On the category of stratifolds
– The Serre-Swan Theorem –

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2017 年度ホモトピー論シンポジウム
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This is joint work with Toshiki Aoki.


Acknowledgements. The second author thanks Takayoshi Aoki and Wakana Otsuka for considerable discussions on stratifolds.

This research was partially supported by a Grant-in-Aid for challenging Exploratory Research 16K13753 from Japan Society for the Promotion of Science.
1 Introduction
   • Definition of a stratifold
   • Examples of stratifolds

2 The category of stratifolds
   • **Stfd** embeds into the category of \( \mathbb{R} \)-algebras (From the talk in Karatsu 2013)
   • The structure sheaf of a stratifold

3 Vector bundles and the Serre-Swan theorem
   • Vector bundles over stratifolds
   • The Serre-Swan theorem for stratifolds

4 Perspective
   • Toward Diffeology for Homotopy Theory of stratifolds
Definition of a stratifold

Definition 1.1 (Differential spaces in the sense of Sikorski (1971))

A differential space is a pair \((S, \mathcal{C})\) consisting of a topological space \(S\) and an \(\mathbb{R}\)-subalgebra \(\mathcal{C}\) of the \(\mathbb{R}\)-algebra \(C^0(S)\) of continuous real-valued functions on \(S\), which is supposed to be locally detectable and \(C^\infty\)-closed.

- **Local detectability**: \(f \in \mathcal{C}\) if and only if for any \(x \in S\), there exist an open neighborhood \(U\) of \(x\) and an element \(g \in \mathcal{C}\) such that \(f|_U = g|_U\).
- **\(C^\infty\)-closedness**: For each \(n \geq 1\), each \(n\)-tuple \((f_1, \ldots, f_n)\) of maps in \(\mathcal{C}\) and each smooth map \(g : \mathbb{R}^n \to \mathbb{R}\), the composite \(h : S \to \mathbb{R}\) defined by \(h(x) = g(f_1(x), \ldots, f_n(x))\) belongs to \(\mathcal{C}\).

The tangent space at \(x \in S\).

\[ T_x S := \text{The vector space of derivations on } \mathcal{C}_x \text{ the germs at } x \]
Definition 1.2 (Kreck (2010))

A stratifold is a differential space \((S, C)\) such that the following four conditions hold:

1. \(S\) is a locally compact Hausdorff space with countable basis;
2. the skeleta \(s_k(S) := \{x \in S \mid \dim T_x S \leq k\}\) are closed in \(S\);
3. for each \(x \in S\) and open neighborhood \(U\) of \(x\) in \(S\), there exists a bump function at \(x\) subordinate to \(U\); that is, a non-negative function \(\rho \in C\) such that \(\rho(x) \neq 0\) and such that the support \(\text{supp} \rho := \{p \in S \mid \rho(p) \neq 0\}\) is contained in \(U\);
4. the strata \(S^k := s_k(S) - s_{k-1}(S)\) are \(k\)-dimensional smooth manifolds such that restriction along \(i : S^k \hookrightarrow S\) induces an isomorphism of stalks

\[
i^* : C_x \xrightarrow{\cong} C^\infty(S^k)_x.
\]

for each \(x \in S^k\).
Introduction

Examples

Let $M$ be a manifold. The open cone of $M$ is defined by

$$CM^\circ := M \times [0, 1)/M \times \{0\} \ni [M \times \{0\}] = *$$

$$C := \left\{ f : CM^\circ \to \mathbb{R} \middle| f_{|M \times (0, 1)} \text{ is smooth, } f_{|U} \text{ is constant for some open } U \ni * \right\}$$

$(CM^\circ, C)$ is a stratifold with non-empty strata $S^{k+1} = M \times (0, 1)$ and $S^0 = *$.

$(S, C)$ a stratifold, $W$ is a manifold with boundary, which has a collar $c : \partial W \times [0, \epsilon) \xrightarrow{\cong} W$. We have a stratifold

$$(S' = S \cup_f W, C'),$$

$$C' = \left\{ g : S' \to \mathbb{R} \middle| g_{|S} \in C, gc(w, t) = gf(w) \text{ for } w \in \partial W \right\}$$

Moreover, we have a sub stratifold and the product of stratifolds.
The category of stratifolds

We assume that all stratifolds are finite-dimensional; $S = sk_n(S)$ for some $n$. Let $(S, C)$ and $(S', C')$ be stratifolds. We call a continuous map $f : S \rightarrow S'$ a morphism of the stratifolds, denoted

$$f : (S, C) \rightarrow (S', C')$$

if $f$ induces a map $f^* : C' \rightarrow C$; that is, $\varphi \circ f \in C$ for each $\varphi \in C'$. Thus we define a category

$$\text{Stfd}$$

of stratifolds.

Theorem 2.1

*The category $\text{Stfd}$ fully faithfully embeds into the category of $\mathbb{R}$-algebras.*
Sketch of the proof

For an $\mathbb{R}$-algebra $\mathcal{F}$, we define

$$|\mathcal{F}| := \text{the set of all morphisms of } \mathbb{R}\text{-algebras from } \mathcal{F} \text{ to } \mathbb{R}$$

Moreover, we define a map $\tilde{f} : |\mathcal{F}| \to \mathbb{R}$ by $\tilde{f}(x) = x(f)$ for any $f \in \mathcal{F}$. Let $\tilde{\mathcal{F}}$ be the $\mathbb{R}$-algebra of maps from $|\mathcal{F}|$ to $\mathbb{R}$ of the form $\tilde{f}$ for $f \in \mathcal{F}$. Then we consider the Gelfand topology on $|\mathcal{F}|$; that is, $|\mathcal{F}|$ is regarded as the topological space with the open basis

$$\{\tilde{f}^{-1}(U) \mid U : \text{open in } \mathbb{R}, \tilde{f} \in \tilde{\mathcal{F}}\}$$

Thus the assignment of a topological space to an $\mathbb{R}$-algebra gives rise to a contravariant functor

$$| \cdot | : \mathbb{R}\text{-Alg} \to \text{Top}$$
Lemma 2.2 (Using a bump function)

Let \((S, \mathcal{C})\) be a stratifold. Then the map \(\theta : S \to |\mathcal{C}|\) defined by \(\theta(p)(f) = f(p)\) is a homeomorphism.

Proposition 2.3

The map \(\theta : S \to |\mathcal{C}|\) gives rise to an isomorphism of continuous spaces

\[ \theta : (S, \mathcal{C}) \to (|\mathcal{C}|, \tilde{\mathcal{C}}) \]

Theorem 2.4

The forgetful functor \(F : \text{Stfd} \to \text{R-Alg}\) defined by \(F(S, \mathcal{C}) = \mathcal{C}\) is fully faithful; that is, the induced map

\[ F : \text{Hom}_{\text{Stfd}}((S, \mathcal{C}), (S', \mathcal{C}')) \to \text{Hom}_{\text{R-Alg}}(\mathcal{C}', \mathcal{C}) \]

is a bijection.
The category of stratifolds

The structure sheaf of a stratifold

- A maximal ideal $m$ of $\mathcal{C}$ real $\overset{\text{def}}{\leftrightarrow}$ the quotient $\mathcal{C}/m$ is isomorphic to $\mathbb{R}$ as an $\mathbb{R}$-algebra.

- $\text{Spec}_r \mathcal{C}$: the real spectrum, i.e. the subset of the prime spectrum $\text{Spec} \mathcal{C}$ of $\mathcal{C}$ consisting of real ideals. We consider $\text{Spec}_r \mathcal{C}$ the subspace of $\text{Spec} \mathcal{C}$ with the Zariski topology.

- A map $u : |\mathcal{C}| \to \text{Spec}_r \mathcal{C}$ defined by $u(\varphi) = \text{Ker} \varphi$ is bijective. Moreover, the map $u$ is continuous. In fact, for an open base $D(f) = \{m \in \text{Spec}_r \mathcal{C} \mid f \notin m\}$ for some $f \in \mathcal{C}$, we see that $u^{-1}(D(f)) = \tilde{f}^{-1}(\mathbb{R}\setminus\{0\})$.

Proposition 2.5

The bijection $u : |\mathcal{C}| \overset{\cong}{\to} \text{Spec}_r \mathcal{C}$ is a homeomorphism.

\[ S \cong |\mathcal{C}| \cong \text{Spec}_r \mathcal{C} \subset \text{Spec} \mathcal{C}. \]
Theorem 2.6

Let $(S, \mathcal{O}_S)$ be a ringed space which comes from a stratifold $(S, C)$ and $i : \text{Spec}_r \mathcal{O}_S(S) \to \text{Spec} \mathcal{O}_S(S)$ the inclusion. Then $(S, \mathcal{O}_S)$ is isomorphic to $i^*(\text{Spec} \mathcal{O}_S(S), \mathcal{O}_S(S))$ as a ringed space, where $(\text{Spec} \mathcal{O}_S(S), \mathcal{O}_S(S))$ is the affine scheme associated with the ring $\mathcal{O}_S(S)$.

Sketch of the proof.

Let $m : S \xrightarrow{\cong} |S| \xrightarrow{\cong} \text{Spec}_r \mathcal{O}_S(S)$. It suffices to show that $(S, \mathcal{O}_S)$ is isomorphic to the structure sheaf $(\text{Spec}_r \mathcal{O}_S(S), \mathcal{O}_S(S))$. To this end, we construct an isomorphism from $\mathcal{O}_S(S)$ to $m_* \mathcal{O}_S$. For an open set $U$ of $\text{Spec}_r \mathcal{O}_S(S)$, we define

$$\alpha_U : M_U^{-1} \mathcal{O}_S(S) \to (m_* \mathcal{O}_S)(U)$$

by $\alpha([f/s]) = f \cdot \frac{1}{s}$, where $M_U := \bigcap_{m \in U} m^c$. □
Definition 3.1 (A vector bundle over a stratifold)

Let \((S, \mathcal{C}_S)\) be a stratifold and \((E, \mathcal{C}_E)\) a differential space. A morphism of differential spaces \(\pi : (E, \mathcal{C}_E) \to (S, \mathcal{C}_S)\) is a vector bundle over \((S, \mathcal{C}_S)\) if the following conditions are satisfied.

1. \(E_x := \pi^{-1}(x)\) is a vector space over \(\mathbb{R}\) for \(x \in S\).

2. There exist an open cover \(\{U_\alpha\}_{\alpha \in J} \) of \(S\) and an isomorphism \(\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n_\alpha}\) of differential spaces for each \(\alpha \in J\). Here \(\pi^{-1}(U_\alpha)\) is regarded as a differential subspace of \((E, \mathcal{C}_E)\) and \(U_\alpha \times \mathbb{R}^{n_\alpha}\) is considered the product of the substratifold \((U_\alpha, \mathcal{C}_{U_\alpha})\) of \((S, \mathcal{C}_S)\) and the manifold \((\mathbb{R}^{n_\alpha}, \mathcal{C}^\infty(\mathbb{R}^{n_\alpha}))\).

3. The diagram \(\pi^{-1}(U_\alpha) \xrightarrow{\phi_\alpha} U_\alpha \times \mathbb{R}^{n_\alpha}\) is commutative,

\[
\begin{array}{ccc}
\pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times \mathbb{R}^{n_\alpha} \\
\downarrow{\pi} & & \downarrow{pr_1} \\
U_\alpha & & 
\end{array}
\]

where \(pr_1\) is the projection onto the first factor.

4. The composite \(pr_2 \circ \phi_\alpha|_{E_x} : E_x \to U_\alpha \times \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_\alpha}\) is a linear isomorphism, where \(pr_2 : U_\alpha \times \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_\alpha}\) denotes the projection onto the second factor.
**Proposition 3.2**

The transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL_n(\mathbb{R})$ are morphisms of stratifolds.

**Proposition 3.3**

Let $\pi : (E, C_E) \rightarrow (S, C_S)$ be a vector bundle in the sense of Definition 3.1. Then the differential space $(E, C_E)$ admits a stratifold structure for which $\pi$ is a morphism of stratifolds.

By virtue of Proposition 3.2, we see that $\pi : \pi^{-1}(S^i) \rightarrow S^i$ is a smooth vector bundle.

\[
\begin{array}{ccc}
(C_E)_x & \xrightarrow{i^*} & C^\infty(\pi^{-1}(S^i))_x \\
\downarrow \cong & & \downarrow \cong \\
(C_{\pi^{-1}(U_\alpha)})_x & \xrightarrow{i^*} & C^\infty(\pi^{-1}(S^i \cap U_\alpha))_x \\
\uparrow \cong & & \uparrow \cong \\
(C_{U_\alpha \times \mathbb{R}^n})_{\phi_\alpha}(x) & \xrightarrow{(i \times 1_{\mathbb{R}^n})^*} & C(S^i \cap U_\alpha \times \mathbb{R}^n)_{\phi_\alpha}(x)
\end{array}
\]

Since $U_\alpha \times \mathbb{R}^n$ is a stratifield, we see that $(i \times 1_{\mathbb{R}^n})^*$ is an isomorphism.
The Serre-Swan theorem for stratifolds

We denote by $\mathbf{VBb}_{(S,\mathcal{C})}$ the category of vector bundles over $(S,\mathcal{C})$ of bounded rank.

**Theorem 3.4**

Let $(S,\mathcal{C})$ be a stratifold. Then the global section functor

$$\Gamma(S, -) : \mathbf{VBb}_{(S,\mathcal{C})} \to \mathbf{Fgp} (\mathcal{C})$$

gives rise to an equivalence of categories, where $\mathbf{Fgp} (\mathcal{C})$ denotes the category of finitely generated projective modules over $\mathcal{C}$.

Let $\mathbf{Lfb}(S)$ be the full subcategory of $\mathcal{O}_S$-$\mathbf{Mod}$ consisting of locally free $\mathcal{O}_S$-modules of bounded rank. We define a functor $\mathbf{L} : \mathbf{VBb}_{(S,\mathcal{C})} \to \mathbf{Lfb}(S)$ by $\mathbf{L}_E : U \leadsto \Gamma(U, E)$, which is fully faithful and essentially surjective.
**Theorem 3.5 (Morye (2013))**

Let \((X, \mathcal{O}_X)\) be a locally ringed space such that \(X\) is a paracompact Hausdorff space of finite covering dimension, and \(\mathcal{O}_X\) is a fine sheaf of rings. Then the Serre-Swan theorem holds for \((X, \mathcal{O}_X)\); that is, the global section functor induces an equivalence of categories between \(\text{Lfb}(X)\) and \(\text{Fgp}(\Gamma(X, \mathcal{O}_X))\).

**Corollary 3.6**

Let \((S, \mathcal{C})\) be a stratifold and \(\mathcal{O}_S\) the structure sheaf. Then the global sections functor \(\Gamma(S, -) : \text{Lfb}(S) \to \text{Fgp}(\mathcal{C})\) is an equivalence.

\[
\begin{array}{c}
\text{VBb}(S, \mathcal{C}) \\
\rightsquigarrow \sim \Gamma(S, -) \\
\sim \mathcal{L} \\
\sim \text{Lfb}(S) \\
\sim \Gamma(S, -) \\
\text{Lfb}(\text{Spec } \mathcal{C})
\end{array}
\]

Serre

\(\text{Fgp}(\mathcal{C})\)
Let **Diffeology** be the category of diffeological spaces. We define a functor

\[ k : \text{Stfd} \rightarrow \text{Diffeology} \]

by \( k(S, C) = (S, D_C) \) and \( k(\phi) = \phi \) for a morphism \( \phi : S \rightarrow S' \) of stratifolds, where

\[
D_C := \left\{ u : U \rightarrow S \mid \begin{array}{l}
U : \text{open in } \mathbb{R}^q, q \geq 0, \\
\phi \circ u \in C^\infty(U) \text{ for any } \phi \in C
\end{array} \right\}.
\]

The functor \( k \) is faithful, but not full; that is, for a continuous map \( f : S \rightarrow S' \), it is more restrictive to be a morphism of stratifolds \( (S, C) \rightarrow (S', C') \) than to be a morphism of diffeological spaces \( (S, D_C) \rightarrow (S', D_{C'}) \).
Haraguchi and Shimakawa are considering a model structure of \textit{Diffeology} with the adjoint pair \((D, S)\). (2013 – )

Christensen and Wu have studied a model structure of \textit{Diffeology} with the adjoint pair \((|D, S^D)\), where \(S^D(X) := \{\mathbb{A}^n \to X : \text{smooth}\}\) and \(\mathbb{A}^n := \{(t_0, ..., t_n) \in \mathbb{R}^n \mid \sum t_i = 1\}\), the “non-compact \(n\)-simplex”. (2014)

Kihara has given a model structure to \textit{Diffeology} with an adjoint pair given by modifying \((|D, S^D)\), more precisely, changing the diffeological structure of \(\Delta^n\). (2016, 2017)
Iwase and Izumida (2015) have considered de Rham theorem in Diffeology using $S_{\square}$ and the cubical differential forms.

Let $\Omega_{DR}^*(X)$ be the de Rham complex of a diffeological space $(X, \mathcal{D}^X)$ in the sense of Iglesias-Zemmour.

$$\Omega_{DR}^p(X) := \left\{ \begin{array}{c} \text{Open} \xrightarrow{\mathcal{D}^X} \text{Sets} \\ \wedge^p \end{array} \right\}$$

$$\wedge^*(U) = \{ U \text{ smooth} \rightarrow \wedge^*(\bigoplus_{i=1}^{\dim U} \mathbb{R} dx_i) \}: \text{the usual de Rham complex on } U$$

Theorem 4.1 (Iwase - Izumida (2015))

For a CW complex $X$, one has isomorphisms

$$H^*(X; \mathbb{R}) \cong H^*(S_{\square}(X)) \cong H(\text{"a cubical de Rham complex" of } X) \cong H(\Omega_{DR}^*(X))$$
For a simplicial set $K$, $C^*(K; \mathbb{R})$ denotes the normalized cochain algebra.

We have two simplicial DGA $C^\Delta := C^*(\Delta[\bullet])$ and $\Omega^\Delta := \Omega^*_{DR}(A^\bullet)$.

Define cochain algebra $A(K) := \text{Sets}^{A^\bullet \text{op}}(K, A^\bullet)$ for a simplicial set $K$ and a simplicial DGA $A^\bullet$.

**Assertion 4.2 (Emoto - K. (Work in progress))**

For a diffeology $(X, D_X)$, one has a commutative diagram

\[
\begin{array}{ccc}
C^*(S^D(X)) & \xrightarrow{\varphi} & (C^\Delta \otimes \Omega^\Delta)(S^D(X)) \\
& \xrightarrow{\psi} & \Omega^\Delta(S^D(X)) \\
& \xleftarrow{\alpha} & \Omega_{DR}(X)
\end{array}
\]

in which $\varphi$ and $\psi$ are quasi-isomorphism of DGAs and $\alpha$ is a DGA map. Moreover, if $(X, D)$ comes from a stratifolds, then $\alpha$ is a quasi-iso. and hence $\int$ is an isomorphism of graded algebras on the cohomology. We get the “de Rham theorem” for stratifolds.
A little more perspective

- **Rational homotopy theory** uses \((A_{PL}^\bullet)\), the simplicial DG algebra of polynomial (rational) differential forms.

- **Real homotopy theory** in the sense of Brown and Szczarba uses \(\Omega^*_\text{de Rham}(\Delta^\bullet)\), the usual de Rham complex on the standard simplexes, which is regarded as the simplicial DG *topological* algebra.

- **Smooth homotopy theory** may use \(\Omega^\Delta := \Omega^*_{DR}(A^\bullet), \Omega^*_{DR}(\Delta^\bullet_{\text{sub}})\) or \(\Omega^*_{DR}(\Delta^\bullet_{\text{Kihara}})\), which is considered a simplicial DG *diffeological* algebra.