A FUNCTION SPACE MODEL APPROACH TO THE RATIONAL EVALUATION SUBGROUPS

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Abstract. This is a summary of the joint work [20] with Y. Hirato and N. Oda, in which we investigate the evaluation subgroups $G_*(U, X; f)$ for a map $f : U \to X$ from a connected nilpotent space U to a connected rational space X. The key device for the study is an explicit Sullivan model for the connected component containing f of the function space of maps from U to X, which is derived from the general theory of such a model due to Brown and Szczarba [5]. This note also contains a brief explanation of the background of our work.

1. Background

One may hope a full subcategory of the category of topological spaces is able to be controlled by a category of appropriate algebraic objects. As algebraic model theories for the study of spaces, in particular*, we can mention rational homotopy theory due to Quillen [33] and Sullivan [36] and p-adic homotopy theory due to Mandell [28]. Let $\mathcal{C}$ be a category with a family of *weak equivalences* and $h(\mathcal{C})$ denote the homotopy category obtained by giving formal inverses of weak equivalences. The correspondences between "spaces" and "algebras" are roughly summarized as follows:

Rational Homotopy Theory, see also [3]. The functor $A_{PL}(\cdot)$ of rational polynomial differential forms on a space and the realization functor $|\cdot|$ give an equivalence

$$h\begin{pmatrix}\text{the category of connected nilpotent rational spaces}\\ \text{of finite } \mathbb{Q}\text{-type}\end{pmatrix}$$
$$|\cdot|\Big\uparrow \cong \Big\downarrow A_{PL}(\cdot)$$
$$h(\text{the category of differential graded algebras over } \mathbb{Q}).$$

p-adic Homotopy Theory. The normalized singular cochain functor $C^*(\cdot\ ; \overline{\mathbb{F}}_p)$ with coefficients in the closure $\overline{\mathbb{F}}_p$ and the realization functor give an equivalence

$$h\begin{pmatrix}\text{the category of connected nilpotent } p\text{-complete spaces}\\ \text{of finite } p\text{-type}\end{pmatrix}$$
$$|\cdot|\Big\uparrow \cong \Big\downarrow C^*(\cdot\ ;\overline{\mathbb{F}}_p)$$
$$h\begin{pmatrix}\text{a full subcategory of the category of algebras}\\ \text{over an } E_\infty\overline{\mathbb{F}}_p\text{ -operad}\end{pmatrix}.$$

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*For other algebraic model theories, we refer the reader to integral homotopy theory [29] and tame homotopy theory [6] [19]. Adams-Hilton models [2] and TV-models [18] are regarded as algebraic models for spaces.

In principle, it seems possible to translate various topological invariants into computable algebraic invariants. However, we often encounter a problem of constructing an explicit model when executing that. We here carry out such translation for the evaluation subgroups, that are topological invariants, within the framework of rational homotopy theory. In ongoing work [22], the rational visibility of a Lie group in the space of self-homotopy equivalences of a homogeneous space is considered by means of tools developed in [20] and [21].

We also expect that our ideas in [20] are applicable in other algebraic model theory in order to understand topological invariants and notions algebraically. In the near future, we shall proceed to the study of function spaces with their operadic models constructed in [7].

2. Results

Let U and X be connected based spaces and $f : U \to X$ a based map. We denote by $\mathcal{F}(U, X; f)$ the connected component in the function space of *free* maps from U to X that contains f. Let $ev : \mathcal{F}(U, X; f) \to X$ be the evaluation map which sends a map $g : U \to X$ to $g(u_0)$, where u_0 is the base point of U. The nth evaluation subgroup for the triple $(U, X; f)$, denoted $G_n(U, X; f)$, is the subgroup of the homotopy group $\pi_n(X)$ defined by

$$G_n(U, X; f) = ev_*(\pi_n(\mathcal{F}(U, X; f), f)).$$

In the special case where $U = X$ and $f = id$ the identity map on X, the nth evaluation subgroup is referred to as the nth Gottlieb group of X and written $G_n(X)$. In what follows, we shall write $G_*(U, X; f)$ for $\oplus_{n\geq 0} G_n(U, X; f)$.

The evaluation subgroups were essentially introduced by Gottlieb [12][14] and were investigated extensively by Woo and Kim [37] [38] and by Woo and Lee [23] [39] [40] [41]. The lack of functoriality in Gottlieb groups makes the study of the subject more difficult. In such a situation, the G-sequence introduced in [40] is one of relevant tools for studying the groups $G_*(X)$ and $G_*(U, X; f)$.

As for rational Gottlieb groups, Félix and Halperin have proved that, for any simply-connected space X with finite rational Lusternik-Schnirelmann category m, the graded Gottlieb group $G_*(X) \otimes \mathbb{Q}$ is concentrated in odd degrees and has dimension at most m ([9, Theorem III]). We stress that the consideration of Gottlieb groups appears in their investigation of rational category. Moreover, from the lecture notes [31] due to Oprea, we can know relationship between Gottlieb groups and transformation groups as well as fixed point theory. In [34], Smith has studied the rational evaluation subgroups by relying on the approach to the study of function spaces due to Federer [8]. Interesting examples of vanishing and non-vanishing evaluation subgroups are given in [34, §5]. Recently, Lupton and Smith [25][26] have considered the exactness of the G-sequence by representing the evaluation subgroups in terms of derivations in Sullivan models and in Quillen models. Especially, in [25, Example 4.1], the non-exactness of a certain G-sequence is captured by calculation of derivations.

The objective of the paper [20] is to investigate the evaluation subgroup of the form

$$G_*(U, X_{\mathbb{Q}}; f),$$

where U is a nilpotent space and $X_{\mathbb{Q}}$ is the localization of a nilpotent space X. We try to consider the rational evaluation subgroup *without drawing on the derivation* *argument*. In fact, the key device for the study is an explicit algebraic model for the function space $\mathcal{F}(U, X_{\mathbb{Q}}; f)$, which we construct in [20] by invoking the general theory of such a model due to Brown and Szczarba [5]; see [20, Section 3].

We here explain our main results briefly. Theorems 2.1 and 2.2 describe sufficient conditions for rational evaluation subgroups to be proper. Theorem 2.6 presents a tractable condition for a fibration to be Gottlieb trivial in the sense of Lupton and Smith [27]. Theorem 2.7 gives a non-trivial upper bound for the dimension of the localization of some subquotient of the first evaluation subgroup. By Theorem 2.9, one can determine the first Gottlieb group of the total space of each S^1-bundle over the n-dimensional torus in non-rational case with knowledge of the classifying map of the bundle.

Unless otherwise explicitly stated, it is assumed that a space is well-based and has the homotopy type of a CW complex with rational homology of finite type. We further suppose that a map is based. We shall say that a space is rational if the space has the homotopy type of the spatial realization of a Sullivan algebra; see [20, Section 2]. Observe that the homotopy group $\pi_n(X)$ of a rational space X is a vector space over $\mathbb{Q}$ for $n > 1$ and that so is the fundamental group $\pi_1(X)$ if the group is abelian. These facts follow from the Sullivan-de Rham equivalence; see for example [3, Theorems 10.1 and 12.2].

In the rest of this section, we state the results more precisely.

Suppose that X is a connected rational space. Then the function space $\mathcal{F}(U, X; f)$ is also a rational space; see [5]. The definition of the evaluation subgroup enables us to obtain a commutative diagram

$$\begin{tikzcd}
\pi_n(\mathcal{F}(U,X;f)) \arrow[d, "ev_*"'] \arrow[dr, "ev_*"] & \\
G_n(U,X;f) \arrow[r, hook] & \pi_n(X)
\end{tikzcd} \tag{2.1}$$

in the category of groups for $n \geq 1$. This is regarded as a diagram in the category of vector spaces for $n > 1$. Let H be a group and let $(\Gamma_1/\Gamma_2)H$ denote the quotient group of H by the commutator subgroup. Put $G^\sharp = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q})$ for an abelian group G. Then we have a commutative diagram

$$\begin{tikzcd}
\Big((\Gamma_1/\Gamma_2)\pi_n(\mathcal{F}(U,X;f))\Big)^\sharp & \\
G_n(U,X;f)^\sharp \arrow[u, tail] & \pi_n(X)^\sharp \arrow[l, two heads] \arrow[ul, "ev_*^\sharp"']
\end{tikzcd}$$

in the category of vector spaces for $n > 1$ and for $n = 1$ if $\pi_1(X)$ is abelian. Recall that, for any connected nilpotent space Y with a minimal model $\wedge Z$, there exists a natural isomorphism $Z^n \cong \pi_n(Y)^\sharp$ for $n > 1$ and for $n = 1$ if $\pi_1(Y)$ is an abelian group; see [3].

Let $\wedge V$ and $\wedge W$ be minimal models for X and $\mathcal{F}(U, X; f)$, respectively. We denote by $Q(\widetilde{ev}) : V \to W$ the linear part of the Sullivan representative $\widetilde{ev} : \wedge V \to \wedge W$ for the evaluation map. Observe that the vector space $\Big((\Gamma_1/\Gamma_2)\pi_n(\mathcal{F}(U,X;f))\Big)^\sharp$ is a subspace of W^n; see [20, Section 8] for details. Suppose that $\pi_1(X)$ is an abelian group. Then we have an isomorphism $G_*(U, X; f)^\sharp \cong V/\mathrm{Ker}Q(\widetilde{ev})$ of vector spaces.

This fact implies, for example, that $G_*(U, X; f)$ is a proper subgroup of $\pi_*(X)$ if and only if $\mathrm{Ker} Q(\widetilde{ev})$ is nontrivial.

In [5], Brown and Szczarba have presented an explicit form of Lannes' division functor in the category of commutative differential graded algebras; see also [4]. By using the functor, they have constructed an algebraic model for a connected component of a function space. Unfortunately, the model is very complicated and not minimal in general. However the linear part δ_0 of the differential of the model for $\mathcal{F}(U, X; f)$, which is needed to construct the minimal model, is comparatively tractable. Moreover an explicit model $\overline{ev}$ for the evaluation map $ev : \mathcal{F}(U, X; f) \to X$ is derived from the consideration in [21, Section 5].

In some cases, we can find a nonzero element in $\mathrm{Im}\ \overline{ev} \cap \mathrm{Im}\ \delta_0$ with knowledge of the terms having the least wordlength in $d(v)$ for an appropriate element $v \in V$. It turns out then that $\mathrm{Ker} Q(\widetilde{ev}) \neq 0$. The dual element in $V^\sharp \cong \pi_*(X)$ to such an element v is said to be *detective*; see [20, Section 4] for the precise definition. With this terminology, one of our main theorems is stated as follows.

Theorem 2.1. [20] *Let $f : U \to X$ be a map from a connected nilpotent space U to a connected rational space X whose fundamental group is abelian. Suppose that $\dim \oplus_{q \geq 0} H^q(U; \mathbb{Q}) < \infty$ or $\dim \oplus_{i \geq 2} \pi_i(X) < \infty$ and that there exists a detective element x in $\pi_*(X)$ with respect to the triple $(U, X; f)$. Then the evaluation subgroup $G_k(U, X; f)$ is a proper subgroup of $\pi_k(X)$ for some $1 \leq k \leq \deg x$.*

While the notion of the detective element is somewhat technical, it does work well when exhibiting the properness of a given evaluation subgroup; see [20, Example 4.6].

We can also detect geometrically an element which is not in the evaluation subgroup. Before describing the result, we recall briefly the higher order Whitehead product set defined by Porter in [32]. Let ι_m denote the generator of $H_m(S^m)$ which is the image of the identity map by the Hurewicz map. Let T be the fat wedge of s spheres S^{n_i}, $1 \leq i \leq s$; that is, the subspace of the product $S^{n_1} \times \cdots \times S^{n_s}$ consisting of all s-tuples with at least one coordinate at the base point. Let μ be the generator of $H_N(\times_{i=1}^s S^{n_i}; \mathbb{Z})$, corresponding to $\iota_{n_1} \otimes \cdots \otimes \iota_{n_s} \in H_*(S^{n_1}) \otimes \cdots \otimes H_*(S^{n_s})$ via the Künneth isomorphism, where $N = \sum n_i$. Since the CW pair $(\times_{i=1}^s S^{n_i}, T)$ is $(N-1)$-connected, we have a sequence

$$H_N(\times_{i=1}^s S^{n_i}) \xrightarrow{j_*} H_N(\times_{i=1}^s S^{n_i}, T) \xleftarrow[\cong]{h} \pi_N(\times_{i=1}^s S^{n_i}, T) \xrightarrow{\partial} \pi_{N-1}(T)$$

and an element $w = \partial h^{-1} j_*(\mu)$, where h is the Hurewicz map and ∂ is the boundary map. In what follows, we do not distinguish between a map and the homotopy class which it represents. Choose elements $x_i \in \pi_{n_i}(X)$ for $1 \leq i \leq s$. These elements define the map $g : \vee_{i=1}^s S^{n_i} \to X$ whose restriction to each S^{n_i} is the map x_i. Then the sth order Whitehead product set $[x_1, ..., x_s] \subset \pi_{N-1}(X)$ (possibly empty) is defined by

$$[x_1, ..., x_s] = \{f_*(w) \mid f : T \to X \text{ an extension of } g\}.$$

We shall say that the set $[x_1, ..., x_s]$ vanishes if it contains only zero.

As a consequence of a geometric property of higher-order Whitehead products in rational spaces, studied in [1], we obtain the following test for non-Gottlieb elements.

Theorem 2.2. [20] *Let U be a connected space and X a simply-connected rational space. Let $f : U \to X$ be a map for which the induced map $f_* : \pi_*(U) \to \pi_*(X)$ is* *an epimorphism. Assume that all Whitehead products of order less than* r *vanish in* $\pi_*(U)$. *If there exist elements* $x_1, \ldots, x_r$ *in* $\pi_*(X)$ *whose* r*th order Whitehead product* $[x_1, \ldots, x_r]$ *contains a nonzero element, then* $x_k \notin G_*(U, X; f)$ *for any* $k \leq r$.

Remark 2.3. The result [1, Corollary 6.5] asserts that, if all Whitehead products of order $< r$ vanish in $\pi_*(X)$ for a simply-connected rational space X, then any rth order Whitehead product sets in $\pi_*(X)$ is non-empty and consists of a single element. Therefore the Whitehead product $[x_1, \ldots, x_r]$ in Theorem 2.2 contains only one element.

Suppose that x_1 is a Gottlieb element in $\pi_*(X)$ for a connected space X which is not necessarily rational. The ordinary Whitehead product $[x_1, x_2]$ is zero for any $x_2 \in \pi_*(X)$ by [14, Proposition 2.3]. Thus Theorem 2.2 is regarded as a generalization of this fact in the context of rational homotopy theory.

It is worthwhile to deal with relationship between detective elements and higher order Whitehead product sets. With the aid of results in [1], we shall show that a nonzero element in a higher order Whitehead product set is detective; see [20, Theorem 6.1].

As described below, the sufficient conditions in Theorems 2.1 and 2.2 give criterions for a map not to be cyclic.

For maps $f : U \to X$ and $g : V \to X$, we write $g \perp f$ if the map $g \vee f : V \vee U \to X$ is extendable to $V \times U$. A map $f : U \to X$ is called *a cyclic map* if $id_X \perp f$. For example, when a topological group G acts on a space X with base point, the orbit map $G \to X$ at the base point is a cyclic map. As is discussed in the last paragraph on page 730 of [14], we see that $G_n(U, X; f) = \{[g] \in \pi_n(X) \mid g \perp f\}$. It is readily seen that $\pi_*(X) = G_*(U, X; f)$ if f is a cyclic map. Observe that if f is a cyclic map, then so is $e_X \circ f$, where $e_X : X \to X_{\mathbb{Q}}$ is the localization map. Thus we have the following corollary.

Corollary 2.4. [20] *Let* $f : U \to X$ *be a map between a connected nilpotent spaces and* $e_X : X \to X_{\mathbb{Q}}$ *the localization map. If the triple* $(U, X_{\mathbb{Q}}; e_X \circ f)$ *satisfies the conditions in Theorem 2.1 or 2.2, then* f *is not a cyclic map.*

We fix some notations and terminology in order to describe further our results.

Let $f : X \to Y$ be a map between nilpotent spaces. Let $\varphi : (\wedge V, d) \to A_{PL}(Y)$ be a minimal model for Y, where $A_{PL}(Y)$ denotes the differential graded algebra of rational polynomial forms on Y. A quasi-isomorphism $m : (\wedge V \otimes \wedge W, \widehat{d}) \to A_{PL}(X)$ is called a Sullivan model for f if $\widehat{d}|_{\wedge V} = d$, $m|_{\wedge V} = A_{PL}(f)\varphi$ and there exists a well-ordered homogeneous basis $\{x_\alpha\}_{\alpha \in \mathcal{I}}$ of W such that $\widehat{d}(1 \otimes x_\alpha) \in \wedge V \otimes \wedge(W_\alpha)$. Here $\wedge(W_\alpha)$ denotes the subalgebra generated by the x_β with $\beta < \alpha$. We further assume, unless otherwise specified, that the model is minimal in the sense that $\deg x_\beta < \deg x_\alpha$ implies $\beta < \alpha$; see [16, 1.1 Definition] and [16, Theorems 6.1 and 6.2] for the existence and the uniqueness of a minimal Sullivan model for a map f. The inclusion $j : (\wedge V, d) \rightarrowtail (\wedge V \otimes \wedge W, \widehat{d})$ is also referred to as a Sullivan model for f. Observe that the DGA $(\wedge V \otimes \wedge W, \widehat{d})$ is a Sullivan algebra; see [10, Proposition 15.5]. For a Sullivan algebra $A = (\wedge V, d)$, let d_0 denote the linear part of the differential d and put

$$\pi^n(A) = H^n(V, d_0).$$

We define the ψ-homotopy space of X, denoted $\pi_\psi^*(X)$, to be the vector space $\pi^*(A)$ for which A is a Sullivan model for X; see [16, Chapter 8]. Observe that π_ψ^* gives rise to a functor from the category of connected spaces with Sullivan models to that of graded vector spaces over $\mathbb{Q}$. Moreover there exists a natural isomorphism $\pi_\psi^*(X) \cong \pi_*(X)^\sharp$ for $* > 1$ and for $* \geq 1$ if $\pi_1(X)$ is abelian; see [3], [16]. For a free algebra $\wedge V$, let $\wedge^{\geq l} V$ denote the ideal generated by elements with word length greater than or equal to l.

We describe an important result concerning a decomposition of an evaluation subgroup. In [24], Woo and Lee show that, for any based spaces F and Y,

$$G_*(F, F \times Y; i) \cong G_*(F) \oplus \pi_*(Y),$$

where $i : F \to F \times Y$ denotes the inclusion into the first factor. This has motivated us to consider its generalization from the rational homotopy theory point of view.

We here introduce a class of maps.

Definition 2.5. A map $p : X \to Y$ is *separable* if there exists a Sullivan model $(\wedge V, d) \to (\wedge V \otimes \wedge W, \widehat{d})$ for p such that

$$\widehat{d}(w) \in \wedge^{\geq 2} V \otimes \wedge W + \mathbb{Q} \otimes \wedge^{\geq 2} W$$

for any $w \in W$. A fibration $p : X \to Y$ is said to be separable if the projection p is separable.

We establish the following theorem.

Theorem 2.6. [20] *Let $F \xrightarrow{i} X \xrightarrow{p} Y$ be a separable fibration of connected rational spaces with $\dim \oplus_{q \geq 0} H^q(F; \mathbb{Q}) < \infty$ or $\dim \oplus_{i \geq 2} \pi_i(X) < \infty$. Suppose that F is simply-connected and $\pi_1(Y)$ acts on $H^i(F; \mathbb{Q})$ nilpotently for any i. Then the sequence*

$$0 \to G_n(F) \xrightarrow{i_\sharp} G_n(F, X; i) \xrightarrow{p_\sharp} \pi_n(Y) \to 0$$

is exact for $n > 1$.

Very recently, Lupton and Smith [27] have proved a similar result to Theorem 2.6. Let $F \xrightarrow{i} X \xrightarrow{p} Y$ be a fibration of simply-connected CW complexes. In the remarkable result [27, Theorem 5.3], a sufficient condition for the sequence

$$0 \to G_*(F) \otimes \mathbb{Q} \xrightarrow{i_\sharp \otimes 1} G_*(F, X; i) \otimes \mathbb{Q} \xrightarrow{p_\sharp \otimes 1} \pi_*(Y) \otimes \mathbb{Q} \to 0 \tag{2.2}$$

to be exact is described in terms of the classifying map of the fibration in the sense of Stasheff [35]. It is important to mention that Theorem 2.6 follows from [27, Theorems 4.2 and 5.3] provided the given fibration is the localization of a fibration $F \to X \to Y$ of *simply-connected* CW complexes of finite type with fibre F *finite*. The fibration which yields the short exact sequence (1.2) is said to be Gottlieb trivial [27]. Theorem 2.6 asserts that the Gottlieb triviality of a fibration follows from the separability.

We turn our attention to the first evaluation subgroup of $\pi_1(X)$ for a nilpotent space X. When considering the subgroup, a detective element can be found with the knowledge of the minimal model for X, in particular, of the quadratic part of the differential if $\pi_1(X)$ is not abelian. This fact enables us to deduce Theorem 2.7 below.

Let G be a nilpotent group with the lower central series

$$G = \Gamma_1 G \supset \Gamma_2 G \supset \cdots \supset \Gamma_j G \supset \cdots,$$

where, for $j \geq 2$, $\Gamma_j G = [G, \Gamma_{j-1}G]$ stands for the subgroup of G generated by the commutators $\{xyx^{-1}y^{-1} \mid x \in G, y \in \Gamma_{j-1}G\}$. The nilpotency class of G, denoted $\mathrm{nil}(G)$, is defined to be the largest integer c such that $\Gamma_c G \neq \{1\}$. We write $(\Gamma_q/\Gamma_{q+1})G$ for the subquotient $\Gamma_q G/\Gamma_{q+1}G$.

Theorem 2.7. [20] *Let $f : U \to X$ be a map between a connected nilpotent spaces. Suppose that* (i) $\pi^1_\psi(f) : \pi^1_\psi(X) \to \pi^1_\psi(U)$ *is a monomorphism and that* (ii) U *is a finite CW complex or X is a rational space with* $\dim \oplus_{i \geq 2} \pi_i(X) < \infty$.
(1) *If $(\Gamma_k/\Gamma_{k+1})\pi_1(X)^\sharp \neq 0$, then for any $i < k$,*

$$\dim\Big(\Gamma_i G_1(U,X;f)\big/\Gamma_{i+1}\pi_1(X) \cap \Gamma_i G_1(U,X;f)\Big) \otimes \mathbb{Q} \leq \dim(\Gamma_i/\Gamma_{i+1})\pi_1(X) \otimes \mathbb{Q} - 1.$$

(2) *If $([\pi_1(X), \pi_1(X)]/\Gamma_3\pi_1(X))^\sharp \neq 0$, then*

$$\dim\Big(G_1(U,X;f)\big/[\pi_1(X), \pi_1(X)] \cap G_1(U,X;f)\Big) \otimes \mathbb{Q} \leq \dim H_1(X;\mathbb{Q}) - 2.$$

We see that the subgroup $\Gamma_i G_1(U,X;f)\big/\Gamma_{i+1}\pi_1(X) \cap \Gamma_i G_1(U,X;f)$ of the quotient group $(\Gamma_i/\Gamma_{i+1})\pi_1(X)$ is proper for any $i \geq 1$ under the assumption in Theorem 2.7.

Corollary 2.8. [20] *If $G_1(U,X;f)$ is abelian and $\Big([\pi_1(X), \pi_1(X)]/\Gamma_3\pi(X)\Big)^\sharp \neq 0$, then*

$$\dim G_1(U,X;f) \otimes \mathbb{Q} \leq \dim([\pi_1(X), \pi_1(X)] \cap G_1(U,X;f)) \otimes \mathbb{Q} + \dim H_1(X;\mathbb{Q}) - 2.$$

If $g : S^1 \to X$ is any map such that $[g] \in G_1(U,X;f)$, then $g \perp f$. Hence, the result [30, Proposition 3.4 (1)] applies to an extension $\mu : S^1 \times U \to X$ of $g \vee f : S^1 \vee U \to X$. It follows that $[g]\cdot f_*(\alpha) = g_*([id_{S^1}])\cdot f_*(\alpha) = f_*(\alpha)\cdot g_*([id_{S^1}]) = f_*(\alpha)\cdot[g]$ in $\pi_1(X)$ for any $\alpha \in \pi_1(U)$. Observe that $G_1(U,X;f)$ is contained in the center of the fundamental group $\pi_1(X)$ if the induced map $f_* : \pi_1(U) \to \pi_1(X)$ is surjective. In particular the Gottlieb group $G_1(X)$ is abelian.

We further give a computational example (Theorem 2.9 below) whose proof illustrates how the elaborate machinery in the paper [20] is relevant in computing Gottlieb groups. Consider the S^1-bundle $S^1 \to X_f \to T^n$ over the n-dimensional torus T^n with the classifying map f which is represented by $\rho_f = \sum_{i<j} c_{ij}t_it_j$ in $H^2(T^n;\mathbb{Z}) \cong [T^n, K(\mathbb{Z},2)]$. Here $\{t_i\}_{1\leq i\leq n}$ is a basis of $H^1(T^n;\mathbb{Z})$. Define an $(n\times n)$-matrix A_f by $A_f = \big(c'_{ij}\big)$, where $c'_{ij} = c_{ij}$ for $i<j$, $c'_{ij} = -c_{ji}$ for $i>j$ and $c_{ii} = 0$. We regard A_f as a matrix with entries in $\mathbb{Q}$. Then the rank of A_f is denoted by $\mathrm{rank}A_f$. We establish the following theorem.

Theorem 2.9. [20] $G_1(X_f) \cong \mathbb{Z}^{\oplus(1+n-\mathrm{rank}A_f)}$.

Since the space X_f is aspherical, it follows from [12, Corollary I.13] that $G_1(X_f)$ coincides with the center of $\pi_1(X_f)$. While we have the central extension

$$0 \to \mathbb{Z} \to \pi_1(X_f) \to \mathbb{Z}^{\oplus n} \to 0$$

from the homotopy exact sequence of the fibration $S^1 \to X_f \to T^n$, in general, it is not easy to determine the center of $\pi_1(X_f)$ by looking at the extension.

As for a more general T^k-bundle over T^n, we have not yet determined exactly the first Gottlieb group of the total space of the bundle. However the same argument as in the proof of Theorem 2.9 does work well to get useful information on the rank of the Gottlieb group. Let $T^k \to Y_g \to T^n$ be a T^k-bundle over T^n with the classifying map $g : T^n \to BT^k = K(\mathbb{Z},2)^{\times k}$. We write $g = f_1 \times \cdots \times f_k$ with maps

$f_i : K(\mathbb{Z},1)^{\times n} \to K(\mathbb{Z},2)$ and define the $(n \times n)$-matrix A_{f_i} for $1 \le i \le k$ as in the paragraph before Theorem 2.9. Let A_g be the $(nk \times n)$-matrix consisting of matrices A_{f_i}. Then we have

Proposition 2.10. $k \le \mathrm{rank}\ G_1(Y_g) \le k + n - \mathrm{rank} A_g$.

We say that a s-torus T^s acts almost freely on a space X if all its isotropy subgroups are finite. Define the *toral rank* of X by

$$\text{toral rank}(X) := \sup\{s |\ T^s \text{ acts almost freely on } X\}.$$

The following result due to Halperin gives an upper bound of the toral rank of a space.

Theorem 2.11. [17] *Let X be a connected finite CW complex with $\pi_i(X) \otimes \mathbb{Q} = 0$ for $i > m$. Then*

$$\text{toral rank}(X) \le \dim(G_1(X) \otimes \mathbb{Q}) - \sum_{i=2}^{m} (-1)^i \dim(\pi_i(X) \otimes \mathbb{Q}).$$

Suppose that the matrix A_g mentioned above is of full rank, for example,

$$A_{f_i} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

for some i in the case n is even. Then we see that rank $G_1(Y_g) = k$. It is readily seen that T^k acts freely on Y_g. Thus by combining Proposition 2.10 with the above theorem, we have

Corollary 2.12. *If the matrix A_g mentioned in Proposition 2.10 is of full rank, then*

$$\text{toral rank}(Y_g) = k.$$

We conclude this manuscript with remarks on models for a function space. One might expect Haefliger's model [19] for the connected component of a function space to work well in considering the evaluation subgroups or, more generally, the homotopy type of $\mathcal{F}(U, X; f)$. However it seems that the differential of the model is complicated in general because of the inductive argument in defining it. On the other hand, the model due to Brown and Szczarba has the advantage that its differential is expressed with an explicit formula; see [5]. This is the reason why we draw on the latter in our study on evaluation subgroups. We also wish to mention that the two models above coincide before minimization, if the function space considered is connected; see [21, Theorem 1.1].

We are convinced that both our explicit model and derivations on Sullivan models used in [25], [26] and [27] are useful tools for the study of rational evaluation subgroups.

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