The string bracket, $\boldsymbol{BV}$-exactness and the Eilenberg–Moore spectral sequence

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- $S^1 \curvearrowright LM := \mathrm{map}(S^1, M)$. The coefficients are in $\mathbb{Q}$.

Theorem 1.1 (KNWY21)

Let M be a simply-connected closed manifold. Assume further that the reduced c-action on $\widetilde{H}_*^{S^1}(LM)$ in the homology Gysin sequence of the bundle $S^1 \to ES^1 \times LM \to ES^1 \times_{S^1} LM$ is trivial. Then there exists a commutative diagram

$$\begin{array}{ccccc} H_*^{S^1}(LM;\mathbb{Q})^{\otimes 2} & \xrightarrow[\cong]{\Psi\otimes\Psi} & (\mathrm{Ker}\,\widetilde{\Delta} \oplus \mathbb{Q}[u])^{\otimes 2} & \xrightarrow{inc.\oplus 0} & H_*(LM;\mathbb{Q})^{\otimes 2} \\ {[\ ,\ ]}\downarrow \text{string bracket} & & \vdots\ \text{Gerstenhaber bracket} & & \text{loop product}\downarrow \bullet \\ H_*^{S^1}(LM;\mathbb{Q}) & \xrightarrow[\cong]{\Psi} & (\mathrm{Ker}\,\widetilde{\Delta} \oplus \mathbb{Q}[u]) & \xleftarrow[\Delta]{} & H_*(LM;\mathbb{Q}). \end{array}$$

Theorem 1.2 (KNWY21)

For a simply-connected space M, the reduced c-action (the reduced S-action) is trivial if and only if M is BV-exact.

Proof of Theorem 1.1 BV-exactness Proof : BV-exactness

Consider the principal bundle $S^1 \to ES^1 \times LM \xrightarrow{p} ES^1 \times_{S^1} LM$.
The bundle gives rise to the homology Gysin sequence

$$\cdots \to H_*(LM) \xrightarrow{p_*} H_*^{S^1}(LM) \xrightarrow{c} H_{*-2}^{S^1}(LM) \xrightarrow{\mathsf{M}} H_{*-1}(LM) \to \cdots .$$

Here c is the cap product of the Euler class of the bundle: $- \cap q^*(u)$, where $u \in H^*(BS^1)$ the generator.

\# We use the S-action $S := - \cup q^*(u) : H^*_{S^1}(LM) \to H^{*+2}_{S^1}(LM)$ to prove Theorems 1.1 and 1.2.

Observe that the S^1-principal bundle above fits in the pullback diagram

$$\begin{array}{ccccc}
 & & S^1 & = & S^1 \\
 & & \downarrow & & \downarrow \\
LX & \longrightarrow & ES^1 \times LX & \longrightarrow & ES^1 \\
\| & & {\scriptstyle p}\downarrow & & \downarrow \\
LX & \longrightarrow & ES^1 \times_{S^1} LX & \underset{q}{\longrightarrow} & BS^1
\end{array}$$

in which the lower sequence is the fibration associated with the universal bundle $ES^1 \to BS^1$.

M : a simply-connected closed manifold M of dimension d. Consider

$$\begin{array}{ccccc} LM & \xleftarrow{\mathrm{Comp}} & LM \times_M LM & \xrightarrow{q} & LM \times LM \\ & & \downarrow & & \downarrow {\scriptstyle (ev_0, ev_0)} \\ & & M & \xrightarrow[\mathrm{Diag}]{} & M \times M, \end{array}$$

where the square is the pull-back of the evaluation map (ev_0, ev_0) defined by $ev_0(\gamma) = \gamma(0)$ along the diagonal map Diag and Comp denotes the concatenation of loops. By definition, the composite

$$q^! \circ (\mathrm{Comp})^* : C^*(LM) \to C^*(LM \times_M LM) \to C^*(LM \times LM)$$

induces Dlp the dual to the loop product on $H^*(LM)$.

- The *loop product* $\bullet$ on $\mathbb{H}_*(LM) := H_{*+d}(LM)$ is defined by

$$a \bullet b = (-1)^{d(\deg a + d)}((\mathrm{Dlp})^\vee)(a \otimes b)$$

  for a and $b \in \mathbb{H}_*(LM)$.

$\#$ We apply this construction to a 'Gorenstein spaces' in the sense of Félix, Halperin and Thomas.

- The *string bracket* [,] on $H_*^{S^1}(LM)$ is defined by

$$[a,b] := (-1)^{(\deg a)-d} p_*(\mathsf{M}(a) \bullet \mathsf{M}(b))$$

for $a, b \in H_*^{S^1}(LM)$. The bracket is of degree $2-d$ and gives a Lie algebra structure to the equivariant homology of LM.

$\#\ \cdots \to H_*(LM) \xrightarrow{p_*} H_*^{S^1}(LM) \xrightarrow{c} H_{*-2}^{S^1}(LM) \xrightarrow{\mathsf{M}} H_{*-1}(LM) \to \cdots.$

- The Batalin–Vilkovisky (BV-)operator:

$\Delta : H_*(LM;\mathbb{Q}) \xrightarrow{-\times[S^1]} H_{*+1}(LM \times S^1;\mathbb{Q}) \xrightarrow{\text{rotation act.}} H_{*+1}(LM;\mathbb{Q})$

- The homology is endowed with a Gerstenhaber algebra structure whose Lie bracket (Gerstenhaber bracket) $\{\ ,\ \}$ is given by

$$\{a,b\} = (-1)^{|a|}(\Delta(a \bullet b) - (\Delta a) \bullet b - (-1)^{|a|} a \bullet (\Delta b))$$

for $a, b \in \mathbb{H}_*(LM)$. If a and b are in the kernel of Δ, then

$$\{a,b\} = (-1)^{|a|}\Delta(a \bullet b).$$

To Theorem 1.1 page 2

Proof of Theorem 1.1

Let Ω be a connected comm. DGA with a differential d of degree -1. Assume that $\Omega = \oplus_{i \le 0} \Omega_i$, a non-positive DGA.

Recall the Hochschild chain complex $C(\Omega) = (\sum_{k=0}^{\infty} \Omega \otimes \overline{\Omega}^{\otimes k}, b)$, where $\overline{\Omega} = \Omega/\mathbb{Q}$. The Connes' B-operator $B \colon C(\Omega) \to C(\Omega)$ of degrees $+1$ is defined by

$$B(w_0, \dots, w_k) = \sum_{i=0}^{k} (-1)^{(\epsilon_{i-1}+1)(\epsilon_k - \epsilon_{i-1})} (1, w_i, \dots, w_k, w_0, \dots, w_{i-1}).$$

The *Hochschild homology* $HH_*(\Omega) := H(C(\Omega), b)$.

With a generator u of degree -2,

- The *negative cyclic homology* $HC_*^-(\Omega) := ((C(\Omega)[[u]], b + uB)$
- The *cyclic homology* $HC_*(\Omega) := (C(\Omega)[u^{-1}], b + uB)$
- The *periodic cyclic homology*
$$HC_*^{\mathrm{per}}(\Omega) := (C(\Omega)[[u, u^{-1}], b + uB).$$

We have exact sequences (Connes' exact sequences).

$$\text{(A)}: \cdots \longrightarrow HC^-_{n+2}(\Omega) \xrightarrow{S=\times u} HC^-_n(\Omega) \xrightarrow{\pi} HH_n(\Omega) \xrightarrow{\beta} HC^-_{n+1}(\Omega) \longrightarrow \cdots$$

$$\text{(B)}: \cdots \longrightarrow HH_{n+1}(\Omega) \xrightarrow{I} HC_{n+1}(\Omega) \xrightarrow{S'} HC_{n-1}(\Omega) \xrightarrow{B_{HH}} HH_n(\Omega) \longrightarrow \cdots$$

$$\text{(C)}: \cdots \longrightarrow HC^-_{n+1}(\Omega) \xrightarrow{\times u} HC^{\text{per}}_{n-1}(\Omega) \xrightarrow{\tilde{\pi}} HC_{n-1}(\Omega) \xrightarrow{B_{HC}} HC^-_n(\Omega) \longrightarrow \cdots,$$

where the maps B_{HH}, β and B_{HC} are induced by Connes' B-map B.

Jones' isomorphisms

$$H^*(LM; \mathbb{Q}) \cong HH_*(A_{PL}(M)), H^*_{S^1}(LM; \mathbb{Q}) \cong HC^-_*(A_{PL}(M))$$

translate the Gysin sequence to (A), where $A_{PL}(M)$ is the polynomial de Rham algebra over $\mathbb{Q}$ of a simply-connected space M.

To prove Theorem 1.1, we also use maps I, B_{HH} in (B) and B_{HC} in (C).

There is a commutative diagram

$$
\begin{array}{ccc}
((\widetilde{HH}_*(\Omega)/\operatorname{Im}\Delta') \oplus \mathbb{K}[u])^{\otimes 2} & \xrightarrow{\Xi\otimes\Xi} & HC_*^-(\Omega)^{\otimes 2} \\
\text{`Cokernel'}\otimes\text{`Cokernel'}\uparrow & & \beta\otimes\beta\uparrow \\
HH_*(\Omega)^{\otimes 2} & & HH_*(\Omega)^{\otimes 2} \\
\bullet^\vee\uparrow & & \bullet^\vee\uparrow \quad [\ ,\ ]^\vee \\
HH_*(\Omega) & & HH_*(\Omega) \\
\Delta'\uparrow & & \pi\uparrow \\
(\widetilde{HH}_*(\Omega)/\operatorname{Im}\Delta') \oplus \mathbb{K}[u] & \xrightarrow{\Xi} & HC_*^-(\Omega).
\end{array}
$$

Here $\Delta' = B_{HH} \circ I$ is the *BV operator* of $HH_*(\Omega)$ and the horizontal isomorphism Ξ is defined by the composite

$$
(\widetilde{HH}_*(\Omega)/\operatorname{Im}\Delta) \oplus \mathbb{K}[u] \xrightarrow{I} \widetilde{HC}_*(\Omega) \oplus \mathbb{K}[u] \xrightarrow[\cong]{B_{HC}}
$$

$$
\widetilde{HC}_*^-(\Omega) \oplus \mathbb{K}[u] \xrightarrow[\cong]{\text{sp}} HC_*^-(\Omega).
$$

If the S-action is trivial, then I is an isomorphism. (K-Yamaguchi, '00) □

Definition 3.1 (K, Naito, Wakatsuki, Yamaguchi '21 (KNWY21))

A simply-connected space M is *Batalin–Vilkovisky (BV-) exact* if

$$\mathbf{Im}\,\widetilde{\boldsymbol{\Delta}} = \mathbf{Ker}\,\widetilde{\boldsymbol{\Delta}}$$

for the reduced BV operator $\widetilde{\boldsymbol{\Delta}} : \widetilde{\boldsymbol{H}}_*(\boldsymbol{LM};\mathbb{Q}) \to \widetilde{\boldsymbol{H}}_{*+1}(\boldsymbol{LM};\mathbb{Q})$.
In general, $\boldsymbol{\Delta}^2 = \mathbf{0}$.

To Theorem 1.2 page 2

Before giving a sketch of the proof of Theorem 1.2, we relate the new homotopy invariant, the BV exactness, to traditional ones in rational homotopy theory,

formality, a positive weight decomposition, ...

Theorem 3.2 (The fundamental theorem in RHT (Sullivan '73, Bousfield–Gugenhaim '76))

There exists an equivalence between the homotopy category of nilpotent rational connected spaces of finite $\mathbb{Q}$-type and that of cofibrant connected commutative differential graded algebras of finite $\mathbb{Q}$-type.

We have an equivalence

$$\text{fN}\mathbb{Q}\text{-Ho}(\textbf{Top}) \underset{|\ |}{\overset{Q\circ A_{PL}(\ )}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \text{f}\mathbb{Q}\text{-Ho}(\textbf{CDGA}^{op}).$$

Here Q denotes the cofibrant replacement. As a consequence, we have a quasi-iso. $(\wedge V = (\text{poly. alg} \otimes \text{exterior alg}), d) \overset{\simeq}{\to} A_{PL}(X)$ for a space X.

- The CDGA $(\wedge V, d)$ is called a *Sullivan (rational) model* for X.
- $(\wedge V, d)$: minimal $\overset{\text{def}}{\Leftrightarrow}$ $d(v)$ is decomposable for $v \in V$.

Definition 3.3

A simply-connected space X is *formal* if there is a quasi-isomorphism from a Sullivan model for X to $H^*(X;\mathbb{Q})$; that is, the *rational homotopy type* is determined by its rational cohomology algebra.

A compact simply-connected Kähler manifolds and toric manifolds are formal.

Definition 3.4 (Body–Douglas '78)

A simply-connected space X admits *positive weights* if the Sullivan minimal model (A, d) for X has a decomposition $A^n = \oplus_{i>0} A^n_{(i)}$ for $n > 0$ and $A^0 = A^0_{(0)}$ with

- $d(A^n_{(i)}) \subset A^{n+1}_{(i)}$
- $A^n_{(i)} \cdot A^m_{(j)} \subset A^{n+m}_{(i+j)}$ for all m, n and i.

Proposition 3.5 ((H-S) Halperin–Stasheff '79)

A simply-connected formal space admits positive weight.

Theorem 3.6 (KNWY21)

A simply-connected space X admitting positive weights is BV exact. In particular, a formal space is BV exact.

A nonformal BV-exact space (manifold). Let $UTS^6 \to S^6$ be the unit tangent bundle over S^6. Then, we have a simply-connected 11-dimensional manifold M which fits in the pullback diagram

$$\begin{array}{ccc} M & \longrightarrow & UTS^6 \\ \downarrow & & \downarrow p \\ S^3 \times S^3 & \xrightarrow{f} & S^6, \end{array}$$

where $f : S^3 \times S^3 \to S^6$ is a smooth map homotopic to the map defined by collapsing the 3-skeleton into a point.

Proposition 3.7 (KNWY21)

The $\mathbf{11}$-dimensional manifold M is nonformal and admits positive weight. (As a consequence, M is BV-exact.)

- The minimal model of M has the form $\mathcal{M} = (\wedge(x, y, z), d)$, where $d(x) = 0 = d(y)$, $d(z) = xy$, $\deg x = \deg y = 3$ and $\deg z = 5$. It is readily seen that M is nonformal since the Massey product $\langle x, x, y \rangle$ does not vanish.
- In the minimal model $\mathcal{M}$, we define weights of x, y and z by 1, 1 and 2, respectively. The model $\mathcal{M}$ for the manifold M admits positive weights.

Proof of Theorem 1.2 and a generalization of the result

For a simply-connected space M, we obtain the cobar type Eilenberg-Moore spectral sequence (EMSS) converges to $H^*_{S^1}(LM;\mathbb{Q})$ with

$$E_2^{*,*} \cong \mathrm{Cotor}^{*,*}_{H^*(S^1;\mathbb{Q})}(H^*(LM;\mathbb{Q}),\mathbb{Q}).$$

We have a decomposition

$$\{E_r^{*,*}, d_r\} = \bigoplus_{N\in\mathbb{Z}} \{{}_{(N)}E_r^{*,*}, d_r\} \oplus \{\mathbb{Q}[u], 0\},$$

where bideg $u = (1,1)$, for which the following assertion holds.

Proposition 3.8 (KNWY21)

There exists an action S (a morphism of SSes) on $\{E_r^{*,*}, d_r\}$ which is compatible with the S-action on $H^*_{S^1}(LM;\mathbb{Q})$ and for each N,

$$S : \{{}_{(N)}E_r^{*,*}, d_r\} \to \{{}_{(N-1)}E_r^{*+1,*+1}, d_r\}.$$

Theorem 3.9 (KNWY21)

The E_r-term ${}_{(0)}E_r^{p,q}$ in $\{{}_{(0)}E_r^{*,*}, d_r\}$ is trivial for any (p, q) if and only if the $(r-1)$ times S-action S^{r-1} on $\widetilde{H}^*_{S^1}(LM; \mathbb{Q})$ is.

Assertion: BV-exactness

Proof of Theorem 1.2 # By using the Sullivan model for LX, we show this fact.

The BV-exactness of a space is equivalent to the condition that the E_2-term of the spectral sequence $\{{}_{(0)}E_r^{*,*}, d_r\}$ is trivial. □

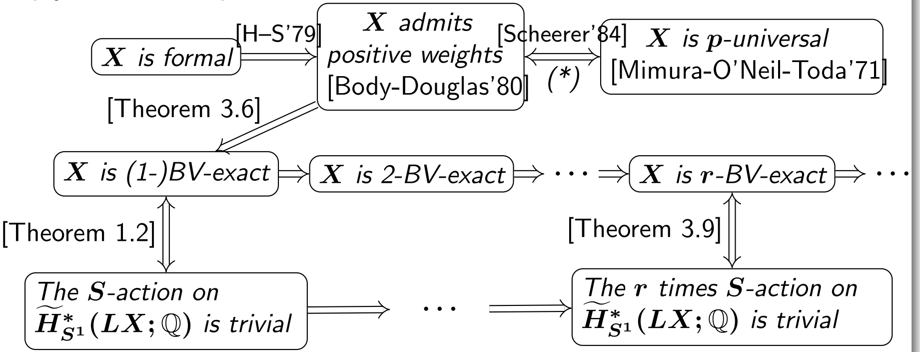
This allows us to propose a *higher version* of the BV-exactness.

Definition 3.10

A simply-connected space X is r-*BV-exact* if the E_{r+1}-term ${}_{(0)}E_{r+1}^{p,q}$ in the spectral sequence $\{{}_{(0)}E_r^{*,*}, d_r\}$ associated with X is trivial for any (p, q).

Theorem 3.11

There are the following implications concerning homotopy invariants for a simply-connected space X.



(*) holds provided X has the homotopy type of a finite CW complex.

Thm 1.1*: M is (1-)BV-exact $\Longrightarrow$ the string bracket for M is reducible to the loop product.*

A geometric description for higer BV-exactness?