooth 2-tensor.

We show that g is compatible with coordinate changes. Let $F\colon V \to U_P$ be a C^∞-map. Then, it follows that

$$g_N(\mathrm{ev}_m \circ (P\circ F))_r(v,w) = g_N(\mathrm{ev}_m \circ P)_{F(r)}(dF_r(v), dF_v(w))$$

Thus, we have

$$\Theta_{g_N}(P\circ F)(v,w)(r,m) = \Theta_{g_N}(P)(dF_r(v), dF_r(w))(F(r), m).$$

Therefore, it follows that

$$g(P\circ F)_r(v,w) = g(P)_{F(r)}(dF_r(v), dF_r(w)) = F^*g(P)_r(v,w).$$

This implies that g is a diffeological covariant 2-tensor field on $C^\infty(M,N)$. Symmetry and positivity follow from the fact that g_N is a metric. Moreover, we have

Theorem 5.6. *Let g be the weak Riemannian metric on $C^\infty(M,N)$ defined as above, where the diffeology is restricted to $\mathcal{D}' := \langle \mathcal{F}_\mathcal{G}\rangle$. Then the metric g is definite with respect to the generating family $\mathcal{F}_\mathcal{G}$ in the sense of Definition 3.7.*

Proof. The inclusion of diffeologies gives rise to a natural transformation $\iota\colon \mathcal{D}' \Longrightarrow \mathcal{D}_{\mathrm{func}}$. By combining ι with the metric g and applying Proposition 3.5, we have a weak Riemannian metric on $C^\infty(M,N)$; see the diagram (3.1).

We show the definiteness of the metric. For a plot P in the generating family $\mathcal{F}_\mathcal{G}$ of $\mathcal{D}'$, assume that $g(P)_r(u,u) = 0$, where $u \in T_rU_p$. Since g_N is positive, we see that $g_N(\mathrm{ev}_m \circ P)_r(u,u) = 0$ for each $m \in M$. Since P satisfies the condition (E), by restricting U_P to $W_{r,m}$, the map $\mathrm{ev}_m \circ P$ is in $\mathcal{G}$. The definiteness of g_N enables us to conclude that $u = 0$. This completes the proof. $\square$

Proposition 5.7. *For the weak Riemannian metric g on $(C^\infty(M,N), \mathcal{D}')$ in* (5.1)*, the pseudodistance d defined by g is indeed a distance provided the pseudodistance defined by g_N is distance on N.*

Proof. Since M is compact, the metric g can be described using a finite set of coordinate neighborhoods $\{(V_i, \varphi_i, (x_1, \dots, x_n))\}_{i=1}^k$ and a partition of unity λ_i subordinate to each V_i as

$$g(P)_r(v,w) = \sum_{i=1}^k \int_{\varphi_i(V_i)} \lambda_i(m) g_N(\mathrm{ev}_m \circ P)_r(v,w)\xi(m)dx_1\cdots dx_n,$$

where $\mathrm{vol}_M = \xi dx_1 \cdots dx_n$ with some smooth function $\xi\colon M \to \mathbb{R}^{>0}$. Now, assume that $d_{C^\infty(M,X)}(f_0, f_1) = \inf_{\gamma\in \mathrm{Path}(C^\infty(M,N); f_0, f_1)} \ell(\gamma) = 0$ for $f_0, f_1 \in C^\infty(M,N)$.

By the definition of the Riemannian distance and the property of the infimum, there exists a sequence of paths $\{\gamma_n\}_{n=1,2,\dots} \subset \mathrm{Path}(C^\infty(M,N); f_0, f_1)$ such that

$$\int_0^1 (g(\gamma_n)_s(1,1))^{\frac{1}{2}}\, ds \leq \frac{1}{n}.$$

Then, we have

$$\int_0^1 \left(\sum_{i=1}^k \int_{\varphi_i(U_i)} \lambda_i(p) g_N(\mathrm{ev}_m \circ \gamma_n)_s(1,1)\xi(m) dx_1 \cdots dx_n \right)^{\frac{1}{2}} ds \leq \frac{1}{n}.$$

Since g_N is positive, this implies that $\lambda_i(m)\xi(m)g_N(\mathrm{ev}_m \circ \gamma_n)_s(1,1) \leq \frac{1}{n^2}$ for any $m \in M$ and s. Therefore, we see that

$$g_N(\mathrm{ev}_m \circ \gamma_n)_s(1,1) \leq \frac{1}{\beta} \cdot \frac{1}{n^2},$$

where $\beta \coloneqq \max_{m\in M, 1\leq i\leq k}\{\lambda_i(m)\xi(m)\}$. For any fixed element $m \in M$, the smooth map $\mathrm{ev}_m \circ \gamma_n$ is regarded as a path on N from $f(m)$ to $g(m)$. Then, we see that

$$\ell(\mathrm{ev}_m \circ \gamma_n) = \int_0^1 (g_N(\mathrm{ev}_m \circ \gamma_n)_s(1,1))^{\frac{1}{2}} ds \leq (\frac{1}{\beta})^{\frac{1}{2}} \cdot \frac{1}{n}.$$

This yields that $d_N(f_0(m), f_1(m)) = 0$ for any $m \in M$. Since d_N is a distance, it follows that $f_0 = f_1$. We have the result. $\square$

We recall the section $s\colon (N, \langle \mathcal{G} \rangle) \to (C^\infty(M,N), \mathcal{D}')$ in Lemma 5.4.

Proposition 5.8. *One has the commutative diagram*

$$\begin{tikzcd}
T_2(C^\infty(M,N), \mathcal{D}')) \arrow[r, "g"] & \mathbb{R} \\
T_2((N, \langle \mathcal{G} \rangle)) \arrow[u, "s_*"] \arrow[ur, "(\int_M \mathrm{vol}_M)\times g_N"'] &
\end{tikzcd}$$

Proof. This result follows from Lemma 5.5 and the definition of the metric g; see (5.1). $\square$

Example 5.9. Let $(N, \langle \mathcal{G} \rangle)$ be a weak Riemannian diffeological space whose metric is definite with respect to $\mathcal{G}$. For example, we can choose the adjunction space in Theorem 4.2 as such a diffeological space. Let M and M' be closed orientable manifolds with $\int_M \mathrm{vol}_M = \int_{M'} \mathrm{vol}_{M'}$. Then, Lemma 5.4, Proposition 5.8 and Theorem 4.2 allows us to obtain a definite weak Riemannian diffeological space of the form $C^\infty(M,N) \coprod_N C^\infty(M', N)$.

Remark 5.10. Consider the case that N is a Riemannian manifold. Suppose $P\colon U_P \to C^\infty(M,N)$ is such that the map $\mathrm{ad}(P)(-,m) = \mathrm{ev}_m \circ P\colon U_P \hookrightarrow N$ is an embedding for any $m \in M$. For example P is such a plot if $U_P \cong \mathrm{Int}\, D^{\dim N}$ and $\mathrm{ad}(P)\colon U_P \times M \to N$ is a framed immersion. Then clearly P itself satisfies the condition (E), and $\mathcal{F}_{\mathcal{G}}$ is not empty if $\mathcal{G}$ contains such a plot.

5.3. **An example: free loop spaces.** For a weak Riemannian diffeological space N, let the free loop space $LN := C^\infty(S^1, N)$ be endowed with the functional diffeologoy.

We regard $S^1 = [0, 2\pi]/(0 \sim 2\pi)$ and define $S^1 \vee S^1$ as the pushout of the diagram

$$\begin{tikzcd} * \arrow[r, "* \mapsto 1"] \arrow[d, "* \mapsto 1"'] & S^1 \arrow[d, "i"] \\ S^1 \arrow[r, "j"'] & S^1 \vee S^1. \end{tikzcd} \tag{5.2}$$

Let $S^1_\star \subset S^1 \vee S^1$ ($\star = \text{left}, \text{right}$) be the image of the first / second copy of S^1 via the subduction $\pi\colon S^1 \coprod S^1 \to S^1 \vee S^1$. The inclusions $i_\star\colon S^1_\star \to S^1 \coprod S^1$ give the smooth map $\eta\colon C^\infty(S^1 \coprod S^1, N) \to LN \times LN$ defined by $\eta(f) = (f \circ i_{\text{left}}, f \circ i_{\text{right}})$. It is readily seen that η is a diffeomorphism.

Moreover, the subduction π gives rise to an induction $\pi^*\colon C^\infty(S^1 \vee S^1, N) \to C^\infty(S^1 \coprod S^1, N)$. This follows from the definition of functional diffeology and the fact that the product preserve a subduction. We will see in Section 6 that a weak Riemannian metric $g \oplus g$ is induced on the product space $LN \times LN$, and by Proposition 3.11 the metric $g \oplus g$ restricts to that on $C^\infty(S^1 \vee S^1, N)$. The restricted metric is denoted by $g_\vee$.

Fix a monotonically increasing smooth function $b\colon \mathbb{R} \to [0, 2\pi]$ (in a weak sense) satisfying $b(s) = 0$ if $s \le \pi/4$ and $b(s) = 2\pi$ if $s \ge 3\pi/4$. Define $p\colon S^1 \to S^1 \vee S^1$ by

$$p(\theta) := \begin{cases} b(\theta) \in S^1_{\text{left}} & 0 \le s \le \pi, \\ b(\theta - \pi) \in S^1_{\text{right}} & \pi \le s \le 2\pi. \end{cases}$$

Then p is smooth. Therefore, we have a smooth map

$$c := p^*\colon C^\infty(S^1 \vee S^1, N) \to C^\infty(S^1, N) \tag{5.3}$$

which is called the *concatenation map*.

Proposition 5.11. *The map c preserves the metrics; that is, $c^* g = g_\vee$.*

Proof. For a plot $P\colon U_P \to C^\infty(S^1 \vee S^1, N)$, we define a plot $P_\star\colon U_P \to C^\infty(S^1, N)$ ($\star = \text{left}, \text{right}$) by $P_\star = (i_\star)^* \circ P$. Then

$$(\mathrm{ev}_\theta \circ (c \circ P))(r) = (\mathrm{ev}_{p(\theta)} \circ P)(r) = P(r)(p(\theta)) = \begin{cases} P(r)_{\text{left}}(b(\theta)) & 0 \le \theta \le \pi, \\ P(r)_{\text{right}}(b(\theta - \pi)) & \pi \le \theta \le 2\pi. \end{cases}$$

Thus for $v, w \in T_r U_P$,

$$\begin{aligned} (c^* g)(P)_r(v, w) &= g(c \circ P)_r(v, w) \\ &= \int_0^\pi g_M(\mathrm{ev}_{p(\theta)} \circ P)_r(v, w) d\theta + \int_\pi^{2\pi} g_M(\mathrm{ev}_{p(\theta)} \circ P)_r(v, w) d\theta \\ &= \int_{S^1_{\text{left}}} g_M(\mathrm{ev}_\theta \circ P_{\text{left}})_r(v, w) d\theta + \int_{S^1_{\text{right}}} g_M(\mathrm{ev}_\theta \circ P_{\text{right}})_r(v, w) d\theta \\ &= g(P_{\text{left}})_r(v, w) + g(P_{\text{right}})_r(v, w) = g_\vee(P)_r(v, w). \end{aligned}$$

Observe that the second equality follows from Lemma 5.5. $\square$

Suppose that N admits a definite weak Riemannian metric g with respect to a generating family $\mathcal{G}$ of the diffeology of N. Lemma 5.2 enables us to deduce that the concatenation map (5.3) is restricted to the smooth map

$$c\colon (C^\infty(S^1 \vee S^1, N), \langle \mathcal{F}_{\mathcal{G}}^{S^1\vee S^1 N}\rangle) \to (C^\infty(S^1, N), \langle \mathcal{F}_{\mathcal{G}}^{S^1 N}\rangle).$$

The proof of Lemma 5.2 implies that that $c \circ P$ is in $\mathcal{F}_{\mathcal{G}}^{S^1 N}$ for each $P \in \mathcal{F}_{\mathcal{G}}^{S^1\vee S^1 N}$. Moreover, the proof of Proposition 5.11 allows us to deduce that $g_\vee(P)_r(v, v) = g(c \circ P)_r(v, v)$ for each plot $P \in \mathcal{F}_{\mathcal{G}}^{S^1\vee S^1 N}$ and $v \in T_r U_P$. By virtue of Theorem 5.6, we have the following result.

Proposition 5.12. *The metric $g_\vee$ on $(C^\infty(S^1 \vee S^1, N), \langle \mathcal{F}_{\mathcal{G}}^{S^1\vee S^1 N}\rangle)$ is definite.*

6. The warped product of Riemannian diffeological spaces

This section introduces the warped product in diffeology. Given weak Riemannian diffeological spaces (X, g_X) and (Y, g_Y), we consider $T_2(X)$, $T_2(Y)$ and $T_2(X \times Y)$. Let $f\colon X \to \mathbb{R}$ be a positive smooth map, $\pi_1\colon T_2(X \times Y) \to T_2(X)$ and $\pi_2\colon T_2(X \times Y) \to T_2(Y)$ the projections. Then, we see that the map

$$g_X \circ \pi_1 + (f \circ \rho \circ \pi_1) \cdot (g_Y \circ \pi_2)\colon T_2(X) \times T_2(Y) \to \mathbb{R}$$

is smooth on $T_2(X) \times T_2(Y)$, where $\rho\colon T_2(X) \to X$ is the natural map. Moreover, by the universality of the product, there exists a smooth map $i\colon T_2(X \times Y) \to T_2(X) \times T_2(Y)$. We define a map $g_{X\times_f Y}\colon T_2(X \times Y) \to \mathbb{R}$ by

$$g_{X\times_f Y} = (g_X \circ \pi_1 + (f \circ \rho \circ \pi_1) \cdot (g_Y \circ \pi_2)) \circ i.$$

It is readily seen that, by the definition, $g_{X\times_f Y}$ is a weak Riemannian metric on $X \times Y$. We shall call $g_{X\times_f Y}$ the *warped product* of weak Riemannian metrics g_X and g_Y with respect to f.

Remark 6.1. The natural map $i\colon T_2(X \times Y) \to T_2(X) \times T_2(Y)$ mentioned above is a diffeomorphism; see [3, Proposition 2.2.12].

Proposition 6.2. *If g_X and g_Y are definite, then so is $g_{X\times_f Y}$.*

Proof. Let $\mathcal{G}_X$ and $\mathcal{G}_Y$ be generating families of the diffeologies on X and Y, respectively. Let $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ be the projections. Then, the product diffeology on $X \times Y$ is generated by the family

$$\mathcal{F} := \{P\colon U_P \to X \times Y \mid \pi_X \circ P \in \mathcal{G}_X, \pi_Y \circ P \in \mathcal{G}_Y\}.$$

This implies that $g_{X\times_f Y}$ is definite with respect to $\mathcal{F}$. $\square$

Example 6.3. By applying Proposition 3.11, we obtain examples of definite weak Riemannian diffeological spaces.

(i) The induction $+ \to \mathbb{R}^2$ (Example 4.5) induces a definite weak Riemannian metric on $+$.

(ii) Moreover, consider a pullback diagram

$$\begin{array}{ccc} Y \times_B X & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & B \end{array}$$

in which X and Y are weak Riemannian diffeological spaces with definite metrics. Then, by combining Proposition 3.11 with Proposition 6.2, we see that the pullback $Y \times_B X$ admits a definite weak Riemannian metric.

Let N be a weak Riemannian diffeological space. We introduce another weak Riemannian metric on $C^\infty(S^1 \vee S^1, N)$ endowed with the functional diffeology.

We recall the smooth maps $i, j\colon S^1 \to S^1 \vee S^1$ in the pushout diagram (5.2), where $S^1 \vee S^1$ denotes the one point union of two circles; see Section 5.3. Then, we have a diagram of the form

$$\begin{tikzcd} C^\infty(S^1 \vee S^1, N) \arrow[r, "\widetilde{l}"] & LN \times_N LN \arrow[r, "q"] \arrow[d] & LN \times LN \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ & N \arrow[r, "\Delta"'] & N \times N, \end{tikzcd} \tag{6.1}$$

where the square is the pull-back of the diagonal map Δ, and $\widetilde{l}$ is defined by $\widetilde{l}(\gamma) = (\gamma(*), \gamma \circ i, \gamma \circ j)$.

Lemma 6.4. *The map $\widetilde{l}$ is a well-defined diffeomorphism with respect to the functional diffeology.*

Since $LN \times_N LN$ is a diffeological subspace of $N \times (LN \times LN)$, it follows from Proposition 3.11 that $C^\infty(S^1 \vee S^1, N)$ admits a weak Riemannian metric $g_N \oplus (g \oplus g)$.

Proof of Lemma 6.4. First, we see that $LN \times_N LN = \{(n, \gamma_1, \gamma_2) \mid \gamma_1(0) = \gamma_2(0) = n\} \subset N \times LN \times LN$ and $Z = \{(\gamma_1, \gamma_2) \mid \gamma_1(0) = \gamma_2(0)\} \subset LN \times LN$ are diffeomorphic. This is because the smooth maps

$$\mathrm{pr}_2 \times \mathrm{pr}_3 \colon LN \times_N LN \to Z, \quad \mathrm{pr}_2 \times \mathrm{pr}_3(n, \gamma_1, \gamma_2) = (\gamma_1, \gamma_2), \text{ and}$$
$$(\mathrm{ev}_0 \circ \mathrm{pr}_1, \mathrm{id}_Z)\colon Z \to LN \times_N LN, \quad (\mathrm{ev}_0 \circ \mathrm{pr}_1, \mathrm{id}_Z)(\gamma_1, \gamma_2) = (\gamma_1(0), \gamma_1, \gamma_2),$$

are clearly inverses of each other. Under this identification, the map $\widetilde{l}$ is interpreted as the map

$$l = i^* \times j^* \colon C^\infty(S^1 \vee S^1, N) \to Z, \quad \gamma \mapsto (\gamma \circ i, \gamma \circ j).$$

Since i and j are smooth, it follows that the maps i^* and j^* are also smooth. Therefore, the map $l = i^* \times j^*$, and hence $\widetilde{l}$, are also smooth.

Moreover, for any $(\gamma_1, \gamma_2) \in Z$, by the universality of the product, there exists a map $\gamma\colon S^1 \vee S^1 \to N$ such that the diagram

$$\begin{tikzcd} S^1 \arrow[r, "i"'] \arrow[dr, bend right, "\gamma_1"'] & S^1 \vee S^1 \arrow[d, dotted, "\exists!\,\gamma"] & S^1 \arrow[l, "j"] \arrow[dl, bend left, "\gamma_2"] \\ & N & \end{tikzcd}$$

commutes. Let $\nu\colon Z \to C^\infty(S^1 \vee S^1, N)$ be the map that assigns to each $(\gamma_1, \gamma_2) \in Z$ its universal morphism γ. Conversely, given a map $\gamma\colon S^1 \vee S^1 \to N$, by the universal property, there exist unique maps $\gamma_1, \gamma_2\colon S^1 \to N$ such that $\gamma_1 = \gamma \circ i$ and $\gamma_2 = \gamma \circ j$. Therefore, we see that ν and l are inverse mappings of each other. We show the smoothness of ν. Consider the adjoint $\mathrm{ad}(\nu)\colon Z \times (S^1 \vee S^1) \to N$ to ν. Then, we have a commutative diagram

$$\begin{tikzcd} Z \times (S^1 \coprod S^1) \arrow[rrr, hook] \arrow[dd, "\mathrm{id}_Z \times \pi"] & & & LN \times LN \times (S^1 \coprod S^1) \arrow[d, "\cong"] \\ & & & LN \times LN \times S^1 \coprod LN \times LN \times S^1 \arrow[d, "\mathrm{pr}_1 \times \mathrm{id}_{S^1} \coprod \mathrm{pr}_2 \times \mathrm{id}_{S^1}"] \\ Z \times (S^1 \vee S^1) \arrow[r, "\mathrm{ad}(\nu)"'] & N & & LN \times S^1 \coprod LN \times S^1. \arrow[ll, "\mathrm{ev} \coprod \mathrm{ev}"] \end{tikzcd}$$

Observe that the quotient map $\pi\colon S^1 \coprod S^1 \to S^1 \vee S^1$ is a subduction and then $\mathrm{id}_Z \times \pi$ is also a subduction. Thus, $\nu\colon Z \to (C^\infty(S^1 \vee S^1, N), \mathcal{D}_{\mathrm{func}})$ is smooth. Hence, the map l is a diffeomorphism with respect to the functional diffeology. □

Remark 6.5. We consider the weak Riemannian metric g on LN described in Section 5.2. Observe that we do not require the definiteness of g. The definition of the metric $g_N \oplus (g \oplus g)$ on $LN \times_N LN$ seems natural but is different from that in §5.3. When we measure the length of a given curve in $C^\infty(S^1 \vee S^1, N)$ with the present metric, the length of the trajectory of the attaching point is added twice. This reflects that the map from $S^1 \vee S^1$ passes through the attaching point twice.

We conclude this section with a problem. Let X and N be diffeological spaces and $\mathcal{G}$ a generating family of the diffeology of N. In Section 5.1, we introduce a subdiffeology $\mathcal{D}'$ of the functional diffeology $\mathcal{D}_{\mathrm{func}}$ of $C^\infty(X, N)$. The diffeology $\mathcal{D}'$ defined by $\mathcal{G}$ with the property (E) in Section 5.1 plays a crucial role when considering the definiteness of a weak Riemannian metric on $C^\infty(X, Y)$. Suppose that N is a Riemannian manifold. Then, we propose question on the subdiffeology.

P1. Is the bijection $l : C^\infty(S^1 \vee S^1, N) \to LN \times_N LN$ in Lemma 6.4 diffeomorphism with respect to the subdiffeology?

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