ON THE LEVELS OF SPACES

KATSUHIKO KURIBAYASHI

Abstract. The level of a module over a differential graded algebra measures the number of steps to build the module in an appropriate triangulated category. We define the levels for spaces and investigate the invariant of spaces over a $\mathbb{K}$-formal space. In particular, the level of the total space of a bundle over the 4-dimensional sphere is computed with the aid of Auslander-Reiten theory over spaces due to Jørgensen. A general method for computing the level of a space is also described.

1. Introduction

In this article, we survey the results obtained in [19] and [20].

Categorical representation theory provides important technical tools and ideas in the study of many areas of mathematics including finite group theory, algebraic geometry and algebraic topology. Triangles and quivers which appear in Auslander-Reiten theory are examples of such tools; see, for example, [8], [9] and [12].

Let A be a simply-connected differential graded algebra over a field of characteristic zero and $\mathrm{D}(A)$ the derived category of differential graded modules over A. Recently, Jørgensen [13] has proved that the full subcategory $\mathrm{D}^c(A)$ of $\mathrm{D}(A)$, which consists of compact objects, has the Auslander-Reiten triangles if the cohomology of A is a Poincaré duality algebra. It is also proved in [14] that each component of the Auslander-Reiten quiver is of the form $\mathbb{Z}A_\infty$.

Very recently, Schmidt [26] has shown that the result on Auslander-Reiten components holds even if the characteristic of the underlying field is positive; see also [15]. Thus the singular cochain complex functor $C^*(\ ;\mathbb{K})$ with coefficients in a field $\mathbb{K}$ makes an appropriate space over a Poincaré space X into an object in $\mathrm{D}^c(C^*(X;\mathbb{K}))$ in which Auslander-Reiten theory is applicable; see Section 4 for more details.

The notion of levels of differential graded modules over a differential graded algebra A was introduced by Avramov, Buchweitz, Iyengar and Miller in [1]. For an object M of $\mathrm{D}(A)$, the level of M counts the number of steps required to build M out of A via triangles in $\mathrm{D}(A)$. We then define *the level of a space Y over a space X* to be that of $C^*(X;\mathbb{K})$-module $C^*(Y;\mathbb{K})$ in $\mathrm{D}(C^*(X;\mathbb{K}))$.

2. The level of a space

To define the level of a space precisely, we begin by recalling from [1] *thickenings* which are full subcategories of a triangulated category $\mathcal{T}$. For a given object G of $\mathcal{T}$,

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Department of Mathematical Sciences, Faculty of Science, Shinshu University, Matsumoto, Nagano 390-8621, Japan e-mail:kuri@math.shinshu-u.ac.jp

we first define the 0th thickening by $\mathrm{thick}^0_{\mathcal{T}}(G) = \{0\}$ and $\mathrm{thick}^1_{\mathcal{T}}(G)$ by the smallest strict full subcategory which contains G and is closed under taking finite coproducts, retracts and all shifts. Moreover for $n > 1$ define inductively the nth thickening $\mathrm{thick}^n_{\mathcal{T}}(G)$ by the smallest strict full subcategory of $\mathcal{T}$ which is closed under retracts and contains objects M admitting a distinguished triangle $M_1 \to M \to M_2 \to \Sigma M_1$ in $\mathcal{T}$ for which M_1 and M_2 are in $\mathrm{thick}^{n-1}_{\mathcal{T}}(G)$ and $\mathrm{thick}^1_{\mathcal{T}}(G)$, respectively. For an object M in $\mathcal{T}$, we define a numerical invariant $\mathrm{level}^G_{\mathcal{T}}(M)$, which is called *the level of* M with respect to G in $\mathcal{T}$, by

$$\mathrm{level}^G_{\mathcal{T}}(M) := \inf\{n \in \mathbb{N} \cup \{0\} \mid M \in \mathrm{thick}^n_{\mathcal{T}}(G)\}.$$

The dimension $\dim \mathcal{T}$ of a triangulated category $\mathcal{T}$ [3], [25] is defined by

$$\dim \mathcal{T} = \inf\{n \in \mathbb{N} \mid \text{there exists an object } G \in \mathcal{T} \text{ with } \mathrm{thick}^{n+1}_{\mathcal{T}}(G) = \mathcal{T}\}.$$

Thus the levels of modules are closely related to the dimension of a triangulated category and to *the ghost lengths* of modules; see [16] and [11].

Let A be a differential graded algebra (abbreviated DGA henceforth) over a field $\mathbb{K}$, with differential decreasing degree by 1. Let $\mathrm{D}(A)$ denote the derived category of differential graded right A-modules, which is viewed as a distinguished triangulated category. Observe that a distinguished triangle in $\mathrm{D}(A)$ comes from a cofibre sequence of the form $M \xrightarrow{f} N \to C_f \to \Sigma M$ in the homotopy category of differential graded modules (abbreviated DG modules) over A, where C_f is the mapping cone and ΣM is the suspension of M defined by $(\Sigma M)^n = M^{m+1}$. In what follows, we denote by $\mathrm{level}_{\mathrm{D}(A)}(M)$ the invariant $\mathrm{level}^A_{\mathrm{D}(A)}(M)$ for any object M in $\mathrm{D}(A)$.

We shall say that a graded vector space M is locally finite if M^i is of finite dimension for any i. Unless otherwise explicitly stated, it is assumed that a space has the homotopy type of a CW complex whose cohomology with coefficients in the underlying field is locally finite. Let B be a simply-connected space and $\mathcal{TOP}/B$ the category of connected spaces over B; that is, objects are maps with the target B and morphisms from $\alpha : X \to B$ to $\beta : Y \to B$ are maps $f : X \to Y$ such that $\beta f = \alpha$. For a given object $\alpha : X \to B$ in $\mathcal{TOP}/B$, the singular cochain complex $C^*(X;\mathbb{K})$ is regarded as a differential graded module (abbreviated DG module) over the DGA $C^*(B;\mathbb{K})$ with the morphism of DGA's induced by α. Thus we have a contravariant functor

$$C^*(\ ;\mathbb{K}) : \mathcal{TOP}/B \to \mathrm{D}(C^*(B;\mathbb{K})). \tag{2.1}$$

Let $X \to B$ be an object in $\mathcal{TOP}/B$. We then write $\mathrm{level}_{\mathrm{D}(C^*(B;\mathbb{K}))}(X)$ for the invariant $\mathrm{level}_{\mathrm{D}(C^*(B;\mathbb{K}))}(C^*(X;\mathbb{K}))$ and refer to it as *the level* of the space X. Observe that if there exists a morphism $f : X \to Y$ in $\mathcal{TOP}/B$ which is a homotopy equivalence, then

$$\mathrm{level}_{\mathrm{D}(C^*(B;\mathbb{K}))}(X) = \mathrm{level}_{\mathrm{D}(C^*(B;\mathbb{K}))}(Y).$$

3. Results and computational examples

Recall that, a space X is $\mathbb{K}$-*formal* if it is simply-connected and there exists a quasi-isomorphism to the cohomology $H^*(X;\mathbb{K})$ from a minimal TV-model for X

in the sense of Halperin and Lemaire [7]. Thus, in the case, we have a sequence of quasi-isomorphisms

$$H^*(X;\mathbb{K}) \xleftarrow[\simeq]{\phi_X} TV_X \xrightarrow[\simeq]{m_X} C^*(X;\mathbb{K}),$$

where $m_X : TV_X \overset{\simeq}{\to} C^*(X;\mathbb{K})$ denotes a minimal TV-model for X; that is, TV_X is a DGA whose underlying $\mathbb{K}$-algebra is the tensor algebra generated by a graded vector space V_X and, for any element $v \in V_X$, the image of v by the differential is decomposable; see also Appendix. Observe that spheres S^d with $d > 1$ are $\mathbb{K}$-formal for any field $\mathbb{K}$ [4][24] and that a simply-connected space whose cohomology with coefficients in $\mathbb{K}$ is a polynomial algebra generated by elements with even degree is also $\mathbb{K}$-formal; see [22, Section 7].

Definition 3.1. Let $q : E \to B$ and $f : X \to B$ be maps between $\mathbb{K}$-formal spaces. The pair (q, f) is relatively $\mathbb{K}$-formalizable if there exists a commutative diagram up to homotopy

$$\begin{tikzcd}
H^*(E;\mathbb{K}) & TV_E \arrow[l, "\phi_E"', "\simeq"] \arrow[r, "m_E", "\simeq"'] & C^*(E;\mathbb{K}) \\
H^*(B;\mathbb{K}) \arrow[u, "H^*(q)"] \arrow[d, "H^*(f)"'] & TV_B \arrow[l, "\phi_B"', "\simeq"] \arrow[r, "m_B", "\simeq"'] \arrow[u, "\widetilde{q}"'] \arrow[d, "\widetilde{f}"] & C^*(B;\mathbb{K}) \arrow[u, "q^*"'] \arrow[d, "f^*"] \\
H^*(X;\mathbb{K}) & TV_X \arrow[l, "\phi_X"', "\simeq"] \arrow[r, "m_X", "\simeq"'] & C^*(X;\mathbb{K}),
\end{tikzcd}$$

in which horizontal arrows are quasi-isomorphisms.

We refer the reader to Appendix for the notion of homotopy in the category of DGA's. We here comment on a map between $\mathbb{K}$-formal spaces. In general, for given quasi-isomorphisms ϕ_E, m_E, ϕ_B and m_B as in Definition 3.1, there exist DGA maps $\widetilde{q}_1$ and $\widetilde{q}_2$ which make the right upper square and left that in the definition homotopy commutative, respectively. However, in general, one cannot choose a map $\widetilde{q}$ which makes upper two squares homotopy commutative simultaneously even if the maps ϕ_E, m_E, ϕ_B and m_B are replaced by other quasi-isomorphisms; see the comments following [18, Theorem 1.1].

The following proposition, which are deduced from the proof of [18, Theorem 1.1], gives examples of relatively $\mathbb{K}$-formalizable pairs of maps.

Proposition 3.2. *A pair (q, f) of maps between $\mathbb{K}$-formal spaces with the same target is relatively $\mathbb{K}$-formalizable if the two maps q and f satisfy any of the following three conditions concerning a map $\pi : S \to T$, respectively.*

(P_1) *$H^*(S;\mathbb{K})$ and $H^*(T;\mathbb{K})$ are polynomial algebras with at most countably many generators in which the operation Sq_1 vanishes when the characteristic of the field $\mathbb{K}$ is 2. Here $Sq_1 x = Sq^{n-1}x$ for x with degree n; see* [22, 4.9].
(P_2) *The homomorphism $BH^*(\pi;\mathbb{K}) : BH^*(T;\mathbb{K}) \to BH^*(S;\mathbb{K})$ defined by $H^*(\pi;\mathbb{K})$ between the bar complexes induces an injective homomorphism on the homology.*
(P_3) $\widetilde{H}^i(S;\mathbb{K}) = 0$ *for any i with* $\dim \widetilde{H}^{i-1}(\Omega T;\mathbb{K}) - \dim(QH^*(T;\mathbb{K}))^i \neq 0$.

Example 3.3. (i) Let G be a connected compact Lie group and K a connected closed subgroup. Suppose that $H_*(G;\mathbb{Z})$ and $H_*(K;\mathbb{Z})$ are p-torsion free. Then the map $Bi : BK \to BG$ between classifying spaces induced by the inclusion $i : K \to G$ satisfies the condition (P_1) with respect to the field $\mathbb{F}_p$. Assume further that rank G = rank K. Let M be the homogeneous space G/K and $\mathrm{aut}_1(M)$ the connected component of function space of all self-maps on M containing the identity map. Then the universal fibration $\pi : M_{\mathrm{aut}_1(M)} \to B_{\mathrm{aut}_1(M)}$ with fibre M satisfies the condition (P_1) with respect to the field $\mathbb{Q}$; see [10] and [21].
(ii) Let $q : E \to B$ be a map between $\mathbb{K}$-formal spaces with a section. Then q satisfies the condition (P_2). This follows from the naturality of the bar construction.
(iii) Consider a map $f : S^4 \to BG$ for which G is a simply-connected Lie group and $H_*(G;\mathbb{Z})$ is p-torsion free. Suppose that $\widetilde{H}^i(S^4;\mathbb{F}_p) \neq 0$, then $i = 4$. One obtains $\dim \widetilde{H}^{4-1}(\Omega BG;\mathbb{F}_p) - \dim(QH^*(BG;\mathbb{F}_p))^4 = 0$. Thus the map $f : S^4 \to BG$ satisfies the condition (P_3).

Let $q : E \to X$ be a fibration over a space X and $f : B \to X$ a map. Let $\mathcal{F}$ denote the pullback diagram

$$\begin{tikzcd} E \times_X B \arrow[r] \arrow[d] & E \arrow[d, "q"] \\ B \arrow[r, "f"'] & X. \end{tikzcd}$$

Our main theorem concerning the level of a space is stated as follows.

Theorem 3.4. [19] *Suppose that the spaces X, B and E in the diagram $\mathcal{F}$ are $\mathbb{K}$-formal and the pair (q, f) is relatively $\mathbb{K}$-formalizable. Then*

$$\mathrm{level}_{\mathrm{D}(C^*(B;\mathbb{K}))}(E \times_X B) = \mathrm{level}_{\mathrm{D}(H^*(B;\mathbb{K}))}(H^*(E;\mathbb{K}) \otimes^{\mathrm{L}}_{H^*(X;\mathbb{K})} H^*(B;\mathbb{K})).$$

In general, the equality in Theorem 3.4 does not hold even if the spaces X, B and E in $\mathcal{F}$ are $\mathbb{K}$-formal; see [19, Example 3.2].
By virtue of Theorem 3.4 and Proposition 3.2, we have

Proposition 3.5. [19] *Let G be a simply-connected Lie group and $G \to E_f \to S^4$ a G-bundle with the classifying map $f : S^4 \to BG$. Suppose that $H^*(BG;\mathbb{K})$ is a polynomial algebra on generators with even degree. Then*

$$\mathrm{level}_{\mathrm{D}(C^*(S^4;\mathbb{K}))}(E_f) = \begin{cases} 2 & \text{if } H^4(f;\mathbb{K}) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 3.6. [19] *Let G be a simply-connected Lie group and H a maximal rank subgroup. Let $G/H \to E_g \to S^4$ be the pullback of the fibration $G/H \to BH \xrightarrow{\pi} BG$ by a map $g : S^4 \to BG$. Suppose that $H^*(BG;\mathbb{K})$ and $H^*(BH;\mathbb{K})$ are polynomial algebras on generators with even degree. Then one has*

$$\mathrm{level}_{\mathrm{D}(C^*(S^4;\mathbb{K}))}(E_g) = 1.$$

Propositions 3.5 and 3.6 make one expect that a 'nice' object in $\mathcal{TOP}/X$ such as the total space of a fibration associated with a bundle is almost of low level. On the other hand, the following result guarantees existence of an object in $\mathcal{TOP}/S^d$ with the level greater than given arbitrary number.

Theorem 3.7. [19] *Suppose that the underlying field $\mathbb{K}$ is of characteristic zero. For any integer $l \geq 1$, there exists an object $P_l \to S^d$ in $\mathcal{TOP}/S^d$ such that*

$$\mathrm{level}_{\mathrm{D}(C^*(S^d;\mathbb{K}))}(P_l) \geq l.$$

4. Jørgensen's results on Auslander-Reiten theory over topological spaces

The original computation in Proposition 3.5 is made relying on Jørgensen's work on Auslander-Reiten theory over topological spaces in [13]. We here give an overview of the result.

We say that an object in a triangulated category $\mathcal{T}$ is indecomposable if it is not a coproduct of nontrivial objects. A triangle $L \xrightarrow{u} M \xrightarrow{v} N \xrightarrow{w} \Sigma L$ in a triangulated category $\mathcal{T}$ is *an Auslander-Reiten triangle* [8], by definition, if the following conditions are satisfied:

(i) L and N are indecomposable.
(ii) $w \neq 0$.
(iii) Each morphism $N' \to N$ which is not a retraction factors through v.

We say that a morphism $f : M \to N$ in $\mathcal{T}$ is *irreducible* if it is neither a section nor a retraction, but satisfies that in any factorization $f = rs$, either s is a section or r is a retraction. The category $\mathcal{T}$ is said to have Auslander-Reiten triangles if, for each object N with local endomorphism ring, there exists an Auslander-Reiten triangle with N as the third term from the left. Recall also that an object K in $\mathcal{T}$ is *compact* if the functor $\mathrm{Hom}_{\mathcal{T}}(K, \ )$ preserves coproducts; see [23, Chapter 4].

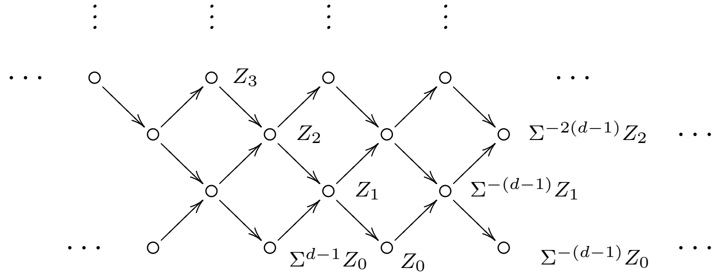
We denote by $\mathrm{D}^c(A)$ the full subcategory of the derived category $\mathrm{D}(A)$ consisting of the compact objects. For a DG module M over A, let DM be the dual $\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ to M.

Suppose that $d := \sup\{i \mid H^iA \neq 0\}$ is finite. One of the main results in [13] asserts that both $\mathrm{D}^c(A)$ and $\mathrm{D}^c(A^{op})$ have Auslander-Reiten triangles if and only if there are isomorphisms of graded HA-modules ${}_{HA}(DHA) \cong {}_{HA}(\Sigma^d HA)$ and $(DHA)_{HA} \cong (\Sigma^d HA)_{HA}$; that is, $H^*(A)$ is a Poincaré duality algebra. Moreover we observe that the condition on A is equivalent to the Gorensteinness of A in the sense of Félix, Halperin and Thomas [5].

Definition 4.1. The Auslander-Reiten quiver of D has as vertices the isomorphism classes $[M]$ of indecomposable objects. It has one arrow from $[M]$ to $[N]$ when there is an irreducible morphism $M \to N$ and no arrow from $[M]$ to $[N]$ otherwise.

The form of the Auslander-Reiten quiver of $\mathrm{D}^c(A)$ is clarified in [13] and [14] for a DGA A whose cohomology is a Poincaré duality algebra. The key lemma [13, Lemma 8.4] to proving results in [13, Section 8] is obtained by using the rational formality of the spheres. Since the spheres are also $\mathbb{K}$-formal for any field $\mathbb{K}$, the assumption of the characteristic of the underlying field can be removed from all the results in [13, Section 8]. In particular, we have

Theorem 4.2. [13, Theorem 8.13][13, Proposition 8.10] *Let S^d be the d-dimensional sphere with $d > 1$ and $\mathbb{K}$ an arbitrary field. Then the Auslander-Reiten quiver of the category $\mathrm{D}^c(C^*(S^d;\mathbb{K}))$ consists of $d-1$ components, each isomorphic to $\mathbb{Z}A_\infty$. The component containing $Z_0 \cong C^*(S^d;\mathbb{K})$ is of the form*



Moreover, the cohomology of the indecomposable object $\Sigma^{-l}Z_m$ has the form

$$H^i(\Sigma^{-l}Z_m) \cong \begin{cases} \mathbb{K} & \text{for } i = -m(d-1)+l \text{ and } d+l, \\ 0 & \text{otherwise.} \end{cases}$$

We can compute the level of a compact object over $C^*(S^d;\mathbb{K})$ with the aid of the following result due to Schmidt.

Lemma 4.3. [26, Proposition 6.6] *Let Z_i be the indecomposable object in $\mathrm{D}^c(C^*(S^d;\mathbb{K}))$ described in Theorem 4.2. Then $\mathrm{level}_{\mathrm{D}(C^*(S^d;\mathbb{K}))}(Z_i) = i+1$.*

5. A general method for computing the level of a space.

We first give a sketch of the original proof of Proposition 3.5. Consider the map $f : S^4 \to BG$ in Proposition 3.5. As mentioned in Example 3.3, the map f satisfies the condition (P_3) in Proposition 3.2. It is immediate that the universal bundle $\pi : EG \to BG$ satisfies the condition (P_3). Thus by Proposition 3.2 we see that the pair (f, π) is relatively $\mathbb{K}$-formalizable. An explicit calculation enables us to conclude that, in the derived category $\mathrm{D}(C^*(S^4;\mathbb{K}))$, the object $C^*(E_f;\mathbb{K})$ is isomorphic to a coproducts of shifts of Z_1 if $f^* \neq 0$; see [19, Section 5]. Moreover we see that $C^*(E_f;\mathbb{K})$ is isomorphic to a coproducts of shifts of $Z_0 = C^*(S^4;\mathbb{K})$ if $f^* = 0$; see [19, Section 5]. Thus the result follows from Lemma 4.3.

One may ask why the level of such a bundle is small. The following general method for computing the level of a space will answer the question.

Theorem 5.1. [20] *Let $\mathcal{F}$ be a pull-back diagram*

$$\begin{tikzcd} E_\varphi \arrow[r] \arrow[d] & E \arrow[d, "q"] \\ B \arrow[r, "\varphi"'] & X \end{tikzcd}$$

in which q is a fibration and the pair (q, φ) is relatively $\mathbb{K}$-formalizable. Suppose that either of the following conditions (i) *and* (ii) *holds.*

(i) *The cohomology* $H^*(X;\mathbb{K})$ *is a polynomial algebra generated by* m *indecomposable elements. Let* Λ *be the subalgebra of* $H^*(X;\mathbb{K})$ *generated by the vector subspace* $\Gamma := \mathrm{Ker}\ \varphi^* \cap QH^*(X;\mathbb{K})$. *Then* $\dim \mathrm{Tor}_*^{\Lambda}(H^*(E;\mathbb{K}),\mathbb{K}) < \infty$.

(ii) *There exists a homotopy commutative diagram*

$$\begin{tikzcd} E \arrow[r, "\simeq"] \arrow[d, "q"'] & X' \arrow[d, "\Delta"] \\ X \arrow[r, "\simeq", "h"'] & X' \times X' \end{tikzcd}$$

in which horizontal arrows are homotopy equivalences and Δ *is the diagonal map. Moreover* $H^*(X';\mathbb{K})$ *is a polynomial algebra generated by* m *indecomposable elements. In this case put* $\Gamma = \mathrm{Ker}\ (\Delta^*|_{QH^*(X'\times X')}) \cap \mathrm{Ker}\ (h\varphi)^*$.
Then one has

$$\mathrm{level}_{\mathrm{D}(C^*(B;\mathbb{K}))}(E_\varphi) \le m - \dim\Gamma + 1.$$

In particular, $\mathrm{level}_{\mathrm{D}(C^*(B;\mathbb{K}))}(E_\varphi) = 1$ *if* $\varphi^* \equiv 0$.

We consider again the G-bundle $E_f \to S^4$ described in Proposition 3.5. Let Γ be the vector space $\mathrm{Ker}\ f^* \cap QH^*(BG;\mathbb{K})$ and Λ the subalgebra of $H^*(BG;\mathbb{K})$ generated by Γ. Suppose that $f^* = 0$. Then it is immediate that $\dim\Gamma = m$. Since Λ is a polynomial algebra, it follows that $\mathrm{Tor}_*^{\Lambda}(H^*(EG;\mathbb{K}),\mathbb{K}) \cong \mathrm{Tor}_*^{\Lambda}(\mathbb{K},\mathbb{K})$ is an exterior algebra and hence $\dim \mathrm{Tor}_*^{\Lambda}(H^*(EG;\mathbb{K}),\mathbb{K}) < \infty$. This yields that the condition (i) in Theorem 5.1 holds. Thanks to the theorem, we see that $\mathrm{level}_{\mathrm{D}(C^*(S^4;\mathbb{K}))}(E_f) = m - m + 1 = 1$.

In the case where $f^* \neq 0$, it is readily seen that $\dim\Gamma = m-1$ and the dimension of $\mathrm{Tor}_*^{\Lambda}(H^*(EG;\mathbb{K}),\mathbb{K})$ is finite. Thus Theorem 5.1 is applicable and hence we have

$$1 \le \mathrm{level}_{\mathrm{D}(C^*(S^4;\mathbb{K}))}(E_f) \le m - (m-1) + 1 = 2.$$

In order to prove that the level of E_f is greater than one, we appeal to a proposition which characterizes a space of level one in terms of spectral sequences.

Proposition 5.2. [20] *Let* $\mathcal{F}' : F \xrightarrow{j} E \to B$ *be a fibration with* B *simply-connected and* F *connected. If* $\mathrm{level}_{\mathrm{D}(C^*(B;\mathbb{K}))}(E) = 1$, *then both the Leray-Serre spectral sequence and the Eilenberg-Moore spectral sequence for* $\mathcal{F}'$ *collapse at the* E_2*-term, where the coefficients of the spectral sequence are in the field* $\mathbb{K}$.

In the Leray-Serre spectral sequence $\{E_r^{*,*}, d_r\}$ for the universal bundle $G \to EG \to BG$, the indecomposable elements of $H^*(G;\mathbb{K}) \cong E_2^{0,*}$ are chosen as transgressive ones. Since $f^* \neq 0$, it follows that the Leray-Serre spectral sequence for the fibration $G \to E_\varphi \to S^4$ does not collapse at the E_2-term. Proposition 5.2 implies that $1 < \mathrm{level}_{\mathrm{D}(C^*(S^4;\mathbb{K}))}(E_f)$. We have the result.

We conclude this section with comments. For a DG module M over a DGA A, the level of M defined in the derived category $\mathrm{D}(A)$ certainly counts the number of steps to build M out of A via triangles in $\mathrm{D}(A)$. As for topological side, what the invariant describes is not clear. We are, however, convinced that the level of a topological space measures complexity of the topological one in some sense. We refer the reader to [20, §3 Examples] for more computations of levels of spaces.

6. Appendix

We recall briefly the TV-model introduced by Halperin and Lemaire [7]. Let TV be the tensor algebra $\sum_{n\geq 0} V^{\otimes n}$ with a graded vector space V over a field $\mathbb{K}$ and $T^{\geq k}V$ denote the ideal $\sum_{n\geq k} V^{\otimes n}$ of the algebra TV, where $V^{\otimes 0} = \mathbb{K}$. As usual, we define the degree of the element $w = v_1 v_2 \cdots v_l \in TV$ by $\deg w = n_1 + \cdots + n_l$ if $v_{n_i} \in V^{n_i}$. Let V' and V'' be copies of V. We write sv for the element of ΣV corresponding to $v \in V$. The cylinder object $TV \wedge I = (T(V' \oplus V'' \oplus \Sigma V), d)$ introduced by Baues and Lemaire [2, §1] is a DGA with differential d defined by

$$dv' = (dv)', \ dv'' = (dv)'' \text{ and } dsv = v'' - v' - S(dv),$$

where $S : TV \to T(V' \oplus V'' \oplus \Sigma V)$ is a map with $Sv = sv$ for $v \in V$ and $S(xy) = Sx \cdot y'' + (-1)^{\deg x} x' \cdot Sy$ for $x, y \in TV$. The inclusions $\varepsilon_0 : TV \to TV \wedge I$ and $\varepsilon_1 : TV \to TV \wedge I$ are defined by $\varepsilon_0(v) = v'$ and $\varepsilon_1(v) = v''$, respectively.

For DGA maps $\phi', \phi'' : TV \to A$ form TV to a DGA A, we say that ϕ' and ϕ'' are homotopic if the DGA map $(\phi', \phi'') : T(V' \oplus V'') \to A$ extends to a DGA map $\Phi : TV \wedge I \to A$; that is $\phi' = \Phi\varepsilon_0$ and $\phi'' = \Phi\varepsilon_1$. We refer the reader to [6, Section 3] for homotopy theory of DGA's.

A *TV-model* for a differential graded algebra (A, d_A) is a quasi-isomorphism $(TV, d) \xrightarrow{\simeq} (A, d_A)$. Moreover the model is called minimal if $d(V) \subset T^{\geq 2}V$. For any simply-connected space whose cohomology with coefficients in $\mathbb{K}$ is locally finite, there exists a minimal TV-model $(TV, d) \xrightarrow{\simeq} C^*(X; \mathbb{K})$ which is unique up to homotopy. Such a model (TV, d) is called *a minimal model for X*. It is known that the vector space V^n is isomorphic to $(\Sigma^{-1}\tilde{H}^*(\Omega X; \mathbb{K}))^n = \tilde{H}^{n-1}(\Omega X; \mathbb{K})$ and the quadratic part of the differential d is the coproduct on $\tilde{H}^*(\Omega X; \mathbb{K})$ up to the isomorphism $V \cong \Sigma^{-1}\tilde{H}^*(\Omega X; \mathbb{K})$. The reader is referred to [7] and [24, Introduction] for these facts and more details of TV-models.

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