

Interleavings of maps via relative Sullivan algebras

based on joint work with

T. Naito, S. Wakatsuki and T. Yamaguchi [KNWY: [arXiv:2501.09257](#)]

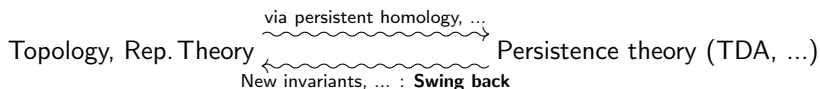
and N- K. Sekizuka -W-Y: [KNSWY: [arXiv:2604.06800](#)]

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§1. Interleaving distances

- $\mathcal{C}^{\mathbb{R}, \leq}$: the functor category for a category $\mathcal{C}$.
- For a real number $\varepsilon \geq 0$, define a functor $T_\varepsilon : (\mathbb{R}, \leq) \rightarrow (\mathbb{R}, \leq)$ by $T_\varepsilon(a) = a + \varepsilon$. Moreover, the *shift functor*

$$(\)^\varepsilon : \mathcal{C}^{\mathbb{R}, \leq} \rightarrow \mathcal{C}^{\mathbb{R}, \leq}$$

is defined by $(\)^\varepsilon(F) = F^\varepsilon := FT_\varepsilon$.

Definition 1.1 (Chazal et al '09, Bubenik et al. '14. Strict interleavings)

Objects F and G in $\mathcal{C}^{\mathbb{R}, \leq}$ are ε -*interleaved* if there exist natural transformations φ and ψ which fit in the commutative diagrams

$$\begin{array}{ccccc} F & \xrightarrow{\quad} & F^\varepsilon & \xrightarrow{\quad} & F^{2\varepsilon} \\ & \searrow & \nearrow & \searrow & \nearrow \\ & \psi & \varphi & \psi^\varepsilon & \varphi^\varepsilon \\ & \nearrow & \searrow & \nearrow & \searrow \\ G & \xrightarrow{\quad} & G^\varepsilon & \xrightarrow{\quad} & G^{2\varepsilon}, \end{array}$$

where horizontal arrows are the natural transformations defined by the structure maps of F and G . Such a pair (φ, ψ) is called an ε -*interleaving* of F and G .

The natural transformations in Definition 1.1 give the commutative diagrams

$$\begin{array}{ccc}
 F(i) & \xrightarrow{F(i \rightarrow i+2\varepsilon)} & F(i+2\varepsilon) \\
 \searrow \varphi(i) & & \nearrow \psi(i+\varepsilon) \\
 & G(i+\varepsilon) &
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 & F(i+\varepsilon) & \\
 \nearrow \psi(i) & & \searrow \varphi(i+\varepsilon) \\
 G(i) & \xrightarrow{G(i \rightarrow i+2\varepsilon)} & G(i+2\varepsilon)
 \end{array}$$

for all $i \in \mathbb{R}$. Observe that F is isomorphic to G in $\mathcal{C}(\mathbb{R}, \leq)$ if and only if F and G are 0-interleaved.

Definition 1.2 (Bubenik and Scott '14)

For objects F and G in $\mathcal{C}(\mathbb{R}, \leq)$, the *interleaving distance* $d_I(F, G)$ between F and G is defined by

$$d_I(F, G) := \inf\{\varepsilon \geq 0 \mid F \text{ and } G \text{ are } \varepsilon\text{-interleaved}\}.$$

Proposition 1.3 (Bubenik and Scott '14)

The interleaving distance d_I is an extended pseudometric (distance) on the class of objects of $\mathcal{C}(\mathbb{R}, \leq)$.

1.1 Interleavings in $\mathbf{Mod}_{\mathbb{K}}^{(\mathbb{R}, \leq)}$ ($\mathbb{K}$ is a field)

- For an interval J ($a, b \in J$ and $a \leq c \leq b$, then $c \in J$), define an *interval module* $\chi_J \in \mathbf{Mod}_{\mathbb{K}}^{(\mathbb{R}, \leq)}$ by

$$\chi_J(x) = \begin{cases} \mathbb{K} & \text{if } x \in J, \\ 0 & \text{otherwise,} \end{cases} \quad \chi_J(x \leq y) = \begin{cases} id_{\mathbb{K}} & \text{if } x, y \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.4 (Botnan and Crawley-Boevey '20)

A locally finite persistence module V (i.e. an object in $\mathbf{Mod}_{\mathbb{K}}^{(\mathbb{R}, \leq)}$ with $\dim V(t) < \infty$ for $t \in \mathbb{R}$) can be decomposed uniquely as a direct sum of interval modules

$$V \cong \bigoplus_J \chi_J.$$

Then, the *barcode* $\mathcal{B}_V$ associated with a locally finite persistence vector space $V \cong \bigoplus_J \chi_J$ is defined by the multiset consisting of the intervals J .

- For multisets A and B , $A_B := A \amalg (\coprod_{|B|} \{\emptyset\})$.
- $f : A \leftrightarrow B$ for a bijection (matching) $f : A_B \rightarrow B_A$.

Theorem 1.5 ((The isometry theorem) Chazal–Cohen–Steiner–Glisse–Guibas–Oudot '09, Bubenik–Scott '14)

Let V and V' be locally finite persistence vector spaces. Then, one has

$$d_I(V, V') = d_B(\mathcal{B}_V, \mathcal{B}_{V'}) := \inf_{f: \mathcal{B}_V \leftrightarrow \mathcal{B}_{V'}} \sup_{I \in \text{dom}(f)} d_I(\chi_I, \chi_{f(I)}),$$

where $\mathcal{B}_V \leftrightarrow \mathcal{B}_{V'}$ denotes a matching between them, $\chi_{\mathbb{R}}$ and $\chi_{\emptyset}$ are the constant functors $\mathbb{K}$ and 0 , respectively.

Remark 1.6

For intervals I and J , the distance $d_I(\chi_I, \chi_J)$ is determined by endpoints and lengths of I and J .

1.2 Interleavings up to homotopy

- $\mathcal{M}$: a cofibrantly generated model category (e.g. **Top**, **Ch $_{\mathbb{K}}$**).
- $\mathcal{M}^{(\mathbb{R}, \leq)}$: the model category endowed with the *projective model structure*.

Definition 1.7 (Blumberg–Lesnick '23, Lanari–Scoccola '23)

- (1) Objects X and Y in $\mathcal{M}^{(\mathbb{R}, \leq)}$ are ε -homotopy interleaved if there exist $X \simeq X'$ and $Y \simeq Y'$ such that X' and Y' are ε -interleaved in $\mathcal{M}^{(\mathbb{R}, \leq)}$. Here $W \simeq W'$ means that there is a zigzag of weak equivalences connecting W and W' .
- (2) Objects X and Y in $\mathcal{M}^{(\mathbb{R}, \leq)}$ are ε -interleaved in the homotopy category if they are ε -interleaved in $\mathrm{Ho}(\mathcal{M}^{(\mathbb{R}, \leq)})$.
- (3) Let $q_* : \mathcal{M}^{(\mathbb{R}, \leq)} \rightarrow \mathrm{Ho}(\mathcal{M})^{(\mathbb{R}, \leq)}$ be the functor induced by the localization functor. Objects X and Y in $\mathcal{M}^{(\mathbb{R}, \leq)}$ are ε -homotopy commutative interleaved if q_*X and q_*Y are ε -interleaved in $\mathrm{Ho}(\mathcal{M})^{(\mathbb{R}, \leq)}$.

Definition 1.8 (Blumberg and Lesnick '23, Lanari and Scoccola '23))

$X, Y \in \mathcal{M}^{(\mathbb{R}, \leq)}$. The *homotopy interleaving distance* and the *homotopy commutative interleaving distance* are defined by

$$d_{\text{HI}}(X, Y) := \inf\{\varepsilon \geq 0 \mid X, Y \text{ are } \varepsilon\text{-homotopy interleaved}\} \text{ and}$$

$$d_{\text{HC}}(X, Y) := \inf\{\varepsilon \geq 0 \mid X, Y \text{ are } \varepsilon\text{-homotopy commutative interleaved}\},$$

respectively. The *interleaving distance in the homotopy category* is defined by

$$d_{\text{IHC}}(X, Y) := \inf\{\varepsilon \geq 0 \mid X, Y \text{ are } \varepsilon\text{-interleaved in the homotopy category}\}.$$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & q_* & & \\
 & \swarrow & & \searrow & \\
 \text{Ho}(\mathcal{M})^{(\mathbb{R}, \leq)} & \xleftarrow{\theta} & \text{Ho}(\mathcal{M})^{(\mathbb{R}, \leq)} & \xleftarrow{\pi} & \mathcal{M}^{(\mathbb{R}, \leq)}
 \end{array} \\
 d_{\text{HC}} & \leq & d_{\text{IHC}} & \leq & d_{\text{HI}} & \leq & 2d_{\text{HC}}. \\
 \text{(All distances are extended pseudometrics)} & & & & \text{[Lanari–Scoccola]}
 \end{array}$$

Theorem 1.9 (Lanari–Scoccola, '23)

Let $\mathbf{Top}$ be the category of topological spaces. Suppose that $d_{\text{HI}} \leq cd_{\text{HC}}$ on $\mathbf{Top}^{(\mathbb{R}, \leq)}$. Then $c \geq \frac{3}{2}$. Hence $d_{\text{HI}} \neq d_{\text{HC}}$ in general.

For some case, $d_{\text{HC}} \not\leq d_{\text{IHC}} \not\leq d_{\text{HI}} \quad ?$

- $\mathbf{Ch}_{\mathbb{K}}$: the category of dg modules over a field $\mathbb{K}$ of arbitrary characteristic. ($\deg(\text{differential}) = +1$, unbounded in general.)

Definition 1.10 (K. -Naito-Wakatsuki-Yamaguchi '25)

The *cohomology interleaving distance* $d_{\text{Cohl}}(X, Y)$ of persistence dg modules X and Y ($\in \mathbf{Ch}_{\mathbb{K}}^{(\mathbb{R}, \leq)}$) is defined by

$$d_{\text{Cohl}}(X, Y) := \sup\{d_l(H^k(X), H^k(Y)) \mid k \in \mathbb{Z}\}.$$

Definition 1.11

An object X in $\mathbf{Ch}_{\mathbb{K}}^{(\mathbb{R}, \leq)}$ is d_{HI} -formal if $d_{\text{HI}}(X, (H(X), 0)) = 0$.

Theorem 1.12 (KNWY '25)

Every object in $\mathbf{Ch}_{\mathbb{K}}^{(\mathbb{R}, \leq)}$ is d_{HI} -formal. As a consequence, one has the equalities

$$d_{\text{Cohl}} = d_{\text{HC}} = d_{\text{IHC}} = d_{\text{HI}}.$$

Proof.

Discretize persistence dg modules $\rightsquigarrow$ The formality of each object in $\mathbf{Ch}_{\mathbb{K}}^{(\mathbb{Z}, \leq)} \rightsquigarrow$ recover persistence dg modules with homotopy interleavings.

The latter half:

$$\begin{aligned} d_{\text{Cohl}}(X, X') &\leq d_{\text{HI}}(X, X') \\ &\leq d_{\text{HI}}(X, H(X)) + d_{\text{HI}}(H(X), H(X')) + d_{\text{HI}}(H(X'), X') \\ &= d_{\text{HI}}(H(X), H(X')) \leq d_1(H(X), H(X')) = d_{\text{Cohl}}(X, X'). \end{aligned}$$



Problem/Question

For some persistence objects in $\mathcal{M}^{(\mathbb{R}, \leq)}$,

$$d_{\text{Cohl}} \not\leq d_{\text{HC}} \not\leq d_{\text{IHC}} \not\leq d_{\text{HI}} \quad ?$$

Consider the problem
in the category of *extended tame* persistence CDGAs.

§2. Interleavings of CDGAs and maps

Definition 2.1 (Chachólski, Giunti and Landi '21, (the tame version))

Let $\mathcal{M}$ be a category. A functor $X : \mathbb{R}_+ := [0, \infty) \rightarrow \mathcal{M}$ is an *extended tame functor* if there is a divergent or finite sequence $\tau_0 < \tau_1 < \dots < \tau_n < \dots$ in $\mathbb{R}_+$ such that $X^{s \leq t} : X^s \rightarrow X^t$ may fail to be an isomorphism only if $s < \tau_a \leq t$ for some $a \in \mathbb{N}$.

$\mathbf{et}(\mathcal{M}^{\mathbb{R}_+})$: the full subcategory of $\mathcal{M}^{\mathbb{R}_+}$ consisting of extended tame functors (objects).

For a morphism $g : X \rightarrow Y$ in $\mathbf{et}(\mathcal{M}^{\mathbb{R}_+})$, consider the diagram

$$\begin{array}{ccc}
 X(\tau_{a-1}) & \xrightarrow{X(\tau_{a-1} < \tau_a)} & X(\tau_a) \\
 g(\tau_{a-1}) \downarrow & & \downarrow \bar{g}(\tau_a) \\
 Y(\tau_{a-1}) & \xrightarrow{\quad} & Q(\tau_a) \\
 & \searrow & \nearrow \hat{g}(\tau_a) \\
 & & Y(\tau_a)
 \end{array}
 \quad \begin{array}{l}
 \text{curved arrow } g(\tau_a) \text{ from } X(\tau_a) \text{ to } Y(\tau_a) \\
 \text{curved arrow } Y(\tau_{a-1} < \tau_a) \text{ from } Y(\tau_{a-1}) \text{ to } Y(\tau_a)
 \end{array}
 \tag{1}$$

consisting of the pushout square. Here $\tau_0 < \tau_1 < \dots < \tau_n < \dots$ is a sequence discretising both X and Y .

Theorem 2.2 (Chachólski, Giunti and Landi '21)

Let $\mathcal{M}$ be a model category. The following choices of weak equivalences, fibrations and cofibrations form a model structure on $\mathbf{et}(\mathcal{M}^{\mathbb{R}_+})$. A morphism $g : X \rightarrow Y$ in $\mathbf{et}(\mathcal{M}^{\mathbb{R}_+})$ is a

- weak equivalence if $g^t : X^t \rightarrow Y^t$ is a weak equivalence for all t ,
- fibration if $g^t : X^t \rightarrow Y^t$ is a fibration for all t ,
- cofibration if $g^t : X^t \rightarrow Y^t$ factors through a cofibration $\hat{g}^t : Q^t \rightarrow Y^t$ obtained by the pushout diagram above for all t .

In what follows, we work in **CDGA** the category of CDGAs / $\mathbb{Q}$.

- $A_{PL} : \mathbf{Top}^{\text{op}} \rightarrow \mathbf{CDGA}$ denotes the *polynomial de Rham functor*.
- For a connected space A , $(\wedge V_A, d) \xrightarrow{\cong} A_{PL}(X)$ denotes the *minimal Sullivan model* for A .
($\wedge V_A$ is the free algebra generated by a vector space V_A and d satisfies *appropriate* filtration condition on a well-ordered basis of V_A .)

Observation 2.3

- $f : X \rightarrow A$: a continuous map between connected spaces.
 $\iota_f : \wedge V_A \rightarrow \wedge V_A \otimes \wedge W$: a *minimal relative Sullivan model* for f ; that is, we have a commutative diagram

$$\begin{array}{ccc} \wedge V_A & \xrightarrow{\iota_f} & \wedge V_A \otimes \wedge W \\ \sim \downarrow & & \downarrow \sim \\ A_{\text{PL}}(A) & \xrightarrow{A_{\text{PL}}(f)} & A_{\text{PL}}(X) \end{array}$$

whose vertical arrows are quasi-isomorphisms. Moreover, by definition, $\exists$ a filtration $\{W^p(r)\}_{r \geq 0}$ for each W^p such that

$$d : 1 \otimes W^p(r) \rightarrow \wedge V_A \otimes \wedge(W^{<p} \oplus W^p(r-1)).$$

- Then, the model gives rise to a persistence CDGA $\theta(\iota)$ defined by $\theta(\iota_f)(n) := \wedge V_A \otimes \wedge(W^{\leq n})$ together with the inclusions

$$\theta(\iota_f)(n < n+1) : \wedge V_A \otimes \wedge(W^{\leq n}) \rightarrow \wedge V_A \otimes \wedge(W^{\leq n+1})$$

as the structure maps. We have $\theta(\iota_f) : \mathbb{N} \cup \{0\} \rightarrow \mathbf{CDGA}$.

- We recall the floor function $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$. Then, we have an extended tame persistence CDGA $\Theta'(f)$ defined by $\Theta'(f) := (\lfloor \cdot \rfloor)^* \theta(\iota_f)$.

Theorem 2.4 (KNSWY '26)

The function Θ' gives rise to a functor

$$\Theta : \text{Func}(I, \mathbf{Top}_0)^{\text{op}} \rightarrow \text{Ho}(\mathbf{et}(\mathbf{CDGA}^{\mathbb{R}_+})),$$

where $\text{Ho}(\mathbf{et}(\mathbf{CDGA}^{\mathbb{R}_+}))$ is the homotopy category of $\mathbf{et}(\mathbf{CDGA}^{\mathbb{R}_+})$ and the poset $I = \{0 < 1\}$ is regarded as a category.

Geometrical description of Θ ?

We recall the Sullivan–de Rham equivalence theorem.

Theorem 2.5 (cf. Bousfield–Gugenheim '76)

Let $\mathcal{C}_0$ be the full subcategory of **CDGA** consisting of connected CDGAs of finite $\mathbb{Q}$ -type. Let $\mathcal{D}_0$ denote the full subcategory of **sSet** consisting of nilpotent path-connected rational simplicial sets of finite $\mathbb{Q}$ -type. Then, the adjoint pair

$$\langle \rangle : \mathbf{CDGA} \rightleftarrows \mathbf{sSet}^{\mathrm{op}} : A_{\mathrm{PL}}()$$

of the realization functor $\langle \rangle$ and $A_{\mathrm{PL}}()$ gives rise to a partial Quillen equivalence for the subcategories $\mathcal{C}_0$ and $\mathcal{D}_0^{\mathrm{op}}$; that is, the adjunction induces the equivalence between the full subcategories $\mathbf{Full}(\mathcal{C}_0)$ and $\mathbf{Full}(\mathcal{D}_0)$ of homotopy categories.

We also recall the Moore–Postnikov tower $\{X(n) \rightarrow B\}_{n \geq 0}$ for a map $f : X \rightarrow Y$ which is a generalization of a Postnikov tower of a space and functorial.

Theorem 2.6 (KNSWY '26)

One has a diagram of categories and functors

$$\begin{array}{ccccc}
 & & \text{mp} := (\lfloor \rfloor)^* \circ (\text{M-P tower}) \circ S_*(\) \circ (\text{Homotopy fibration}) & & \\
 & \nearrow & & \searrow & \\
 \text{Func}(\mathbf{I}, \mathbf{Top})^{\text{op}} & & \text{et}(\mathbf{CDGA}^{\mathbb{R}_+}) & \xrightleftharpoons[\perp]{\langle \rangle_*} & \text{et}((\mathbf{sSet}^{\text{op}})^{\mathbb{R}_+}) \\
 \uparrow & & \downarrow & & \downarrow \\
 \text{Func}(\mathbf{I}, \mathbf{Top}_0)^{\text{op}} & \xrightarrow{\Theta} & \text{Ho}(\text{et}(\mathbf{CDGA}^{\mathbb{R}_+})) & \xrightleftharpoons[\perp]{A_{\text{PL}}(\)_*} & \text{Ho}(\text{et}((\mathbf{sSet}^{\text{op}})^{\mathbb{R}_+})) \\
 \uparrow & \nearrow \Theta_{\text{res}} & \uparrow & \nwarrow R_{\text{APL}}(\)_* & \uparrow \\
 \mathcal{F}_{\text{sc}} & \xrightarrow{\Theta_{\text{res}}} & \text{Full}(\text{et}(\mathcal{C}_0^{\mathbb{R}_+})) & \xrightleftharpoons[\simeq]{L\langle \rangle_*} & \text{Full}(\text{et}((\mathcal{D}_0^{\text{op}})^{\mathbb{R}_+})) \\
 & \searrow \Theta_{\text{res}} & & \nwarrow R_{\text{APL}}(\)_* & \\
 & & (Rationalization) \circ (\text{mp}) =: (\text{mp})_{\mathbb{Q}} & &
 \end{array}$$

in which the diagram starting at $\mathcal{F}_{\text{sc}}$ (the full-subcat. of maps between 1-conn. spaces) and ending at $\text{Ho}(\text{et}(\mathbf{CDGA}^{\mathbb{R}_+}))$ via $\text{et}((\mathbf{sSet}^{\text{op}})^{\mathbb{R}_+})$ is commutative. As a consequence, the pointwise localization $(\text{mp})_{\mathbb{Q}}$ of the mp factors through the functor Θ_{res} .

Theorem 2.7

Let X and Y be connected spaces. The function $d : [X, Y] \times [X, Y] \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined by

$$d([f], [g]) = d_{\text{IHC}}(\Theta(f), \Theta(g))$$

is a well-defined pseudodistance on $[X, Y]$. Assume further that X and Y are simply connected. Then, $d([f], [g]) = 0$ if and only if there exist self homotopy equivalences α and β on $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$, respectively, such that $\beta \circ f_{\mathbb{Q}} \simeq g_{\mathbb{Q}} \circ \alpha$.

$$\begin{array}{ccc} X_{\mathbb{Q}} & \xrightarrow{f_{\mathbb{Q}}} & Y_{\mathbb{Q}} \\ \alpha \downarrow & & \downarrow \beta \\ X_{\mathbb{Q}} & \xrightarrow{g_{\mathbb{Q}}} & Y_{\mathbb{Q}} \end{array}$$

§3. Formalities of persistence CDGAs

Definition 3.1

A persistence CDGA F in $\mathbf{et}(\mathbf{CDGA}^{\mathbb{R}_+})$ is *H-formal* if there exists a zig-zag sequence of weak equivalences between F and $(H(F), 0)$ in $\mathbf{et}(\mathbf{CDGA}^{\mathbb{R}_+})$.

In order to define other formalities of extended tame persistence CDGAs, we consider the commutative diagram

$$\begin{array}{ccc} \mathbf{et}(\mathbf{CDGA}^{\mathbb{R}_+}) & \xrightarrow{j} & \mathbf{CDGA}^{\mathbb{R}_+} \\ \pi \downarrow & & \downarrow q_* \\ \mathrm{Ho}(\mathbf{et}(\mathbf{CDGA}^{\mathbb{R}_+})) & \xrightarrow{j_*} & \mathrm{Ho}(\mathbf{CDGA}^{\mathbb{R}_+}), \end{array}$$

where π and q are the localization maps, j is the embedding and j_* is unique map induced by j .

Proposition 3.2

For objects F and G in $\mathbf{et}(\mathcal{M}^{\mathbb{R}_+})$,

$$d_{\mathrm{HC}}(j_*(F), j_*(G)) \leq d_{\mathrm{IHC}}(F, G) \leq d_{\mathrm{HI}}(F, G) \leq d_{\mathrm{I}}(F, G). \quad (2)$$

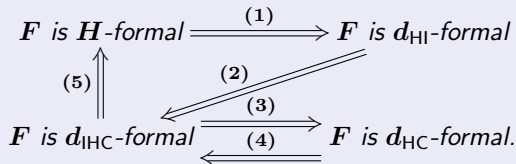
Definition 3.3

Let F be an extended tame persistence CDGA.

- (1) F is d_{HI} -formal if $d_{\text{HI}}(F, H(F)) = 0$.
- (2) F is d_{IHC} -formal if $d_{\text{IHC}}(F, H(F)) = 0$.
- (3) F is d_{HC} -formal if $d_{\text{HC}}(j_*(F), j_*(H(F))) = 0$.

Proposition 3.4

For an extended tame persistence CDGA F , consider the following implications



- (i) The implications (1), (2) and (3) hold in general.
- (ii) Let $\{\tau_n\}_{n \geq 0}$ be a sequence which discretises F . Suppose that there exists a positive number α such that $\tau_{n+1} - \tau_n \geq \alpha$ for each n . Then, the implications (4) and (5) hold.

As in $\mathbf{Ch}_{\mathbb{K}}^{\mathbb{R}}$, we have

Lemma 3.5

Suppose that extended tame persistence CDGAs F and G are d_{HI} -formal. Then

$$d_{\mathrm{HI}}(F, G) = d_{\mathrm{CohI}}(F, G) := d_{\mathrm{I}}(HF, HG).$$

A partial answer of the question:

Example 3.6

$\exists f, g \in \pi_3((\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})_{\mathbb{Q}}) \cong \mathbb{Q} \oplus \mathbb{Q}$ such that

$$0 = d_{\mathrm{CohI}}(\Theta(f), \Theta(g)) \not\leq d_{\mathrm{IHC}}(\Theta(f), \Theta(g)) = 1.$$

Remark 3.7

- The map f and g mentioned above are *not* d_{HI} -formal.
- On $\mathbb{C}$, we see that $0 = d_{\mathrm{CohI}}(\Theta(f), \Theta(g)) = d_{\mathrm{IHC}}(\Theta(f), \Theta(g))$.

Summary, future work and perspective

In $\mathcal{M}^{(\mathbb{R}, \leq)}$, $d_{\text{HC}} \leq d_{\text{IHC}} \leq d_{\text{HI}}$ in general.

In $\mathbf{Ch}_{\mathbb{K}}^{(\mathbb{R}, \leq)}$, $d_{\text{Cohl}} = d_{\text{HC}} = d_{\text{IHC}} = d_{\text{HI}}$ in general.

In $\mathbf{et}(\mathbf{CDGA}^{\mathbb{R}_+})$, $d_{\text{Cohl}} \not\leq d_{\text{IHC}}$ for some case.

- $\mathbf{CDGA} \rightsquigarrow \mathbf{DGLie}$, $A_{\text{PL}}(\) \rightsquigarrow \mathcal{L}$: the Quillen Lie model functor.
- In the paper "Magnitude meets persistence", Otter introduces a persistence simplicial set, which is an enriched nerve and gives rise to the *magnitude*, *path* and the *reachability* (co)chain complexes. Then, we may develop the theory of interleavings (interleaving distances etc.) for generalized metric spaces in

$\mathbf{sSet}^{\mathbb{R}_+}$.