# On the category of stratifolds

## – The Serre-Swan Theorem -

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1. Introduction
   - Definition of a stratifold
   - Examples of stratifolds
2. The category of stratifolds
   - **Stfd** embeds into the category of $\mathbb{R}$-algebras (From the talk in Karatsu 2013)
   - The structure sheaf of a stratifold
3. Vector bundles and the Serre-Swan theorem
   - Vector bundles over stratifolds
   - The Serre-Swan theorem for stratifolds
4. Perspective
   - Toward Diffeology for Homotopy Theory of stratifolds

# Definition of a stratifold

**Definition 1.1 (Differential spaces in the sense of Sikorski (1971))**

A *differential space* is a pair $(S, \mathcal{C})$ consisting of a topological space $S$ and an $\mathbb{R}$-subalgebra $\mathcal{C}$ of the $\mathbb{R}$-algebra $C^0(S)$ of continuous real-valued functions on $S$, which is supposed to be *locally detectable* and $C^\infty$*-closed*.

- Local detectability : $f \in \mathcal{C}$ if and only if for any $x \in S$, there exist an open neighborhood $U$ of $x$ and an element $g \in \mathcal{C}$ such that $f|_U = g|_U$.
- $C^\infty$-closedness : For each $n \geq 1$, each $n$-tuple $(f_1, \ldots, f_n)$ of maps in $\mathcal{C}$ and each smooth map $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, the composite $h : S \longrightarrow \mathbb{R}$ defined by $h(x) = g(f_1(x), \ldots., f_n(x))$ belongs to $\mathcal{C}$.

*The tangent space* at $x \in S$.

$$T_xS := \text{The vector space of derivations on } \mathcal{C}_x \text{ the germs at } x$$

## Definition 1.2 (Kreck (2010))

A *stratifold* is a differential space $(S, \mathcal{C})$ such that the following four conditions hold:

1. $S$ is a locally compact Hausdorff space with countable basis;
2. the *skeleta* $sk_k(S) := \{x \in S \mid \dim T_x S \leq k\}$ are closed in $S$;
3. for each $x \in S$ and open neighborhood $U$ of $x$ in $S$, there exists a *bump function* at $x$ subordinate to $U$; that is, a non-negative function $\rho \in \mathcal{C}$ such that $\rho(x) \neq 0$ and such that the support $\operatorname{supp} \rho := \overline{\{p \in S \mid \rho(p) \neq 0\}}$ is contained in $U$;
4. the *strata* $S^k := sk_k(S) - sk_{k-1}(S)$ are $k$-dimensional smooth manifolds such that restriction along $i : S^k \hookrightarrow S$ induces an isomorphism of stalks
$$i^* : \mathcal{C}_x \xrightarrow{\cong} C^\infty(S^k)_x.$$
for each $x \in S^k$.

# Examples

- Let $M$ be a manifold. The open cone of $M$ is defined by

$$CM^\circ := M \times [0,1)/M \times \{0\} \ni [M \times \{0\}] = *$$

$$\mathcal{C} := \left\{ f : CM^\circ \to \mathbb{R} \;\middle|\; \begin{array}{l} f_{|M\times(0,1)} \text{ is smooth, } f_{|U} \text{ is} \\ \text{constant for some open } U \ni * \end{array} \right\}$$

  $(CM^\circ, \mathcal{C})$ is a stratifold with non–empty strata $S^{k+1} = M \times (0,1)$ and $S^0 = *$.

- $(S, \mathcal{C})$ a stratifold, $W$ is a manifold with boundary, which has a collar $c : \partial W \times [0,\epsilon) \xrightarrow{\cong} W$. We have a stratifold

$$(S' = S \cup_f W, \mathcal{C}'),$$

$$C' = \{\, g : S' \to \mathbb{R} \mid g_{|S} \in \mathcal{C}, gc(w,t) = gf(w) \text{ for } w \in \partial W \,\}$$

- Moreover, we have a sub stratifold and the product of stratifolds.

# The category of stratifolds

We assume that all stratifolds are finite-dimensional; $S = sk_n(S)$ for some $n$. Let $(S, \mathcal{C})$ and $(S', \mathcal{C}')$ be stratifolds. We call a continuous map $f : S \to S'$ a *morphism of the stratifolds*, denoted

$$f : (S, \mathcal{C}) \to (S', \mathcal{C}')$$

if $f$ induces a map $f^* : \mathcal{C}' \to \mathcal{C}$; that is, $\varphi \circ f \in \mathcal{C}$ for each $\varphi \in \mathcal{C}'$. Thus we define a category

**Stfd**

of stratifolds.

Theorem 2.1

*The category* **Stfd** *fully faithfully embeds into the category of* $\mathbb{R}$*-algebras.*

# Sketch of the proof

For an $\mathbb{R}$-algebra $\mathcal{F}$, we define

$$|\mathcal{F}| := \text{the set of all morphisms of } \mathbb{R}\text{-algebras from } \mathcal{F} \text{ to } \mathbb{R}$$

Moreover, we define a map $\widetilde{f} : |\mathcal{F}| \to \mathbb{R}$ by $\widetilde{f}(x) = x(f)$ for any $f \in \mathcal{F}$. Let $\widetilde{\mathcal{F}}$ be the $\mathbb{R}$-algebra of maps from $|\mathcal{F}|$ to $\mathbb{R}$ of the form $\widetilde{f}$ for $f \in \mathcal{F}$. Then we consider the Gelfand topology on $|\mathcal{F}|$; that is, $|\mathcal{F}|$ is regarded as the topological space with the open basis

$$\{\widetilde{f}^{-1}(U) \mid U : \text{open in } \mathbb{R}, \widetilde{f} \in \widetilde{\mathcal{F}}\}$$

Thus the assignment of a topological space to an $\mathbb{R}$-algebra gives rise to a contravariant functor

$$|\ \ | : \mathbb{R}\text{-}\mathbf{Alg} \to \mathbf{Top}$$

### Lemma 2.2 (Using a bump function)

*Let* $(S, \mathcal{C})$ *be a stratifold. Then the map* $\theta : S \to |\mathcal{C}|$ *defined by* $\theta(p)(f) = f(p)$ *is a homeomorphism.*

### Proposition 2.3

*The map* $\theta : S \to |\mathcal{C}|$ *gives rise to an isomorphism of continuous spaces*

$$\theta : (S, \mathcal{C}) \to (|\mathcal{C}|, \widetilde{\mathcal{C}})$$

### Theorem 2.4

*The forgetful functor* $F$ : **Stfd** $\to$ $\mathbb{R}$-**Alg** *defined by* $F(S, \mathcal{C}) = \mathcal{C}$ *is fully faithful; that is, the induced map*

$$F : \mathrm{Hom}_{\mathbf{Stfd}}((S, \mathcal{C}), (S', \mathcal{C}')) \to \mathrm{Hom}_{\mathbb{R}\text{-}\mathbf{Alg}}(\mathcal{C}', \mathcal{C})$$

*is a bijection.*

# The structure sheaf of a stratifold

- A maximal ideal $\mathbf{m}$ of $\mathcal{C}$ real $\overset{\text{def}}{\Longleftrightarrow}$ the quotient $\mathcal{C}/\mathbf{m}$ is isomorphic to $\mathbb{R}$ as an $\mathbb{R}$-algebra.
- $\mathrm{Spec}_r\ \mathcal{C}$ : the *real spectrum*, i.e. the subset of the prime spectrum $\mathrm{Spec}\ \mathcal{C}$ of $\mathcal{C}$ consisting of real ideals. We consider $\mathrm{Spec}_r\ \mathcal{C}$ the subspace of $\mathrm{Spec}\ \mathcal{C}$ with the Zariski topology.
- A map $u : |\mathcal{C}| \longrightarrow \mathrm{Spec}_r\ \mathcal{C}$ defined by $u(\varphi) = \mathrm{Ker}\ \varphi$ is bijective. Moreover, the map $u$ is continuous. In fact,
  for an open base $D(f) = \{\mathbf{m} \in \mathrm{Spec}_r\ \mathcal{C} \mid f \notin \mathbf{m}\}$ for some $f \in \mathcal{C}$, we see that $u^{-1}(D(f)) = \widetilde{f}^{-1}(\mathbb{R}\backslash\{0\})$.

**Proposition 2.5**

*The bijection $u : |\mathcal{C}| \xrightarrow{\cong} \mathrm{Spec}_r\ \mathcal{C}$ is a homeomorphism.*

$$S \cong |\mathcal{C}| \cong \mathrm{Spec}_r\ \mathcal{C} \subset \mathrm{Spec}\ \mathcal{C}.$$

## Theorem 2.6

*Let $(S, \mathcal{O}_S)$ be a ringed space which comes from a stratifold $(S, \mathcal{C})$ and $i$ : $\mathrm{Spec}_r\mathcal{O}_S(S) \longrightarrow \mathrm{Spec}\,\mathcal{O}_S(S)$ the inclusion. Then $(S, \mathcal{O}_S)$ is isomorphic to $i^*(\mathrm{Spec}\,\mathcal{O}_S(S), \widetilde{\mathcal{O}_S(S)})$ as a ringed space, where $(\mathrm{Spec}\,\mathcal{O}_S(S), \widetilde{\mathcal{O}_S(S)})$ is the affine scheme associated with the ring $\mathcal{O}_S(S)$.*

## Sketch of the proof.

Let $m : S \xrightarrow{\cong} |S| \xrightarrow{\cong} \mathrm{Spec}_r\mathcal{O}_S(S)$. It suffices to show that $(S, \mathcal{O}_S)$ is isomorphic to the structure sheaf $(\mathrm{Spec}_r\mathcal{O}_S(S), \widehat{\mathcal{O}_S(S)})$. To this end, we construct an isomorphism from $\widehat{\mathcal{O}_S(S)}$ to $m_*\mathcal{O}_S$. For an open set $U$ of $\mathrm{Spec}_r\mathcal{O}_S(S)$, we define

$$\alpha_U : M_U^{-1}\mathcal{O}_S(S) \to (m_*\mathcal{O}_S)(U)$$

by $\alpha([f/s]) = f \cdot \frac{1}{s}$, where $M_U := \bigcap_{\mathfrak{m} \in U} \mathfrak{m}^c$. □

## Definition 3.1 (A vector bundle over a stratifold)

Let $(S, \mathcal{C}_S)$ be a stratifold and $(E, \mathcal{C}_E)$ a differential space. A morphism of differential spaces $\pi : (E, \mathcal{C}_E) \to (S, \mathcal{C}_S)$ is a *vector bundle* over $(S, \mathcal{C}_S)$ if the following conditions are satisfied.

1. $E_x := \pi^{-1}(x)$ is a vector space over $\mathbb{R}$ for $x \in S$.
2. There exist an open cover $\{U_\alpha\}_{\alpha \in J}$ of $S$ and an isomorphism $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n_\alpha}$ of differential spaces for each $\alpha \in J$. Here $\pi^{-1}(U_\alpha)$ is regarded as a differential subspace of $(E, \mathcal{C}_E)$ and $U_\alpha \times \mathbb{R}^{n_\alpha}$ is considered the product of the substratifold $(U_\alpha, \mathcal{C}_{U_\alpha})$ of $(S, \mathcal{C}_S)$ and the manifold $(\mathbb{R}^{n_\alpha}, C^\infty(\mathbb{R}^{n_\alpha}))$.
3. The diagram
$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\phi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^{n_\alpha} \arrow[dl, "pr_1"] \\ & U_\alpha & \end{tikzcd}$$
is commutative, where $pr_1$ is the projection onto the first factor.
4. The composite $pr_2 \circ \phi_\alpha|_{E_x} : E_x \to U_\alpha \times \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_\alpha}$ is a linear isomorphism, where $pr_2 : U_\alpha \times \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_\alpha}$ denotes the projection onto the second factor.

## Proposition 3.2

*The transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{R})$ are morphisms of stratifolds.*

## Proposition 3.3

*Let $\pi : (E, \mathcal{C}_E) \longrightarrow (S, \mathcal{C}_S)$ be a vector bundle in the sense of Definition 3.1. Then the differential space $(E, \mathcal{C}_E)$ admits a stratifold structure for which $\pi$ is a morphism of stratifolds.*

By virtue of Proposition 3.2, we see that $\pi : \pi^{-1}(S^i) \longrightarrow S^i$ is a smooth vector bundle.

$$\begin{array}{ccc}
(\mathcal{C}_E)_x & \xrightarrow{\quad i^* \quad} & C^\infty(\pi^{-1}(S^i))_x \\
{\scriptstyle res^*}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle res^*} \\
(\mathcal{C}_{\pi^{-1}(U_\alpha)})_x & \xrightarrow{\quad i^* \quad} & C^\infty(\pi^{-1}(S^i \cap U_\alpha))_x \\
{\scriptstyle \phi_\alpha^*}\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow{\scriptstyle \phi_\alpha^*} \\
(\mathcal{C}_{U_\alpha \times \mathbb{R}^n})_{\phi_\alpha(x)} & \xrightarrow{(i \times 1_{\mathbb{R}^n})^*} & \mathcal{C}(S^i \cap U_\alpha \times \mathbb{R}^n)_{\phi_\alpha(x)}
\end{array}$$

Since $U_\alpha \times \mathbb{R}^n$ is a stratifold, we see that $(i \times 1_{\mathbb{R}^n})^*$ is an isomorphism.

# The Serre-Swan theorem for stratifolds

We denote by $\textbf{VBb}_{(S,\mathcal{C})}$ the category of vector bundles over $(S,\mathcal{C})$ of bounded rank.

Theorem 3.4

*Let $(S,\mathcal{C})$ be a stratifold. Then the global section functor*

$$\Gamma(S,-) : \textbf{VBb}_{(S,\mathcal{C})} \to \textbf{Fgp}(\mathcal{C})$$

*gives rise to an equivalence of categories, where $\textbf{Fgp}(\mathcal{C})$ denotes the category of finitely generated projective modules over $\mathcal{C}$.*

Let $\textbf{Lfb}(S)$ be the full subcategory of $\mathcal{O}_S$-$\textbf{Mod}$ consisting of locally free $\mathcal{O}_S$-modules of bounded rank. We define a functor $\mathcal{L} : \textbf{VBb}_{(S,\mathcal{C})} \longrightarrow \textbf{Lfb}(S)$ by $\mathcal{L}_E : U \rightsquigarrow \Gamma(U,E)$, which is fully faithful and essentially surjective.

### Theorem 3.5 (Morye (2013))

*Let $(X, \mathcal{O}_X)$ be a locally ringed space such that $X$ is a paracompact Hausdorff space of finite covering dimension, and $\mathcal{O}_X$ is a fine sheaf of rings. Then the Serre-Swan theorem holds for $(X, \mathcal{O}_X)$; that is, the global section functor induces an equivalence of categories between* $\mathbf{Lfb}(X)$ *and* $\mathbf{Fgp}(\Gamma(X, \mathcal{O}_X))$.

### Corollary 3.6

*Let $(S, \mathcal{C})$ be a stratifold and $\mathcal{O}_S$ the structure sheaf. Then the global sections functor* $\Gamma(S, -) : \mathbf{Lfb}(S) \to \mathbf{Fgp}(\mathcal{C})$ *is an equivalence.*

$$\begin{tikzcd}
\mathbf{VBb}_{(S,\mathcal{C})} \arrow[rr, "\Gamma(S,-)", "\text{Theorem 3.4}"'] \arrow[dr, "\simeq", "\mathcal{L}"'] & & \mathbf{Fgp}(\mathcal{C}) & \\
& \mathbf{Lfb}(S) \arrow[ur, "\simeq", "\Gamma(S,-)"'] & & \mathbf{Lfb}(\mathrm{Spec}\ \mathcal{C}) \arrow[ul, "\text{Serre}"', "\simeq"]
\end{tikzcd}$$

# Perspective – Toward Diffeology for Homotopy Theory of stratifolds –

Let **Diffeology** be the category of diffeological spaces. We define a functor

$$k : \mathbf{Stfd} \longrightarrow \mathbf{Diffeology}$$

by $k(S, \mathcal{C}) = (S, \mathcal{D}_{\mathcal{C}})$ and $k(\phi) = \phi$ for a morphism $\phi : S \longrightarrow S'$ of stratifolds, where

$$\mathcal{D}_{\mathcal{C}} := \left\{ u : U \to S \;\middle|\; \begin{array}{l} U : \text{open in } \mathbb{R}^q, q \geq 0, \\ \phi \circ u \in C^\infty(U) \text{ for any } \phi \in \mathcal{C} \end{array} \right\}.$$

The functor $k$ is faithful, but not full; that is, for a continuous map $f : S \longrightarrow S'$, it is more restrictive to be a morphism of stratifolds $(S, \mathcal{C}) \longrightarrow (S', \mathcal{C}')$ than to be a morphism of diffeological spaces $(S, \mathcal{D}_{\mathcal{C}}) \longrightarrow (S', \mathcal{D}_{\mathcal{C}'})$.

$$\begin{tikzcd}
 & & \mathbf{Sets}^{\square^{op}} & \\
\mathbf{Mfd} \arrow[r, "\text{fully faithful}", "j"'] \arrow[rr, bend left, "\ell\text{:fully faithful}"] & \mathbf{Stfd} \arrow[r, "k"'] & \mathbf{Diffeology} \arrow[u, "S_\square"'] \arrow[r, shift left=2, "D"] \arrow[r, phantom, "\perp"] \arrow[d, shift left=2, "S^D"] \arrow[d, phantom, "\dashv"] & \mathbf{Top}, \arrow[l, shift left=2, "C"] \\
 & & \mathbf{Sets}^{\Delta^{op}} \arrow[u, shift left=2, "|\ |_D"] &
\end{tikzcd}$$

- Haraguchi and Shimakawa are considering a model structure of **Diffeology** with the adjoint pair $(D, S)$. (2013 – )
- Christensen and Wu have studied a model structure of **Diffeology** with the adjoint pair $(|\ |_D, S^D)$, where $S^D(X) := \{\mathbb{A}^n \to X : \text{smooth}\}$ and $\mathbb{A}^n := \{(t_0, \ldots, t_n) \in \mathbb{R}^n \mid \sum t_i = 1\}$, the "non-compact $n$-simplex". (2014)
- Kihara has given a model structure to **Diffeology** with an adjoint pair given by modifying $(|\ |_D, S^D)$, more precisely, changing the diffeological structure of $\Delta^n$. (2016, 2017)

- Iwase and Izumida (2015) have considerd de Rham theorem in **Diffeology** using $S_\square$ and the cubical differential forms

Let $\Omega^*_{DR}(X)$ be the de Rham complex of a diffeological space $(X, \mathcal{D}^X)$ in the sense of Iglesias-Zemmour.

$$\Omega^p_{DR}(X) := \left\{ \begin{tikzcd} \textbf{Open} \arrow[r, bend left, "\mathcal{D}^X"] \arrow[r, bend right, "\wedge^p"'] \arrow[r, phantom, "\Downarrow\omega"] & \textbf{Sets} \end{tikzcd} \;\middle|\; \text{natural trans.} \right\}$$

$\wedge^*(U) = \{U \xrightarrow{\text{smooth}} \wedge^*(\oplus_{i=1}^{\dim U}\mathbb{R}dx_i)\}$: the usual de Rham complex on $U$

Theorem 4.1 (Iwase - Izumida (2015))

*For a CW complex $X$, one has isomorphisms*

$$H^*(X;\mathbb{R}) \cong H^*(S_\square(X)) \cong H(\text{"a cubical de Rhma complex" of } X) \cong H(\Omega^*_{DR}(X))$$

- For a simplicial set $K$, $C^*(K;\mathbb{R})$ denotes the normalized cochain algebra.
- We have two simplicial DGA $C^{\Delta} := C^*(\Delta[\bullet])$ and $\Omega^{\Delta} := \Omega^*_{DR}(\mathbb{A}^{\bullet})$. Define cochain algebra $A(K) := \mathbf{Sets}^{\Delta^{\mathrm{op}}}(K, A_\bullet)$ for a simplicial set $K$ and a simplicial DGA $A_\bullet$.

### Assertion 4.2 (Emoto - K. (Work in progress))

*For a diffeology $(X, \mathcal{D}_X)$, one has a commutative diagram*

$$\begin{tikzcd}
C^*(S^D(X)) \arrow[r, "\simeq", "\varphi"'] \arrow[dr, "="'] & (C^{\Delta}\otimes\Omega^{\Delta})(S^D(X)) \arrow[d, "\mathrm{mult}\circ(1\otimes\int)"'] & \Omega^{\Delta}(S^D(X)) \arrow[l, "\simeq"', "\psi"] & \Omega_{DR}(X) \arrow[l, "\alpha"'] \arrow[dll, "\text{“integration” }\int"] \\
 & C^*(S^D(X)) & &
\end{tikzcd}$$

*in which $\varphi$ and $\psi$ are quasi-isomorphism of DGAs and $\alpha$ is a DGA map. Moreover, if $(X, \mathcal{D})$ comes from a stratifolds, then $\alpha$ is a quasi-iso. and hence $\int$ is an isomorphism of graded algebras on the cohomology. We get the “de Rham theorem” for stratifolds.*

## A little more perspective

- Rational homotopy theory uses $(A^*_{PL})_\bullet$ the simplicial DG algebra of polynomial (rational) differential forms.
- Real homotopy theory in the sense of Brown and Szczarba uses $\Omega^*_{\text{de Rham}}(\Delta^\bullet)$ the usual de Rham complex on the standard simplexes, which is regarded as the simplicial DG *topological* algebra.
- Smooth homotopy theory may use $\Omega^\Delta := \Omega^*_{DR}(\mathbb{A}^\bullet)$, $\Omega^*_{DR}(\Delta^\bullet_{\text{sub}})$ or $\Omega^*_{DR}(\Delta^\bullet_{\text{Kihara}})$, which is considered a simplicial DG *diffeological* algebra.

E. Wu, Homological algebra for diffeological vector spaces, Homology Homotopy Appl. **17** (2015), 339–376.