THE DE RHAM THEOREM IN DIFFEOLOGY

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1. Introduction

This note is prepared for an intensive course at University of Tokyo from 13 June to 17 June, 2022. The aims of the lectures are to explain the de Rham theorem for diffeological spaces and its related topics starting with a crash course in diffeology.

Contents

2. An overview of diffeology

The papers [25, 9, 10] are references for the topics in this section. In the lecture, we may denote the references [25] and [9] by [IZ] and [CSW], respectively.

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2.1. **A crash course in diffeology.** We begin by recalling the definition of a diffeological space. Afterward, some examples of diffeological spaces are given.

Definition 2.1. ([42]) Let X be a set. A set $\mathcal{D}$ of functions $U \to X$ for each open subset U in $\mathbb{R}^n$ and for each $n \in \mathbb{N}$ is a *diffeology* of X if the following three conditions hold:

(1) (Covering) Every constant map $U \to X$ for each open subset $U \subset \mathbb{R}^n$ is in $\mathcal{D}$;
(2) (Compatibility) If $U \to X$ is in $\mathcal{D}$, then for any smooth map $V \to U$ from an open subset V of $\mathbb{R}^m$, the composite $V \to U \to X$ is also in $\mathcal{D}$;
(3) (Locality) If $U = \cup_i U_i$ is an open cover and $U \to X$ is a map such that each restriction $U_i \to X$ is in $\mathcal{D}$, then the map $U \to X$ is in $\mathcal{D}$.

In what follows, we may call an open subset of $\mathbb{R}^n$ a *domain*. A *diffeological space* (diff-space for short) $(X, \mathcal{D})$ is comprised of a set X and a diffeology $\mathcal{D}$ of X. A map from a domain to X and an element of a diffeology $\mathcal{D}$ are called a *parametrization* and a *plot* of X, respectively. Let $(X, \mathcal{D}^X)$ and $(Y, \mathcal{D}^Y)$ be diffeological spaces. A map $X \to Y$ is *smooth* if for any plot $p \in \mathcal{D}^X$, the composite $f \circ p$ is in $\mathcal{D}^Y$. All diffeological spaces and smooth maps form a category Diff.

We may write $C^\infty(X, Y)$ for a hom-set of the form $\mathrm{Hom}_{\mathsf{Diff}}(X, Y)$ in the category Diff.

Example 2.2. (1) Let M be a manifold*. Then, the underlying set and the *standard diffeology* $\mathcal{D}^M$ give rise to a diffeological space $(M, \mathcal{D}^M)$, where $\mathcal{D}^M$ is defined to be the set of all smooth maps $U \to M$ from domains to M in the usual sense. We have a functor $m : \mathsf{Mfd} \to \mathsf{Diff}$ from the category of manifolds defined by $m(M) = (M, \mathcal{D}^M)$.

(2) For a diffeological space $(X, \mathcal{D}^X)$ and a subset A of X, we define $\mathcal{D}(A)$ by $\mathcal{D}(A) := \{p : U \to A \mid U \text{ is a domain and } i \circ p \in \mathcal{D}^X\}$, where $i : A \to X$ is the inclusion. Then, the set $\mathcal{D}(A)$ is a diffeology of A, which is called the *sub-diffeology*. We call $(A, \mathcal{D}(A))$ a *diffeological subspace* of X.

(3) Let $\{(X_i, \mathcal{D}_i)\}_{i \in I}$ be a family of diffeological spaces. Then, the product $\Pi_{i \in I} X_i$ has a diffeology $\mathcal{D}$, called the *product diffeology*, defined to be the set of all parameterizations $p : U \to \Pi_{i \in I} X_i$ such that $\pi_i \circ p$ are plots of X_i for each $i \in I$, where $\pi_i : \Pi_{i \in I} X_i \to X_i$ denotes the canonical projection.

(4) More general, the *initial diffeology* $\mathcal{D}^Y$ for maps $h_i : Y \to (X_i, \mathcal{D}_i)$ for $i \in I$ is defined by $\mathcal{D}^Y := \{p : U \to Y \mid h_i \circ P \in \mathcal{D}_i \text{ for } i \in I\}$. This is the largest diffeology on Y making all h_i smooth.

(5) Let $(X, \mathcal{D}^X)$ and $(Y, \mathcal{D}^Y)$ be diffeological spaces. We consider the set $C^\infty(X, Y)$ of all smooth maps from X to Y. The *functional diffeology* is defined to be the set of parametrizations $p : U \to C^\infty(X, Y)$ whose adjoints $ad(p) : U \times X \to Y$ are smooth; see Proposition 2.11.

Example 2.3. (1) Let $\mathcal{F} := \{f_i : Y_i \to X\}_{i \in I}$ be a set of maps from diffeological spaces $(Y_i, \mathcal{D}^{Y_i})$ $(i \in I)$ to a set X. Then a diffeology $\mathcal{D}^X$ of X is defined to be the set of parametrizations $p : U \to X$ for each of which for each $r \in U$, (i) there exists an open neighborhood V_r of r in U and a plot $p_i \in \mathcal{D}^{Y_i}$ with $p|_{V_r} = f_i \circ p_i$, or (ii) there exists exists an open neighborhood V_r of r in U such that $p|_{V_r}$ is constant. We

*Note that "manifold" means Hausdorff, second countable finite dimensional one without boundary unless otherwise specified.

call $\mathcal{D}^X$ the *final diffeology* of X with respect to $\mathcal{F}$. This is the smallest diffeollogy on X making all f_i smooth. Observe that the condition (i) is only required in the definition if $X = \cup_{i\in I}\mathrm{Im} f_i$.

A surjective map $\pi : X \ \to Y$ between diff-spaces is called a *subduction* if the diffeology of Y coincides with the final diffeology defined by π.

(2) For a family of diffeological spaces $\{(X_i, \mathcal{D}_i)\}_{i\in I}$, the coproduct $\coprod_{i\in I} X_i$ has a diffeology $\mathcal{D}$, called the *sum diffeology*, defined by the final diffeology with respect to the set of canonical inclusions.

(3) Let $(X, \mathcal{D})$ be a diffeological space with an equivalence relation $\sim$. Then a difffeology of $X/\sim$, called the *quotient diffeology*, is defined by the final diffeology with respect to the quotient map $X \to X/\sim$.

With the constructions in Examples 2.2 and 2.3, we have

Theorem 2.4. ([2, Theorem 5.25]) *The category* Diff *is complete, cocommplete and cartesian closed.*

In fact, limits and colimits in Diff are defined by those in the category of sets with the sub-diffeology and the quotient diffeology, respectively.

Proposition 2.5. *For a diff-space* $(X, \mathcal{D}^X)$*,* $\mathrm{colim}_{P\in\mathcal{D}^X} U_P \cong X$ *as a diffeological space.*

Proof. We regard $\mathcal{D}^X$ as a category whose morphisms are smooth maps $f : U_P \to U_Q$ with $Q \circ f = P$. Then the colimit is regarded as the quotient of the coproduct $\coprod_{P\in\mathcal{D}^X} U_P$ by the equivalence relation generated by relations $u \sim f(u)$ for $u \in U_P$ and $f : U_P \to U_Q$. Observe that the colimit is endowed with the quotient diffeology. The universaliry of the colimit gives rise to a smooth map $\Psi : Z := \mathrm{colim}_{P\in\mathcal{D}^X} U_P \to X$ with $\Psi(u) = Q(u)$ for $u \in U_Q$.

We define a map $\Phi : X \to Z$ by $\Phi(x) = (*, i_x)$, where $i_x : \{*\} \to X$ is the 0-plot to x in X. We show that the map Φ is smooth. Suppose that $P : U_P \to X$ is a plot. Then, since $(u, P) \sim (*, i_{P(u)})$ for $u \in U_P$, the map $\Phi \circ P$ is thought as of the composite $U_P \to \coprod_{P\in\mathcal{D}^X} U_P \to Z$ of the canonical injection and the quotient. By the definition of the quotient diffeology, we see that $\Phi \circ P$ is smooth and hence so is Φ. It is immediate that $\Psi \circ \Phi = id_X$ and $\Phi \circ \Psi = id_Z$. We have the result. □

2.2. **More examples of diffeological spaces.** A group object G in Diff is called a *diffeological group*. By definition, the product $G \times G \to G$ and the inverse operation on G are smooth. A Lie group is a typical example of a diffeological group†. Moreover, we have an crucial example. Let M be a diffeological space and $\mathrm{Diff}(M)$ the group of diffeomorphisms on M.

Proposition 2.6. *Suppose that M is a finite dimensional manifold. Then* $\mathrm{Diff}(M)$ *is a diffeological group with the sub-diffeology of* $C^\infty(M, M)$ *endowed with functional diffeology.*

First, we deal with a general case.

Example 2.7. Let $(M, \mathcal{D}^M)$ be a diffeological space. Consider the inclusion $i : \mathrm{Diff}(M) \to C^\infty(M, M) =: F$ and the map $\mathrm{Inv} : \mathrm{Diff}(M) \to \mathrm{Diff}(M)$ defined by $\mathrm{Inv}(g) = g^{-1}$. With the initial diffeology $\mathcal{D} := i^*(\mathcal{D}^F) \cap (\mathrm{Inv})^*(\mathcal{D}^F)$, we have a diffeological group $\mathrm{Diff}(M)$, where $\mathcal{D}^F$ denotes the functional diffeology of F. In

†Observe that the functor m : Mfd $\to$ Diff preserves products.

fact, in order to prove that $\mathcal{D}$ is a diffeology, it suffices to show that the composite $m : \mathrm{Diff}(M)\times\mathrm{Diff}(M) \to \mathrm{Diff}(M)$ is smooth under the functional diffeology. To see this, let α and β be plots of $\mathrm{Diff}(M)$. Consider the composite $\alpha\times\beta : U_\alpha \times U_\beta \to \mathrm{Diff}(M)\times\mathrm{Diff}(M) \stackrel{m}{\to} \mathrm{Diff}(M)$ and its adjoint $ad(m\circ(\alpha\times\beta)) : U_\alpha\times U_\beta\times M \to M$. For any plot γ in $\mathcal{D}^M$, with adjoints $ad(\alpha) : U_\alpha\times M \to M$ and $ad(\beta) : U_\beta\times M \to M$, we see that

$$ad(m\circ(\alpha\times\beta))\circ(1\times 1\times\gamma) = ad(\alpha)\circ(1\times ad(\beta))\circ(1\times 1\times\gamma)$$

as a map from $U_\alpha \times U_\beta \times U_\gamma$ to M. The right-hand side is indeed smooth. Thus the composite is smooth map.

Proof of Proposition 2.6. Thanks to the argument in Example 2.7, it suffices to show that $(\mathrm{Inv})^*(\mathcal{D}^F) \supset i^*(\mathcal{D}^F)$; that is, the map Inv is smooth with respect to the functional diffeology. To this end, we prove that for a plot $\alpha : U_\alpha \to \mathrm{Diff}(M)$, the composite

$$ad(\mathrm{Inv}\circ\alpha) = ad(\mathrm{Inv})\circ(\alpha\times 1) : U_\alpha\times M \to \mathrm{Diff}(M)\times M \to M$$

is smooth. We define $\varphi : U_\alpha\times M \to U_\alpha\times M$ by $\varphi(x,m) := (x, ad(\alpha)(x,m))$. For each x_0, the map $ad(\alpha)(x_0,-) : M \to M$ is a diffeomorphism by assumption. Then the inverse function theorem yields that φ is a diffeomorphism in the usual sense. Moreover, we see that $\varphi^{-1}(x,n) = (x, ad(\mathrm{Inv})(\alpha\times 1)(x,n))$. It turns out that Inv is smooth. □

Assertion 2.8. ([2, 2.1 Example]) *The functor* $m : \mathsf{Mfd} \to \mathsf{Diff}$ *defined in Example 2.2(1) is a fully faithful embedding.*

Proof. A plot in the standard diffeology of a manifold factors through locally a chart of the manifold. Then the assignment m is also well defined on the morphisms. We show that the map $m : \mathrm{Hom}_{\mathsf{Mfd}}(M,N) \to \mathrm{Hom}_{\mathsf{Diff}}(m(M),m(N))$ defined by $m(f) = f$ is bijective. Let $f : m(M) \to m(N)$ be a smooth map in Diff. Then for a map $\varphi_i^{-1} : \varphi_i(U_i) \to m(M)$ defined by a chart, the composite $f\circ\varphi_i^{-1}$ is a plot of $m(N)$ and hence it is smooth. This implies that $f\circ\varphi_i^{-1}$ is regarded locally a chart of N followed by a smooth map. We see that f is smooth in Mfd. □

The category Diff is related to Top with adjoint functors. Let X be a topological space. Then the *continuous diffeology* is defined by the family of continuous parametrizations $U \to X$. This yields a functor $C : \mathsf{Top} \to \mathsf{Diff}$.

For a diffeological space $(M, \mathcal{D}_M)$, we say that a subset A of M is *D-open* if for every plot $P \in \mathcal{D}_M$, the inverse image $P^{-1}(A)$ is an open subset of the domain of P equipped with the standard topology. The family of D-open subsets of M defines a topology of M. Thus, by giving the topology to each diffeological space, we have a functor $D : \mathsf{Diff} \to \mathsf{Top}$ which is the left adjoint to C; see [41] for more details. The topology for a diffeological space M is called the *D-topology* of M[‡].

Assertion 2.9. *Let M be a manifold. Then U is a D-open subset of $m(M)$ if and only if U is an open subset of M. Thus, the composite $D\circ m$ is nothing but the forgetful functor U.*

[‡]Let X be a diff-space. Then we see that A be a D-open subset of $X/\sim$ with respect to the quotient diffeology if and only if A is an open subset with respect to the quotient topology of $D(X) \to X/\sim$; see [25, 2.12]

Proof. Let V be a D-open subset of $m(M)$ and $\{U_i, \varphi_i\}_i$ an atlas of M. We observe that the inverse of each chart is a plot of $m(M)$. By definition, the preimage $(\varphi_i^{-1})^{-1}(V) = \varphi_i(V \cap U_i)$ is open in $\varphi_i(U_i)$. Since $V = \cup_i(\varphi_i^{-1})(\varphi_i(V \cap U_i))$, it follows that V is an open subset of M.

Suppose that V is an open subset of M. Each plot $P : U_P \to M$ of $m(M)$ is smooth and then continuous. We see that the preimage $P^{-1}(V)$ is open in U_P. □

Lemma 2.10. ([10, Lemma 4.1]) *Let X and Y be two diffeological spaces. Suppose that $D(X)$ is locally compact Hausdorff. Then the natural bijection $D(X \times Y) \to D(X) \times D(Y)$ is a homeomorphism.*

Proof. For an open subsets U and V of Euclidian spaces, we see that (*): $D(U \times V) = D(U) \times D(V)$ because the D-topology of such an open subset is the usual topology; see Assertion 2.9 above. The functors $D : \mathsf{Diff} \to \mathsf{Top}$, $Z \times - : \mathsf{Diff} \to \mathsf{Diff}$ are the left adjoints. Moreover, the functor $W \times - : \mathsf{Top} \to \mathsf{Top}$ is the left adjoint if W is locally compact Hausdorff. Then these functors preserve the colomit. Thus we have

$$\begin{aligned} D(X \times Y) &= D(X \times \operatorname{colim}_{Q \in \mathcal{D}^Y} U_Q) = \operatorname{colim}_{Q \in \mathcal{D}^Y} \operatorname{colim}_{P \in \mathcal{D}^X} D(U_P \times U_Q) \\ &= \operatorname{colim}_{Q \in \mathcal{D}^Y} D(X) \times D(U_Q) = D(X) \times D(Y). \end{aligned}$$

Here the third and fourth equalities follow from (*) and the fact that $D(U_Q)$ and $D(X)$ are locally compact Hausdorff. □

Comparisons of the D-topology on function spaces and other topology are considered in [9, Section 4 and Appendix]. Notably, we have

Proposition 2.11. ([9, Proposition 4.2]) *The D-topology on the function space of diff-spaces contains the compact-open topology, that is, $\mathcal{O}(D(C^\infty(X,Y)))$ contains $\mathcal{O}_{CO}(C^\infty(X,Y))$ the relative topology of $C^\infty(X,Y)$ to Map$(D(X), D(Y))$ endowed with the compact-open topology.*

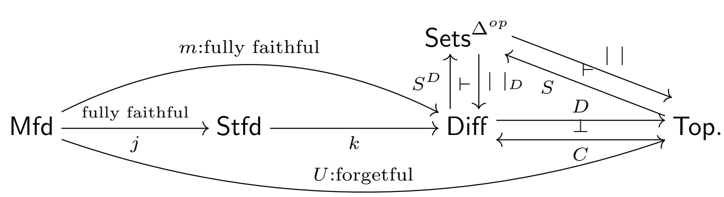
Proof. Let $B(K,W) := \{f \in C^\infty(X,Y) \mid f(K) \subset W\}$ be a subbasis of the compact-open topology, where K is a compact subset in $D(X)$ and W is an open subset of $D(Y)$. Let $\phi : U \to C^\infty(X,Y)$ be a plot. The the adjoint $ad(\phi) : U \times X \to Y$ is smooth and then $ad(\phi) : D(U \times X) \to D(Y)$ is continuous. For any $u \in U$, we see that $\{u\} \times K \subset ad(\phi)^{-1}(W)$ is an open subset of $D(U \times X) \cong D(U) \times D(X)$. The diffeomorphism follows from Lemma 2.10. The compactness of K implies that there exists an open neighborhood V of u such that $V \subset \phi^{-1}(A(K,W))$. □

2.3. **The category of diffeological spaces and its related categories.**

References for the material in this section are the papers [10, 29, 30, 41].

The following diagram assembles categories and functors which we address in this lecture.

(2.1)



Observe that the composite $k \circ j$ coincides with the embedding m. We refer the reader to [10, 29, 30, 41] for the functors C and D and their fundamental properties.

In particular, the adjointness for C and D follows from [41, Proposition 3.1]. It is immediate that $D = D \circ C \circ D$ and $C \circ D \circ C = C$; see Remark B.2.

The functors S^D and $| \ |_D$ are introduced by Christensen and Wu in [10]§; see Section 3.2. The adjointness of the pair is verified by the same proof as that for the singular simplex functor S and the realization functor $| \ |$ between the categories $\mathsf{Sets}^{\Delta^{op}}$ and Top. In fact, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Diff}}(|K|_D, Y) &\cong \lim_{\Delta[n]\to K} \mathrm{Hom}_{\mathsf{Diff}}(\mathbb{A}^n, Y) \\
&\cong \lim_{\Delta[n]\to K} S^D_n(Y) \quad \text{(by the definition of } S^D_n(Y)) \\
&\cong \lim_{\Delta[n]\to K} \mathrm{Hom}_{\mathsf{Sets}^{\Delta^{op}}}(\Delta[n], S^D(Y)) \quad \text{(by the Yoneda lemma)} \\
&\cong \mathrm{Hom}_{\mathsf{Sets}^{\Delta^{op}}}(K, S^D(Y)).
\end{aligned}$$

We observe that $|K|_D = \mathrm{colim}_{\Delta[n]\to K} \mathbb{A}^n$ by definition, where $\Delta[n]_\bullet$ denotes the standard simplex. The book [15] of Goerss and Jardine is a good reference for simplicial arguments in this lecture.

We conclude this section with comments on *concrete sheaves*. Let Euc be the category of open subsets of the Euclidian space $\mathbb{R}^N$ for each $N \geq 0$. Morphisms in Euc are usual smooth maps.

Definition 2.12. A presheaf $\widehat{X}$ on Euc values in sets is *concrete* if the map

$$\alpha : \widehat{X}(U) \to \mathrm{Hom}_{\mathsf{Sets}}(\mathrm{Hom}_{\mathsf{Euc}}(\mathbb{R}^0, U), \widehat{X}(\mathbb{R}^0))$$

defined by $\alpha(x)(P) := P^*(x)$ (induced by the structure maps) is injective. A *concrete sheaf* is a sheaf and a concrete presheaf.

Given a *non-empty* diff-space $(X, \mathcal{D}^X)$, we can define a presheaf $\widehat{X}$ by $\widehat{X}(U) := \mathcal{D}^X(U) = C^\infty(U, X)$. The axioms (2) and (3) of diff-spaces indeed allow us to deduce that $\widehat{X}$ is a concrete sheaf. The axiom (1) implies that each $\widehat{X}(U)$ is non-empty.

Let $\widehat{X}$ be a concrete sheaf. Then, we define a set $\mathcal{D}^X$ of parametrization of $X := \widehat{X}(\mathbb{R}^0)$ by $\mathcal{D}^X(U) = \alpha(\widehat{X}(U))$. We then have a diff-space $(X, \mathcal{D}^X)$. In fact, it is immediate that the axioms of sheaves give rise to the axioms (2) and (3). Suppose that $\widehat{X}(\mathbb{R}^0)$ is non-empty. In order to show that every constant map on $U \in \mathsf{Euc}$ is in $\mathcal{D}^X(U)$, we consider a sequence

$$\widehat{X}(\mathbb{R}^0) \xrightarrow{u^*} \widehat{X}(U) \xrightarrow{\alpha} \mathrm{Hom}_{\mathsf{Sets}}(\mathrm{Hom}_{\mathsf{Euc}}(\mathbb{R}^0, U), \widehat{X}(\mathbb{R}^0)) \cong \mathrm{Hom}_{\mathsf{Sets}}(U, \widehat{X}(\mathbb{R}^0)),$$

where $u : U \to \mathbb{R}^0$ is the trivial map. Then the naturality of the presheaf with respect to maps in Euc yields that for $y \in U$,

$$\alpha(u^*(x))(y) = y^*(u^*(x)) = (u \circ y)^*(x) = (\mathrm{id}_{\mathbb{R}^0})^*(x) = x.$$

The set $\mathcal{D}^X(U)$ contains the constant map at y. Moreover, we have

Proposition 2.13. ([2, Propositions 4.13 and 4.15]) *The category* Diff *is equivalent to the category of concrete sheaves.*

§The results [10, Propositions 4.10 and 4.11] show that the right-hand side triangle in (2.1) is commutative up to weak equivalence when regarding $\mathsf{Sets}^{\Delta^{op}}$ and Top as the target and the source.

3. The de Rham theorem in diffeology

The section is a main part of the lectures. Our main references are the papers [26, 33].

The topics in this subsection are the Souriau–de Rham comeplex Ω^*, the tautological map, Iglesias-Zemmour's integration map $\int^{\mathrm{IZ}}$, the singular de Rham complex A^*_{DR}, the factor map (natural transformation) $\alpha : \Omega^* \to A^*_{DR}$ and their fundamental properties.

3.1. **The Souriau–de Rham complex.** We here recall the de Rham complex $\Omega^*(X)$ of a diffeological space $(X, \mathcal{D}^X)$ in the sense of Souriau [42]. For an open set U of $\mathbb{R}^n$, let $\mathcal{D}^X(U)$ be the set of plots with U as the domain and $\Lambda^*(U) = \{h : U \longrightarrow \wedge^*(\oplus_{i=1}^n \mathbb{R}dx_i) \mid h \text{ is smooth}\}$ the usual de Rham complex of U. We can regard $\mathcal{D}^X(\ )$ and $\Lambda^*(\ )$ as functors from $\mathsf{Euc}^{\mathrm{op}}$ to Sets the category of sets. A *p-form* is a natural transformation from $\mathcal{D}^X(\ )$ to $\Lambda^*(\ )$. Then the de Rham complex $\Omega^*(X)$ is the cochain algebra consisting of p-forms for $p \geq 0$; that is, $\Omega^*(X)$ is the direct sum of

$$\Omega^p(X) := \left\{ \mathsf{Euc}^{\mathrm{op}} \underset{\Lambda^p}{\overset{\mathcal{D}^X}{\Downarrow \omega}} \mathsf{Sets} \;\middle|\; \omega \text{ is a natural transformation} \right\}$$

with the cochain algebra structure induced by that of $\Lambda^*(U)$ pointwisely. The de Rham complex defined above is certainly a generalization of the usual de Rham complex of a manifold.

Remark 3.1. Let M be a manifold and $\Lambda^*(M)$ the usual de Rham complex of M. We recall the *tautological map* $\theta : \Lambda^*(M) \to \Omega^*(M)$ defined by

$$\theta(\omega) = \{p^*\omega\}_{p \in \mathcal{D}^M}.$$

Then it follows that θ is an isomorphism of cochain algebras; see [20, Section 2].

In what follows, we may write ω for the assignment $\omega_U : \mathcal{D}^X(U) \to \wedge^p(U)$ for a differential form $\omega \in \Omega^p(X)$. For a smooth map $f : X \to Y$ in Diff, we define the *pullback* $f^*(\omega) \in \Omega^p(X)$ of a differential form $\omega \in \Omega^p(Y)$ by $f^*(\omega)(P) := \omega(f \circ P)$ for any plot P of X.

To get used to dealing with differential forms, we consider very carefully 0-forms on a diff-space.

Proposition 3.2. ([25, 6.31]) *Let X be a diffeological space. Then, as vector spaces $\Omega^0(X) \cong C^\infty(X, \mathbb{R})$ and $H^0_{\text{de Rham}}(X) \cong \mathrm{Maps}(\pi_0(X), \mathbb{R})$, where $\mathrm{Maps}(\pi_0(X), \mathbb{R})$ denotes the subspace of $C^\infty(X, \mathbb{R})$ consisting of constant maps on each smooth path-connected component.*

Proof. We define a linear map $\xi : \Omega^0(X) \to C^\infty(X, \mathbb{R})$ by $\xi(\omega)(x) := \omega([0 \mapsto x])(0)$, where $[0 \mapsto x]$ denotes the 0-plot of X valued at x. The map ξ is well defined. In fact, we have to verify that for a plot $P : U \to X$, the composite $\xi(\omega) \circ P : U \to \mathbb{R}$ is smooth. For any $r \in U$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}(U) & \xrightarrow{\omega_U} & \Lambda^0(U) = C^\infty(U, \mathbb{R}) \\ {\scriptstyle \mathcal{D}(\rho)}\downarrow & & \downarrow{\scriptstyle \Lambda^0(\rho) =: \rho^*} \\ \mathcal{D}(V) & \xrightarrow[\omega_V]{} & \Lambda^0(V) = C^\infty(V, \mathbb{R}), \end{array}$$

where $\rho : \{0\} = V \to U$ is defined by $\rho(0) = r$. This follows from the definition of the differential form. Thus, we see that

$$\begin{aligned}(\xi(\omega) \circ P)(r) &= (\omega([0 \mapsto P(r)])(0) \\ &= (\omega(P \circ [0 \mapsto r]))(0) \\ &= \rho^*(\omega(P))(0) \qquad \text{(by the commutativity of the diagram above)} \\ &= \omega(P)(r).\end{aligned}$$

This implies that $\xi(\omega) \circ P$ is smooth. The inverse map $\eta : C^\infty(X, \mathbb{R}) \to \Omega^0(X)$ is defined by $\eta(f)(P) = f \circ P$ for a plot P of X. In fact, it follows from the computation above that

$$(\eta(\xi(\omega)))(P) = \xi(\omega) \circ P = \omega(P).$$

Moreover, we have $\xi(\eta(f))(x) = \eta(f)([0 \mapsto x])(0) = f(x)$.

As for the second assertion, the definition of the differential $d : \Omega^p(X) \to \Omega^{p+1}(X)$ enables us to deduce that (*) : $d(\eta(f))(P) = d(\eta(f)(P)) = d(f \circ P)$ for a map $f \in C^\infty(X, \mathbb{R})$ and each plot $P : U \to X$. Then, we see that $d(\eta(f)) = 0$ if $f \in \mathrm{Maps}(\pi_0(X), \mathbb{R})$. This implies that the restriction $\eta_| : \mathrm{Maps}(\pi_0(X), \mathbb{R}) \to \mathrm{Ker}\, d$ is a well-defined injective map. Suppose that $d(\eta(f)) = 0$. Let $\gamma : \mathbb{R} \to X$ be a smooth path connecting points x and x'. Then the fact (*) yields that $d(f \circ \gamma) = 0$ and then f is in $\mathrm{Maps}(\pi_0(X), \mathbb{R})$. We see that the restriction $\eta_|$ is surjective. □

The following is one of important properties of the Souriau–de Rham complex.

Proposition 3.3. ([25, 6.39]) *Suppose that $\pi : X \to Y$ is a subduction; see Example 2.3 (1). Then $\Omega^*(\pi) : \Omega^*(Y) \to \Omega^*(X)$ is injective.*

Proof. Suppose that $\pi^*(\omega) = 0$ for $\omega \in \Omega^p(Y)$. We show that $\omega(P) = 0$ for any plot $P : U_P \to Y$ of Y. Since π is a subduction, it follows that P lifts locally to X; that is, for any $r \in U_P$, there exist an open neighborhood V of r in U_P and a plot $Q : V \to X$ such that $\pi \circ Q = P|_V$. By the definition of a differential form, namely a natural transformation, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}(V) & \xrightarrow{\omega_V} & \Lambda^p(V) \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle i^*} \\ \mathcal{D}(U_P) & \xrightarrow[\omega_{U_P}]{} & \Lambda^p(U_P) \end{array}$$

for the inclusion $i : V \to U_P$. Then we have $\omega(P|_V) = \omega(P)|_V$. Moreover, we see that $\omega(P|_V) = \omega(\pi \circ Q) = \pi^*(\omega)(Q) = 0$. This yields the result. □

Assertion 3.4. ([25, Excercise 105]) *Let T_Γ be the irrational torus $\mathbb{R}^n/\Gamma$, where Γ is a totally disconnected*[†] *dense subgroup of $\mathbb{R}^n$. Let $\pi : \mathbb{R}^n \to T_\Gamma$ be the canonical projection. Then the map $\pi^* : \Omega^*(T_\Gamma) \to \Omega^*(\mathbb{R}^n)$ induced by π gives rise to an isomorphism $\pi^* : \Omega^*(T_\Gamma) \xrightarrow{\cong} (\wedge^*_{ext}\mathbb{R}^n, d \equiv 0)$ of CDGA's, where $\wedge^*_{ext}V$ denotes the exterior algebra generated by a vector space V.*

Proof. We show that $a := \pi^*(\alpha)$ is a constant form for each $\alpha \in \Omega^*(T_\Gamma)$. Here we identify the element $a := \pi^*(\alpha)$ with the element $\theta^{-1}(a) = a(id_{\mathbb{R}^n})$ in the usual de Rham complex $\Lambda^p(\mathbb{R}^n)$ via the tautological map θ in Remark 3.1.

[†] Each connected component is a singleton.

For $\gamma \in \Gamma$, let $\gamma_* : \mathbb{R}^n \to \mathbb{R}^n$ be the map defined by $\gamma_*(x) = x + \gamma$. The commutative diagram

$$\begin{array}{ccc} \mathcal{D}(\mathbb{R}^n) & \xrightarrow{a} & \Lambda^p(\mathbb{R}^n) \\ {\scriptstyle (\gamma_*)^*}\uparrow & & \uparrow{\scriptstyle (\gamma_*)^*} \\ \mathcal{D}(\mathbb{R}^n) & \xrightarrow[a]{} & \Lambda^p(\mathbb{R}^n) \end{array}$$

enables us to deduce that $(\gamma_*)^*(a(id_{\mathbb{R}^n})) = (\pi^*)(\alpha((\gamma_*))) = \alpha(\pi\circ\gamma_*) = \alpha(\pi\circ id_{\mathbb{R}^n}) = a(id_{\mathbb{R}^n})$. This implies that $a(id_{\mathbb{R}^n})$ is invariant under the translation $\gamma \in \Gamma$.

We write

$$a(id_{\mathbb{R}^n})(x) = \sum_{i_1,\dots,i_k} a_{i_1\dots i_k}(x) e_{i_1} \wedge \cdots \wedge e_{i_k},$$

where $\{e_{i_j}\}$ denotes the canonical basis. Since $a(id_{\mathbb{R}^n})$ is Γ-invariant and $\gamma^*(e_{i_j}) = e_{i_j}$, it follows that $a_{i_1\dots i_k}(x+\gamma) = a_{i_1\dots i_k}(x)$ for $\gamma \in \Gamma$ and each $i_1, ..., i_k$. By assumption, the subgroup Γ is dense in $\mathbb{R}^n$. Then we see that each $a_{i_1\dots i_k}$ is constant. This yields that the image of π^* is contained in $(\wedge^*\mathbb{R}^n, d \equiv 0)$. Moreover, Proposition 3.3 yields that $\pi^* : \Omega^*(T_\Gamma) \to (\wedge^*\mathbb{R}^n, d \equiv 0)$ is injective. We observe that the target is regarded as a CDG subalgebra of $\Omega^*(\mathbb{R}^n)$.

We prove that the map π^* is an epimorphism by using the following lemma.

Lemma 3.5. ([25, 6.38] Pushing forms onto quotients) *Let $\pi : X \to X'$ be a subduction. Then $\alpha = \pi^*(\beta)$ for some $\beta \in \Omega^*(X')$ if and only if $\alpha(P) = \alpha(Q)$ for any plots P and Q of X with* $\mathrm{dom}(P) = \mathrm{dom}(Q)$ *and $\pi \circ P = \pi \circ Q$.*

Thus we show that $a(P) = a(P')$ for a p-form $a \in \wedge^p_{ext}(\mathbb{R}^n)$ and plots $P, P' : U \to \mathbb{R}^n$ with $\pi \circ P = \pi \circ P'$. Then we see that $P(r) - P'(r) \in \Gamma$ for any $r \in U$. Since the subgroup Γ is totally disconnected, it follows that $P - P'$ is a locally constant smooth map on U. Then there exists an element γ_j in Γ for each connected component U_j of U such that $(*)$: $P(r) = P'(r) + \gamma_j$ for any $r \in U_j$. This yields that

$$a(P) = P^*(a(id_{\mathbb{R}^n})) = (P')^*(a(id_{\mathbb{R}^n})) = a(P').$$

The second equality follows from $(*)$ above. In fact, we see that $P^*(dx_i) = (P')^*(dx_i)$ for any differential 1-form dx_i on $\mathbb{R}^n$. We observe that a is a constant p-form. Lemma 3.5 yields that π^* is an epimorphism. □

Remark 3.6. The de Rham theorem does *not* hold for the Souriau–de Rham complex in general. To see this, we first recall the irrational torus. Let $\pi : \mathbb{R}^2 \to S^1 \times S^1 = \mathbb{T}^2$ be the canonical projection and $\Delta_\theta := \{(x, \theta x) \mid x \in \mathbb{R}\} \cong \mathbb{R}$ be the subgroup of $\mathbb{R}^2$, where $\theta \in \mathbb{R}\backslash\mathbb{Q}$. Then, the irrational torus $\mathbb{T}_\theta$ is defined by the diff-space $\mathbb{T}^2/R_\theta$ with the quotient diffeology, where $R_\theta := \pi(\Delta_\theta)$. We observe that the quotient map π induces a diffeomorphims $\Delta_\theta \cong R_\theta$; see [25, 1.49]. Moreover, we have a diffeomorphism $\eta : \mathbb{T}_\theta \to \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$ defined by $\eta(x, y) = y - \theta x$. The inverse η' to η is indeed defined by $\eta'(x) = (0, x)$. Observe that $\mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$ is also endowed with the quotient diffeology.

We consider a diffeological bundle (see Section 4 for the definition) of the form $\mathbb{R} \to \mathbb{T}^2 \to \mathbb{T}_\theta$. Proposition 4.5 below ([10, Proposition 4.28]) allows us to obtain a Kan fibration $S^D(\mathbb{R}) \to S^D(\mathbb{T}^2) \to S^D(\mathbb{T}_\theta)$. By making use of the Leray–Serre spectral sequence for the Kan fibration ([28, Theorem 29.1] and [15, IV 5.1.]), we see that $H^*(S^D(\mathbb{T}_\theta); \mathbb{R}) \cong H^*(S^D(\mathbb{T}^2); \mathbb{R})$ as an algebra. Moreover, it follows from our main theorem (Theorem 3.11 below) that $H^*(S^D(\mathbb{T}^2); \mathbb{R}) \cong H^*(\mathbb{T}^2) \cong \wedge(t_1, t_2)$

as algebras. On the one hand, as mentioned above, there exists a diffeomorphism $\mathbb{T}_\theta \cong \mathbb{R}/(\mathbb{Z}+\theta\mathbb{Z})$. We observe that $\mathbb{Z}+\theta\mathbb{Z}$ is a totally disconnected dense subgroup of $\mathbb{R}$. Then Assertion 3.4 allows us to deduce that $H^*(\Omega^*(\mathbb{T}_\theta)) \cong \wedge^*_{ext}(\mathbb{R})$ as an algebra.

In order to prove the homotopy invariance of the de Rham cohomology, we recall part of the Cartan-de Rham calculus developed in the book [25][‡].

Let M be a diffeological space and $h : \mathbb{R} \to \mathrm{Diff}(M)$ a smooth map with $h(0) = \mathrm{id}_M$. Here $\mathrm{Diff}(M)$ denotes the diffeomorphism group endowed with the functional diffeology. We define a linear map $\mathcal{L}_h : \Omega^p(M) \to \Omega^p(M)$, which is called the *Lie derivation*, by

$$(\mathcal{L}_h(\alpha)(P)(s))(v_1, ..., v_p) := \frac{d}{dt}\alpha(h(t)\circ P)(s)(v_1, ..., v_p)|_{t=0}$$

for a p-form $\alpha \in \Omega^p(M)$, an n-plot P of M, $s \in \mathrm{dom}P$ and vectors v_l in $\mathbb{R}^n$; see [25, 6.54] for the differentiability of the function. Moreover, we define an integration operator $\Phi : \Omega^p(M) \to \Omega^p(\mathrm{Paths}(M))$ by

$$\Phi(\alpha)(P)(s)(v_1, ..., v_p) := \int_0^1 \alpha(ev_t \circ P)(s)(v_1, ..., v_p)dt \tag{3.1}$$

for a plot $P : U \to \mathrm{Paths}(M)$, where $\mathrm{Paths}(M)$ denotes the diff-space of smooth paths on M with the functional diffeology and $ev_t : \mathrm{Paths}(M) \to M$ is the evaluation map at t. We observe that the integration Φ is a cochain map; see [25, 6.79].

Let $\tau : \mathbb{R} \to \mathrm{Diff}(\mathrm{Paths}(M))$ be a map defined by $\tau(u)(\gamma) = \gamma \circ T_u$, where $T_u(t) = t + u$. Then, the map τ is a well-defined smooth map; see [25, 6.81]. Furthermore, we have

Proposition 3.7. *The diagram*

$$\begin{array}{ccc} \Omega^p(\mathrm{Paths}(M)) & \xrightarrow{\mathcal{L}_\tau} & \Omega^p(\mathrm{Paths}(M)) \\ \uparrow{\scriptstyle \Phi} & \nearrow_{(ev_1)^*-(ev_0)^*} & \\ \Omega^p(M) & & \end{array}$$

is commutative; that is, $\mathcal{L}_\tau(\Phi(\alpha)) = (ev_1)^*\alpha - (ev_0)^*\alpha$ *for any p-form α.*

Proof. For a plot $P : U \to \mathrm{Paths}(M)$ and $s \in U$, omitting the vectors $v_1, ..., v_p$, we have

$$\begin{aligned} (\mathcal{L}_\tau(\Phi(\alpha))(P)(s)) &= \frac{d}{du}\Phi(\alpha)(\tau(u)\circ P)(s)|_{u=0} = \frac{d}{du}\Big(u \mapsto \Phi(\alpha)(\tau(u)\circ P)(s)\Big)|_{u=0} \\ &= \frac{d}{du}\Big(u \mapsto \int_0^1 \alpha(ev_t \circ \tau(u) \circ P)(s)dt\Big)|_{u=0} \\ &= \frac{d}{du}\Big(u \mapsto \int_u^{u+1} \alpha(ev_{t'} \circ P)(s)dt'\Big)|_{u=0} \quad (\text{using } t' = t+u) \\ &= \alpha(ev_1 \circ P)(s) - \alpha(ev_0 \circ P)(s) \\ &= ((ev_1)^*\alpha(P))(s) - ((ev_0)^*\alpha(P))(s). \end{aligned}$$

[‡]Izumida gives a very thoughtful exposition on the homotopy invariance in his Master's thesis; see [23, Sections 2, 3 and 4]

Observe that we apply the naturality of the differential form α to show the last equality. $\square$

Here, for a smooth map $F : (-\varepsilon, \varepsilon) \to \mathrm{Diff}(M)$, we introduce the *contraction* $i_F : \Omega^p(M) \to \Omega^{p-1}(M)$ defined by

$$i_F(\alpha)(P)(s)(v_1, ..., v_{p-1}) := \alpha\Big(ad(F) \circ (1 \times P)\Big)(0, s)(\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, ... \begin{pmatrix} 0 \\ v_{p-1} \end{pmatrix})$$

By using a *variation* of the integral $\int_\sigma \alpha$ of a differential p-form α on a cubic simplex $\sigma \in C^\infty(\mathbb{R}^p, X)$ ([25, 6.70–6.71]), we obtain *Cartan's magic formula* for the Lie derivative and the contraction in diffeology. The integration $\int_\sigma \alpha$ due to Iglesias-Zemmour is recalled in Section 3.3.

Proposition 3.8. ([25, 6.72])(the Cartan–Lie formula) *Let $F : \mathbb{R} \to \mathrm{Diff}(M)$ be a smooth map. One has*

$$\mathcal{L}_F = [d, i_F](:= d \circ i_F + i_F \circ d).$$

Thus we have a lemma which enables us to obtain the homotopy invariance of the de Rham cohomology.

Lemma 3.9. *Let $\tau : \mathbb{R} \to \mathrm{Diff}(\mathrm{Paths}(M))$ be the smooth map defined in the paragraph before Proposition 3.7. Then, the map $K := i_\tau \circ \Phi$ defined by the contraction and the integration operator Φ in (3.1) gives a homotopy between chain maps $ev_1^*, ev_0^* : \Omega^*(M) \to \Omega^*(\mathrm{Paths}(M))$; that is, $K \circ d + d \circ K = ev_1^* - ev_0^*$.*

Proof. By virtue of Proposition 3.8, we see that

$$\mathcal{L}_\tau(\Phi(\alpha)) = i_\tau(d\Phi(\alpha)) + d(i_\tau\Phi(\alpha)) = i_\tau(\Phi(d\alpha)) + d(i_\tau\Phi(\alpha)) = K(d\alpha) + dK(\alpha)$$

for any $\alpha \in \Omega^p(M)$. Then, Proposition 3.7 allows us to obtain the result. $\square$

Theorem 3.10. *Let $f_0, f_1 : X \to Y$ be smooth homotopic smooth maps. Then the cochain maps $f_0^*, f_1^* : \Omega^*(Y) \to \Omega^*(X)$ are homotopic.*

Proof. Let $H : X \times \mathbb{R} \to Y$ be a smooth homotopy between f_0 and f_1. The adjoint $\varphi : X \to \mathrm{Paths}(Y)$ is smooth with $ev_i \circ \varphi = f_i$ for $i = 0$ and 1. Then, it follows from Lemma 3.9 that $f_1^* - f_0^* = \varphi^* \circ (ev_1)^* - \varphi^* \circ (ev_1)^* = \varphi^*(K \circ d + d \circ K) = (\varphi^* K) \circ d + d \circ (\varphi^* K)$. $\square$

3.2. **The singular de Rham complex.** We begin by recalling certain simplicial cochain algebras which formulate the de Rham theorem in diffeology.

Let $\mathbb{A}^n := \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i = 1\}$ be the affine space equipped with the sub-diffeology of $\mathbb{R}^{n+1}$. Let Δ^n_{sub} denote the diffeological space, whose underlying set is the standard n-simplex Δ^n, equipped with the sub-diffeology of the affine space $\mathbb{A}^n$. Let $(A^*_{DR})_\bullet$ be the simplicial cochain algebra defined by $(A^*_{DR})_n := \Omega^*(\mathbb{A}^n)$ for each $n \geq 0$.

Let Δ be the category which has posets $[n] := \{0, 1, ..., n\}$ for $n \geq 0$ as objects and non-decreasing maps $[n] \to [m]$ for $n, m \geq 0$ as morphisms. By definition, a simplicial set is a contravariant functor from Δ to Sets the category of sets. For a diffeological space $(X, \mathcal{D}^X)$, let $S^D_\bullet(X)$ be the simplicial set defined by

$$S^D_\bullet(X) := \{\{\sigma : \mathbb{A}^n \to X \mid \sigma \text{ is a } C^\infty\text{-map}\}\}_{n \geq 0}.$$

We mention that $S^D_\bullet(\text{-})$ gives the *smooth singular functor* defined in [10]. Moreover, let $S^D_\bullet(X)_{\text{sub}}$ denote the simplicial set defined by

$$S^D_\bullet(X)_{\text{sub}} := \{\{\sigma : \Delta^n_{\text{sub}} \to X \mid \sigma \text{ is a } C^\infty\text{-map}\}\}_{n\geq 0}.$$

We observe that the inclusion $j : \Delta^n_{\text{sub}} \to \mathbb{A}^n$ induces a morphism $j^* : S^D_\bullet(X) \to S^D_\bullet(X)_{\text{sub}}$; see [19] for the study of the simplicial set $S^D_\bullet(X)_{\text{sub}}$ in diffeology.

Let K be a simplicial set. We denote by $C^*(K)$ the cochain complex of maps from K_p to $\mathbb{R}$ in degree p and vanishing on degenerate simplices. The simplicial structure gives rise to the cochain algebra structure on $C^*(K)$; see, for example, [12, 10 (d)] for more detail. In particular, the multiplication on $C^*(K)$ is the *cup product* defined by $(f \cup g)(\sigma) = (-1)^{pq} f(d_{p+1}\cdots d_{p+q}\sigma)\cdot g(d_0\cdots d_0\sigma)$ for $f \in C^p(K)$ and $g \in C^q(K)$, where $\sigma \in K_{p+q}$ and d_i denotes the ith face map of K. We also recall the simplicial cochain algebra $(C^*_{PL})_\bullet := C^*(\Delta[\bullet])$, where $\Delta[n] = \hom_\Delta(-, [n])$ is the standard simplicial set.

For a simplicial cochain algebra $A_\bullet$, we denote by $A(K)$ the cochain algebra

$$\mathsf{Sets}^{\Delta^{\text{op}}}(K, A_\bullet) := \left\{ \Delta^{\text{op}} \underset{A_\bullet}{\overset{K}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \omega}}}} \mathsf{Sets} \;\middle|\; \omega \text{ is a natural transformation} \right\}$$

whose cochain algebra structure is induced by that of $A_\bullet$. Observe that, for a simplicial set K, the map $\nu : C^p_{PL}(K) \to C^p(K)$ defined by $\nu(\gamma)(\sigma) = \gamma(\sigma)(id_{[p]})$ for $\sigma \in K_p$ gives rise to a natural isomorphism $C^*_{PL}(K) \overset{\cong}{\to} C^*(K)$ of cochain algebras; see [12, Lemma 10.11]. Moreover, we have a cochain algebra of the form $A^*_{DR}(S^D_\bullet(X))$ for a diffeological space X. This is regarded as a diffeological variant of Sullivan's polynomial simplicial form for a topological space; see [43].

We introduce a map $\alpha : \Omega^*(X) \to A^*_{DR}(S^D_\bullet(X))$ of cochain algebras defined by

$$\alpha(\omega)(\sigma) = \sigma^*(\omega).$$

The maps α is called the *factor map* for X.

3.3. **The de Rham theorem in Diff.** As seen in Remark 3.6, *unfortunately*, the de Rham theorem concerning the Souriau–de Rham cohomology and the singular (cubic) cohomology does not hold for a diffeological space in general. By changing the de Rham complex to the singular one, we have a good situation. The following is de Rham's theorem in diffeology.

Theorem 3.11. ([33, Theorem 2.4]*, cf.* [22, Theorem 9.7], [17, Théorèmes 2.2.11, 2.2.14, 2.2.18]) *For a diffeological space* $(X, \mathcal{D}^X)$*, one has a homotopy commutative diagram*

$$\begin{array}{ccccccc}
C^*(S^D_\bullet(X)) & \underset{\simeq}{\xrightarrow{\ \varphi\ }} & (C^*_{PL}\otimes A^*_{DR})(S^D_\bullet(X)) & \underset{\simeq}{\xleftarrow{\ \psi\ }} & A^*_{DR}(S^D_\bullet(X)) & \xleftarrow{\ \alpha\ } & \Omega^*(X) \\
& {\scriptstyle =}\searrow & \downarrow{\scriptstyle \mathrm{mult}\circ(1\otimes\int)} & \swarrow{\scriptstyle \text{an ``integration'' }\int} & & \swarrow{\scriptstyle \int^{\mathrm{IZ}}} & \\
& & C^*(S^D_\bullet(X)) & \underset{\simeq}{\xleftarrow{\quad l\quad }} & C^*_{\text{cube}}(X) & &
\end{array}$$

in which φ *and* ψ *are quasi-isomorphisms of cochain algebras and the integration map* $\int$ *is a morphism of cochain complexes. Here* mult *denotes the multiplication on the cochain algebra* $C^*(S^D_\bullet(X))$*. Moreover, the factor map* α *is a quasi-isomorphism if* $(X, \mathcal{D}^X)$ *stems from a parametrized stratifold via the functor* k *in the diagram 2.1* *or is a finite dimensional smooth CW-complex in the sense of Iwase and Izumida* [22].

Remark 3.12. As seen in the proof of Theorem 3.11 below, we use the cochain complex $C^*(S^D_\bullet(X)_{\rm sub})$ for showing the existence of the quasi-isomorphism l; see Lemmas 3.21, 3.22 and the subsequent discussion I).

The quasi-isomorphisms in the first sequence of the diagram above are in the category of DGA's. Thus, we have the following corollary, which is not induced immediately by a Mayer–Vietoris exact sequence argument.

Corollary 3.13. *For every diffeological space* $(X, \mathcal{D}^X)$*, the integration map*

$$\int : A^*_{DR}(S^D_\bullet(X))) \to C^*(S^D_\bullet(X))$$

in Theorem 3.11 induces an isomorphism of algebras on the cohomology.

We need to explain some parts of the theorem above in more detail. As for the cubic cohomology, it is defined as follows. For a diff-space X, let $C_p(X)$ be an Abelian group defined by $C_p(X) := \{\sum^{\rm finite}_{\sigma \in C^\infty(\mathbb{R}^p, X)} n_\sigma \sigma \mid n_\sigma \in \mathbb{Z}\}$. A generator $\sigma \in C^\infty(\mathbb{R}^p, X)$ is called a *p-cube*. Define $\partial : C_p(X) \to C_{p-1}(X)$ by

$$\partial(\sigma) = \sum_{i=1}^{p+1} (-1)^i (\varepsilon^i_0(\sigma) - \varepsilon^i_1(\sigma)),$$

where $\varepsilon^i_s(\sigma)(t_1, ..., t_p) = \sigma(t_1, ..., t_{i-1}, s, t_i, ..., t_p)$. A direct calculation shows that $\partial^2 \equiv 0$. We call p-cube σ is degenerate if there exist a projection $pr : \mathbb{R}^p \to R^q$ with $q < p$ and a q-cube τ such that $\sigma = \tau \circ pr$. Let $C^p_{\rm cube}(X)$ be the vector space consisting of homomorphisms $C_p(X) \to \mathbb{R}$ which vanishes on *degenerate* p-cubes. Thus, we have a cochain complex of the form $(C^*_{\rm cube}(X), \delta :=$ the dual to $\partial)$. The cochain map $\int^{\rm IZ} : \Omega^*(X) \to C^*_{\rm cube}(X)$ in Theorem 3.11 is defined by

$$\int^{\rm IZ}_\sigma \omega = \int_{I^p} \omega(\sigma).$$

We observe that, by definition, $\omega(\sigma)$ is a p-form on $\mathbb{R}^p$. This integration is introduced by Iglesias-Zemmour in [26].

3.4. **The extendability of A^*_{DR}.** ([12, 33]) The fact that each morphism in the left-hand side triangles is a quasi-isomorphism follows from the extendability of the simplicial CDGAs that we use.

Definition 3.14. A simplicial DGA $A_\bullet$ is *extendable* if for any n, every subset set $\mathcal{I} \subset \{0, 1, ..., n\}$ and any elements $\Phi_i \in A_{n-1}$ for $i \in \mathcal{I}$ which satisfy the condition that $\partial_i \Phi_j = \partial_{j-1} \Phi_i$ for $i < j$, there exists an element $\Phi \in A_n$ such that $\Phi_i = \partial_i \Phi$ for $i \in \mathcal{I}$.

Proposition 3.15. ([12, Proposition 10.4]) *If a simplicial DGA* $A_\bullet$ *is extendable, then for an inclusion* $i : L \to K$ *of simplicial sets induces a surjective map.*

Proof. For any $\Psi \in A(L)$, we inductively construct $\Phi \in A(K)$ that restricts Ψ. Suppose that we have elements Φ_σ for $\sigma \in K_k$ and $k < n$ such that $\Phi_\sigma = \Psi_\sigma$ if $\sigma \in L_k$, (I) $\Phi_{\partial_i \sigma} = \partial_i \Phi_\sigma$ and (II) $\Phi_{s_j \tau} = s_j \Phi_\tau$ for $\tau \in K_m$ and $m < n - 1$. Then, we define Φ_σ for $\sigma \in K_n$ as follows. For $\sigma \in L_n$, define $\Phi_\sigma := \Psi_\sigma$. If $\sigma = s_j \tau$ for some $\tau \in K_{n-1}$, then the element $s_j\Phi_\tau$ is independent of the choice of j and τ; that is, if $s_j\tau = s_i\tau'(= \sigma)$ then $s_j\Phi_\tau = s_i\Phi_{\tau'}$. In fact, one of the simplicial identities shows that

$$\tau = \partial_j s_j = \partial_j s_i \tau' = \begin{cases} s_{i-1}\partial_j\tau' & (j < i) \\ s_i\partial_{j-1}\tau' & (j > i+1). \end{cases}$$

Say $j < i$. We have

$$\begin{aligned} s_j\Phi_\tau = s_j\Phi_{\partial_j s_i\tau'} &= s_j\Phi_{s_{i-1}\partial_j\tau'} \\ &= s_js_{i-1}\Phi_{\partial_j\tau'} = s_js_{i-1}\partial_j\Phi_{\tau'} \quad \text{by (II) (I)} \\ &= s_j\partial_js_i\Phi_{\tau'} = s_i\partial_{\tau'}. \end{aligned}$$

Thus we define $\Phi_\sigma := s_j\Phi_\tau$. Suppose that $\sigma \in K_n - L_n$ is non-degenerate. Then the condition (I) and one of the simplicial identities allow us to deduce that $\partial_i(\Phi_{\partial_j\sigma}) = \partial_{j-1}(\Phi_{\partial_i\sigma})$ for $i < j$. Since A is extendable, there exists $\Phi_\sigma \in A_n$ such that $\partial_i\Phi_\sigma = \Phi_{\partial_i\sigma}$ for $0 \leq i \leq n$. □

Proposition 3.15 gives rise to the following important result.

Proposition 3.16. ([12, Proposition 10.5]) *Let $\theta : D \to E$ be a morphism of simplicial DGA's. Assume that (i) $H(\theta_n) : H(D_n) \to H(E_n)$ is an isomorphism for $n \geq 0$ and that (ii) D and E are extendable. Then for every simplicial set K, $\theta(K) : D(K) \to E(K)$ is a quasi-isomorphism.*

With the same notations and assumptions as in Proposition 3.15, we have a differential ideal $A(K, L)$ of $A(K)$ which fits in the exact sequence $0 \to A(K, L) \to A(K) \to A(L) \to 0$. Define

$$\gamma : A(K(n), A(n-1)) \to \prod_{\sigma \in (NK)_n} A(\Delta[n], \Delta[n-1])$$

by $\gamma(\Phi) = \{A(\sigma)\Phi\}_{\sigma\in(NK)_n}$, where $(NK)_n$ is the subset of non-degenerate n-simplicies and $K(n)$ denotes the simplicial subset of K generated by K_i for $i \leq n$ and degenerate i-simplicies for $i > n$. If A is extendable, the map γ is an isomorphism; see [12, Lemma 10.6].

Sketch of the proof of Proposition 3.16. Use induction on n for $\partial\Delta[n]$, $(\Delta[n], \partial\Delta[n])$, $(K(n), K(n-1))$ and $K(n)$. Observe that $D_n = D(\Delta[n])$ and $E_n = E(\Delta[n])$. Comparing the exact sequences for D and E with the isomorphisms γ, we proceed the induction. □

Lemma 3.17. [33, Lemma 3.2]) *The simplicial CDGA $(A^*_{DR})_\bullet$ is extendable.*

Proof. Let $\mathcal{I}$ be a subset of $\{0, 1, ..., n\}$ and Φ_i an element in $(A^*_{DR})_{n-1}$ for $i \in \mathcal{I}$. We assume that $\partial_i\Phi_j = \partial_{j-1}\Phi_i$ for $i < j$. We define inductively elements $\Psi_r \in (A^*_{DR})_n$ for $-1 \leq r \leq n$ which satisfy the condition that (*): $\partial_i\Psi_r = \Phi_i$ if $i \in \mathcal{I}$ and $i \leq r$. Put $\Psi_{-1} = 0$ and suppose that Ψ_{r-1} is given with (*). Define a smooth map $\varphi : \mathbb{A}^n - l_r \to \mathbb{A}^{n-1}$ by

$$\varphi(t_0, t_1, ..., t_n) = \Big(\frac{t_0}{1-t_r}, .., \frac{t_{r-1}}{1-t_r}, \frac{t_{r+1}}{1-t_r}, ..., \frac{t_n}{1-t_r}\Big),$$

where l_r denotes the hyperplane $\{(t_0, t_1, ..., t_n) \in \mathbb{A}^n \mid t_r = 1\}$ in $\mathbb{A}^n$. The map φ induces a morphism $\varphi^* : \Omega^*(\mathbb{A}^{n-1}) \to \Omega^*(\mathbb{A}^n - l_r)$ of cochain algebras. If r is not in $\mathcal{I}$, we define Ψ_r by Ψ_{r-1}. In the case where $r \in \mathcal{I}$, we consider the element $\Phi_r - \partial_r\Psi_{r-1}$ in $\Omega^*(\mathbb{A}^{n-1})$.

Let $k_r : \mathbb{A}^n \to \mathbb{A}$ be the projection in the rth factor and ρ a *cut-off function* with $\rho(0) = 1$ and $\rho(1) = 0$. We observe that $(\rho \circ k_r)$ is in $\Omega^0(\mathbb{A}^n)$. Then the action of $(\rho \circ k_r)$ on $\Omega^*(\mathbb{A}^n - l_r)$ defined by the pointwise multiplication gives rise to a linear map $(\rho \circ k_r) \star \text{-} : \Omega^*(\mathbb{A}^n - l_r) \to \Omega^*(\mathbb{A}^n)$. We see that the map $(\rho \circ k_r) \star \text{-}$ fits in the commutative diagram

$$\begin{array}{ccccc} \Omega^*(\mathbb{A}^{n-1}) & \xrightarrow{\varphi^*} & \Omega^*(\mathbb{A}^n - l_r) & \xrightarrow{(\rho\circ k_r)\star} & \Omega^*(\mathbb{A}^n) \\ {\scriptstyle \partial_i}\downarrow & & {\scriptstyle \partial_i}\downarrow & & \downarrow{\scriptstyle \partial_i} \\ \Omega^*(\mathbb{A}^{n-2}) & \xrightarrow[\varphi^*]{} & \Omega^*(\mathbb{A}^{n-1} - l_{r-1}) & \xrightarrow[(\rho\circ k_{r-1})\star]{} & \Omega^*(\mathbb{A}^{n-1}) \end{array}$$

for $i < r$. Define $\Psi \in (A^*_{DR})_n$ by $\Psi := (\rho \circ k_r) \star \varphi^*(\Phi_r - \partial_r \Psi_{r-1})$. Since $\partial_i(\Phi_r - \partial_r \Psi_{r-1}) = \partial_{r-1}(\Phi_i - \partial_i \Psi_{r-1}) = 0$ by assumption for $i < r$, it follows from the commutative diagram above that $\partial_i \Psi = 0$ for $i < r$. Moreover, We see that $\partial_r(\rho \circ k_r) = 1$ and $\varphi \circ \partial^r = id_{\mathbb{A}^n}$. The facts enable us to deduce that the diagram

$$\begin{array}{ccccc} \Omega^*(\mathbb{A}^{n-1}) & \xrightarrow{\varphi^*} & \Omega^*(\mathbb{A}^n - l_r) & \xrightarrow{(\rho\circ k_r)\star} & \Omega^*(\mathbb{A}^n) \\ & \searrow^{id} & {\scriptstyle \partial_r}\downarrow & & \downarrow{\scriptstyle \partial_r} \\ & & \Omega^*(\mathbb{A}^{n-1}) & \xrightarrow[id]{} & \Omega^*(\mathbb{A}^{n-1}) \end{array}$$

is commutative. Thus we have $\partial_r \Psi = \Phi_r - \partial_r \Psi_{r-1}$. It turns out that $\partial_j(\Psi + \Psi_{r-1}) = \Phi_j$ for $j \in \mathcal{I}$ and $j \le r$. This completes the proof. $\square$

The affine space $\mathbb{A}^n$ is smoothly contractible. Then, Proposition 3.2 and Theorem 3.10 enable us to deduce the following result.

Lemma 3.18. *One has $H^*((A_{DR})_n) = \mathbb{R}$ for any $n \ge 0$.*

We moreover have

Lemma 3.19. ([12, Lemma 10.12)][18, 12.37]) *The simplicial cochain algebras C_{PL} and $C_{PL} \otimes A_{PL}$ are extendable and acyclic.*

Proposition 3.20. *Let K be a simplicial set. Then there is a sequence of quasi-isomorphisms*

$$C^*(K) \xleftarrow[\nu]{\cong} C^*_{PL}(K) \xrightarrow[\varphi]{\simeq} (C_{PL} \otimes A_{DR})^*(K) \xleftarrow[\psi]{\simeq} A^*_{DR}(K),$$

where φ and ψ are defined by $\varphi(\gamma) = \gamma \otimes 1$ and $\psi(\omega) = 1 \otimes \omega$, respectively.

This follows from Proposition 3.16; see [12, Section 10] for more details.

3.5. **An integration map.** We define a map $\int : (A^*_{DR})_\bullet \to (C^*_{PL})_\bullet = C^*(\Delta[\bullet])$ by

$$(\int \gamma)(\sigma) = \int_{\Delta^p} \sigma^* \gamma \tag{3.2}$$

for $\gamma \in (A^p_{DR})_n$, where $\sigma : \Delta^p \to \Delta^n$ is the affine map induced by a non-decreasing map $\sigma : [p] \to [n]$. Since the affine map σ is extended to an affine map σ from $\mathbb{A}^p$ to $\mathbb{A}^n$, it follows that $\sigma^*\gamma$ is in $(A^p_{DR})_p$. Then, the map $\int$ is a cochain map. This follows from Stokes' theorem for a manifold; see, for example, [4, V. Sections 4 and 5]. As a consequence, we see that $\int$ is a morphism of simplicial differential graded *modules.*

Let 1 be the unit of $(A^*_{DR})_\bullet$, which is in $C^\infty(\mathbb{A}^n, \mathbb{R}) = \Omega^0_{DR}(\mathbb{A}^n) = (A^0_{DR})_n$. Then we see that $\int 1 = 1$ in $(C^0_{PL})_n$ for $n \geq 0$. This yields the commutative diagram

$$\begin{array}{ccccc} (C^*_{PL})_\bullet & \xrightarrow{\ \varphi\ } & (C_{PL} \otimes A_{DR})^*_\bullet & \xleftarrow{\ \psi\ } & (A^*_{DR})_\bullet \\ & {\searrow}^{=} & \downarrow{\scriptstyle \mathrm{mult}\circ(1\otimes\int)} & {\swarrow}_{\int} & \\ & & (C^*_{PL})_\bullet . & & \end{array} \tag{3.3}$$

We refer the reader to [12, Remark, page 130] for the same triangles as above for the polynomial de Rham complex A^*_{PL}.

3.6. **The acyclic model theorem for a (co)chain complex.** In order to prove the homotopy commutativity of the right-hand side square in Theorem 3.11, the methods of acyclic models in [11, Definition, page 189] and [3] are applied.

Lemma 3.21. ([33, Lemma 4.1]) *If X is a convex subset of $\mathbb{R}^k$ with sub-diffeology, then the nth homology $H_n(S^D_\bullet(X))$ is trivial for $n > 0$ .The same assertion is valid for the functor $\mathbb{Z}S^D_\bullet(\text{-})_{\text{sub}}$.*

Lemma 3.22. *Let $\mathcal{M}$ be the set of convex subsets of $\mathbb{R}^k$ for $k \geq 0$. Then the two functors $\mathbb{Z}S^D_n(\text{-})$ and $\mathbb{Z}S^D_n(\text{-})_{\text{sub}}$ are representable for $\mathcal{M}$ in the sense of Eilenberg–Mac Lane for each n; see* [11, Definition, page 189].

Proof. Let $\widetilde{\mathbb{Z}S^D_n}(X)$ be the free abelian group generated by $\amalg_{M\in\mathcal{M}}(\mathbb{Z}S^D_n(M) \times \mathrm{Hom}_{\mathsf{Diff}}(M, X))$. Define a map $\Psi : \mathbb{Z}S^D_n(X) \to \widetilde{\mathbb{Z}S^D_n}(X)$ by $\Psi(m) = (id_{\mathbb{A}^n}, m)$. It is readily seen that $\Phi \circ \Psi = id$. Therefore, the functor $\mathbb{Z}S^D_n(-)$ is representable for $\mathcal{M}$. Since the identity map $id_{\Delta^n_{\text{sub}}}$ belongs to $\mathbb{Z}S^D_n(\Delta^n_{\text{sub}})_{\text{sub}}$, it follows from the same argument as above that the functor $\mathbb{Z}S^D_n(-)_{\text{sub}}$ is representable for $\mathcal{M}$. $\square$

I) The method of acyclic models [11, Section 8] implies that there exists a chain homotopy equivalence $\widetilde{l} : \mathbb{Z}S^D_\bullet(X)_{\text{sub}} \xrightarrow{\simeq} C_{\text{cube}*}(X)$. The dual to $\widetilde{l}$ yields a cochain homotopy equivalence $l : C^*_{\text{cube}}(X) \xrightarrow{\simeq} C^*(S^D_\bullet(X)_{\text{sub}})$, which induces a morphism of algebras on cohomology; see also [39, Section 8.2] for an acyclic model theorem for contravariant monoidal functors.

II) The restriction map $j^* : \mathbb{Z}S^D_\bullet(X) \to \mathbb{Z}S^D_\bullet(X)_{\text{sub}}$ has a homotopy inverse h in the category of chain complexes. This follows from the method of acyclic models [11, Theorems Ia and Ib] with Lemmas 3.21 and 3.22. Then the map $h^* : C^*(S^D_\bullet(X)) \to C^*(S^D_\bullet(X)_{\text{sub}})$ induces an isomorphism of *algebras* on the homology. In fact, the inverse induced by $(j^*)^* : C^*(S^D_\bullet(X)_{\text{sub}}) \to C^*(S^D_\bullet(X))$ is a morphism of algebras.

In order to prove Theorem 3.11, more observations concerning the cochain complexes in the theorem are now given. We prove III) the right-hand side square in Theorem 3.11 is homotopy commutative. To this end, We recall the acyclic model theorem due to Bousfield and Gugenheim [3].

Definition 3.23. Let $\mathcal{C}$ be a category and $\mathrm{Ch}^*(\mathbb{K})$ the category of cochain complexes over a field $\mathbb{K}$. A contravariant functor $K : \mathcal{C} \to \mathrm{Ch}^*(\mathbb{K})$ admits a *unit* if for each object X in $\mathcal{C}$, there exists a morphism $\eta_X : \mathbb{K} \to K(X)$ in $\mathrm{Ch}^*(\mathbb{K})$. Let $\mathcal{M}$ be a set of objects in $\mathcal{C}$, which is called *models*. A functor K with unit is *acyclic on models* $\mathcal{M}$ if for any M in $\mathcal{M}$, there exists a morphism $\varepsilon_M : K(M) \to \mathbb{K}$ such that $\varepsilon_M \circ \eta_M = id$ and $\eta_M \circ \varepsilon_M \simeq id$.

Let $F : \mathcal{C} \to \mathbb{K}\text{-Mod}$ be a functor from a category with models $\mathcal{M}$ to the category of vector spaces over $\mathbb{K}$. Then we define a contravariant functor $\widehat{F} : \mathcal{C} \to \mathbb{K}\text{-Mod}$ by

$$\widehat{F}(X) := \prod_{M \in \mathcal{M}, \sigma \in \mathcal{C}(M,X)} (F(M) \times \{\sigma\}),$$

where for a morphism $f : X \to Y$ in $\mathcal{C}$, the morphism $\widehat{F}(f) : \widehat{F}(Y) \to \widehat{F}(X)$ is defined by $\widehat{F}(f)\{m_\sigma, \sigma\} = \{m_{f\tau}, \tau\}$. Moreover, we define a natural transformation $\Phi : F \to \widehat{F}$ by $\Phi_X(u) = \{F(x)u, x\}$. We say that F is *corepresentable* on the models $\mathcal{M}$ if there exists a natural transformation $\Psi : \widehat{F} \to F$ such that $\Psi \circ \Phi = id_F$.

Theorem 3.24. [3, 2.4 Proposition][§] *Let $\mathcal{C}$ be a category with models $\mathcal{M}$. Let K_1 and K_2 be contravariant functors from $\mathcal{C}$ to* $\mathrm{Ch}^*(\mathbb{K})$ *with units $\eta : \mathbb{K} \to K_1^0, K_2^0$. Here $\mathbb{K}$ denotes the constant functor defined by $\mathbb{K}(X) = \mathbb{K}$. Suppose that K_1 is acyclic on models $\mathcal{M}$ and $U_k \circ K_2$ is corepresentable on the models for any k, where U_k denotes the forgetful functor to* $\mathbb{K}$-Mod *on the degree k. Then (i) there exists a natural transformation $T : K_1 \to K_2$ which preserves the unit. (ii) Moreover any two such natural transformations are naturally homotopic.*

To prove theorem above, we moreover need an extension of a natural transformation. Let $T : F \to G$ be a natural transformation. Then, we define a natural transformation $\widehat{T} : \widehat{F} \to \widehat{G}$ by $\widehat{T}(X)(\{m_\sigma, \sigma\}) = \{T(M)m_\sigma, \sigma\}$. We see that $\widehat{T}\Phi = \Phi T$.

Let F be a functor which is *acyclic on models* $\mathcal{M}$. Then by definiton, there exists a morphism $\varepsilon_M : F(M) \to \mathbb{K}$ such that $\varepsilon_M \circ \eta_M = id$ and $\eta_M \circ \varepsilon_M \simeq id$. Then we have a morphism $h_M : F^p(M) \to F^{p-1}(M)$ for each p and $M \in \mathcal{M}$ such that $dh_M + h_M d = 1 - \eta_M \varepsilon_M$. In this case, we define $\widetilde{h} : \widehat{F} \to \widehat{F}$ by

$$\widetilde{h}(\{m_\sigma, \sigma\}) = \{h_M(m_\sigma), \sigma\}.$$

Observe that $\widetilde{h}$ is a natural transformation. Similarly, we can define natural transformations $\widetilde{\varepsilon} : \widehat{F} \to \widehat{\mathbb{K}}$ and $\widetilde{\eta} : \widehat{\mathbb{K}} \to \widehat{F}$ with $\widetilde{d}\widetilde{h} + \widetilde{h}\widetilde{d} = 1 - \widetilde{\eta}\widetilde{\varepsilon}$ and $0 = 1 - \widetilde{\varepsilon}\widetilde{\eta}$.

Lemma 3.25. *Let $F : \mathcal{C} \to$ $Ch^*(\mathbb{K})$ be an acyclic functor on models and $G : \mathcal{C} \to$* $\mathbb{K}$-Mod *a corepresentable functor. Let $T : F^p \to G$ be a transformation of functors with $Td = 0$ and $T(M)\eta = 0$ on models. Then there exists a transformation of functor $T' : F^{p+1} \to G$ such that $T = T'd$.*

Proof. It is immediate that $\widehat{T}\widehat{d} = 0$ and $\widehat{T}\widetilde{\eta} = 0$. Since F is a functor, it follows that $\Phi d = \widetilde{d}\phi$. With the notations above, we define $T' := \Psi \circ \widehat{T} \circ \widetilde{h} \circ \Phi : F \to \widehat{F} \to \widehat{F} \to \widehat{G} \to G$. It follows that $T'd = \Psi\widehat{T}\widetilde{h}\Phi d = \Psi\widehat{T}\widetilde{h}\widetilde{d}\Phi = \Psi\widehat{T}(1 - \widetilde{\eta}\widetilde{\varepsilon} - \widetilde{d}\widetilde{h})\Phi = \Psi\widehat{T}\Phi = \Psi\Phi T = T$. □

Proof of Theorem 3.24. An induction on the degree of a cochain complex is used. (i) We define $\overline{T}^p : K_1^p(X) \to K_2^{p+1}(X)$ by $d \circ T^p$. By Lemma 3.25, we have $\overline{T}^{p+1} : K_1^{p+1}(X) \to K_2^{p+1}(X)$. (ii) For two such natural transformations T and T', define $L := T_p - T'_p - dL_p$. Then we see that $Ld = 0$ by the assumption of the induction. Lemma 3.25 enables us to obtain L_{p+1} with $L_{p+1}d = L$. □

[§]The original assertion is for the functors from the category of simplicial sets with models $\{\Delta[n]\}_n$ to $\mathrm{Ch}^*(\mathbb{K})$. However, the proof is valid for more general functors form a category with models.

Now, we are ready to prove III). In Theorem 3.24, we choose the category Diff as $\mathcal{C}$ and then put $K_1 = \Omega^*(\text{-})$ and $K_2 = C^*(S^D_\bullet(\text{-}))$. Let $\mathcal{M}$ be the subset of objects in $\mathcal{C}$ consisting of the affine spaces $\mathbb{A}^n$ for any $n \geq 0$. Then the category Diff is regarded as a category with models $\mathcal{M}$. The Poincaré lemma for diffeological spaces implies that the functor $\Omega^*(\text{-})$ is acyclic for $\mathcal{M}$. We observe that the *chain operator* K in Lemma 3.9 is thought of as a natural transformation.

For a non-negative integer $k \geq 0$, we define a map

$$\Psi_X : \widehat{C^k(S^D_\bullet(X))} := \prod_{\mathbb{A}^n \in \mathcal{M}, \sigma \in C^\infty(\mathbb{A}^n, X)} (C^k(S^D_\bullet(\mathbb{A}^n)) \times \{\sigma\}) \to C^k(S^D_\bullet(X))$$

by $\Psi_X(\{m_\sigma, \sigma\})(\tau) = m_\tau(id_{\mathbb{A}^k})$, where $\tau \in S^D_k(X)$. Then Ψ_- is a natural transformation. In fact, we see that for a smooth map $f : X \to Y$ and $u \in S^D_k(X)$,

$$\Psi_X(\widehat{C^k(S^D_k}(f))\{m_\sigma, \sigma\})(u) = \Psi_X\{m_{f\tau}, \tau\}(u) = m_{fu}(id_{\mathbb{A}^k}) \text{ and}$$
$$((C^k S^D_\bullet)(f))(\Psi_Y\{m_\sigma, \sigma\})(u) = \Psi_Y\{m_\sigma, \sigma\}(fu) = m_{fu}(id_{\mathbb{A}^k}).$$

Since $\Phi_X(u) = \{C^k(S^D_\bullet(\sigma))u, \sigma\}$ for $u \in C^k(S^D_\bullet(X))$ by definition, it follows that

$$\begin{aligned}(\Psi_X \Phi_X(u))(\tau) &= \Psi_X(\{C^k(S^D_\bullet(\sigma))u, \sigma\})(\tau) = C^k(S^D_\bullet(\tau))u(id_{\mathbb{A}^k}) \\ &= u(\tau \circ id_{\mathbb{A}^k}) = u(\tau)\end{aligned}$$

for $\tau \in S^D_k(X)$. Then we have $\Psi\Phi = id$ and hence $C^k(S^D_\bullet(\text{-}))$ is corepresentable. Theorem 3.24 enables us to deduce the homotopy commutativity of the right-hand side square in Theorem 3.11. This completes the proof of III).

Proof of the first assertion in Theorem 3.11. Proposition 3.20, the considerations in I), II), III) and the commutative diagram (3.3) allow us to deduce the first part. □

Sketch of the proof of the latter half of Theorem 3.11. Suppose that $(X, \mathcal{D}^X)$ is a manifold. Then the argument in [4, V. §9], in which the Mayer–Vietoris exact sequences are used, is applicable in order to deduce the map $v := \int \circ\alpha$ is a quasi-isomorphism.

By definition, a parametrized stratifold $(S, \mathcal{C})$ is constructed from manifolds with boundaries via an attaching procedure; see Appendix B. In general, a stratifold admits a partition of unity; see [31, Proposition 2.3]. Moreover, we see that an open set of the underlying topological space S is a D-open set of the diffeology $k(S, \mathcal{C})$; see Lemma A.3. Thus the induction argument with the Mayer–Vietoris sequence works well to show that $H(v)$ is an isomorphism. In fact, let S' be the parametrized stratifold mentioned in Appendix A, which is obtained from a stratifold S and a manifold W with collared boundary ∂W by using an attaching map $f : \partial W \to S$. In the inductive step, we can use open sets of $S' = S \cup_f W$. $U := S \cup_f (\partial W \times [0, \varepsilon))$ and $V := W - (\partial W \times [0, \varepsilon/2])$. Observe that U is smoothly homotopy equivalent to S and V is a manifold without boundary. Hence we have the result for a parametrized stratifold. □

We consider the excision axiom for the homology of $S^D_\bullet(X)_{\text{sub}} = \mathsf{Diff}(\Delta^n_{\text{sub}}, X)$. Let $D : \mathsf{Diff} \to \mathsf{Top}$ and $C : \mathsf{Top} \to \mathsf{Diff}$ be the functors mentioned in Appendix B, which give an adjoint pair. Kihara's proof of [30, Proposition 3.1] enables us to regard the chain complex $\mathbb{Z}S^D_\bullet(X)_{\text{sub}}$ as a subcomplex of the singular chain complex $C_*(DX)$, where $D : \mathsf{Diff} \to \mathsf{Top}$ denotes the functor mentioned in Section 2.3. In fact, [9, Lemma 3.16] implies that $D(\Delta^n_{\text{sub}})$ is the simplex Δ^n with the standard topology. We observe that for the diffeology $\mathbb{R}^n$ with smooth plots, $D(\mathbb{R}^n)$ is Euclidian space. Thus for a diffeological space X, the unit $id : X \to CDX$ yields the sequence of inclusions

$$\mathsf{Diff}(\Delta^n_{\text{sub}}, X) \to \mathsf{Diff}(\Delta^n_{\text{sub}}, CDX) \cong \mathsf{Top}(D(\Delta_{\text{sub}}), DX) = \mathsf{Top}(\Delta^n, DX);$$

see Remark B.3 for a discussion of when the composite of the maps in (4.1) is bijective. Then we can prove the excision axiom by applying a barycentric subdivision argument. Indeed, the subdivision map $Sd : S^D_n(X)_{\text{sub}} \to S^D_n(X)_{\text{sub}}$ is defined by restricting the usual one for the singular chain complex, which is chain homotopic to the identity. It turns out that the relative homology $H^D_*(X, A)$ satisfies the excision axiom for the D-topology; that is, the inclusion $i : (X - U, A - U) \to (X, U)$ induces an isomorphism on the relative homology if the closure of U is contained in the interior of A with respect to the D-topology of X; see [4, IV, Section 17] for details. Thus we have the Mayer–Vietoris exact sequences for the homology and cohomology of $S^D_\bullet(X)_{\text{sub}}$.

Remark 3.26. As seen in above, the homology $H^*(S^D_\bullet(X)_{\text{sub}})$ admits the Mayer–Vietoris exact sequence. Then, it follows from II) that so does $H^*(S^D_\bullet(X))$.

4. The LSSS, the EMSS and Chen's iterated integrals in diffeology

This section is devoted to a survey of the Leray–Serre spectral sequence, the Eilenberg–Moore spectral sequence and Chen's iterated integrals in diffeology. Each tool for developing computation of cohomology is obtained by modifying the original one in algebraic topology.

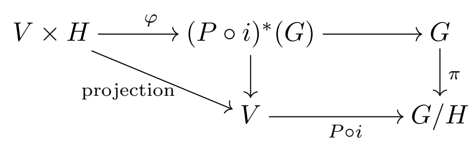
We begin with recalling one of equivalent definitions of a diffeological bundle; see [25, Chapter 8].

Definition 4.1. ([25, 8.9][10, Definition 3.14]) A smooth surjective map $p : X \to Y$ is a *diffeological bundle of fibre type* F if the pullback of p along any plot of Y is locally trivial of fibre type F.

We have an important example of a diffeological bundle.

Proposition 4.2. *Let G be a diffeological group; see Section 2.2, and H a subgroup with the sub-diffeology. Then the projection $G \to G/H$ is a diffeological bundle of fibre type H, where G/H is endowed with the quotient diffeology.*

Proof. For any plot $P : U \to G/H$ and a point $r \in U$, there exists a (local) plot $Q : V \to G$ such that $\pi \circ Q = P|_V$. This follows from the definition of the quotient diffeology. We define a map φ from $V \times H$ to the pullback of π



by $\varphi(r, h) := (r, Q(r)h)$. This is a well-defined smooth map. Moreover, we define $\varphi' : (P \circ i)^*(G) \to V \times H$ by $\varphi'(r, g) := (r, Q(r)^{-1}g)$. The map φ' is also well defined and smooth. It is readily seen that φ' is the inverse to φ. $\square$

4.1. **Spectral sequences.** The Leray–Serre spectral sequence [37, 16][†] and the Eilenberg–Moore spectral sequence [16, 44] are considered. We begin by setting up a situation to which we can apply the computational tools.

Definition 4.3. [10, Definition 4.7] A smooth map $X \to Y$ is a *fibration* if $S^D(X) \to S^D(Y)$ is a Kan fibration; that is, the simplicial map $S^D(X) \to S^D(Y)$ in $\mathsf{Sets}^{\Delta^{op}}$ has the right lifting property with respect to the inclusions $\Lambda[n]_k \to \Delta[n]$. Here $\Lambda[n]_k$ denotes the subfunctor of morphisms $[m] \to [n]$ factor through $\delta_i : [n-1] \to [n]$ for $0 \leq i \leq n$ and $i \neq k$.

We call a diffeological space X *fibrant* if $S^D(X)$ is a Kan complex; that is, the trivial map $S^D(X) \to *$ is a fibration. In general, a simplicial group is a Kan complex; see, for example, [15, I, Lemma 3.4]. Since the functor $S^D(-)$ is the right adjoint; see Section 2.3, it preserves limits, especially, products. Thus, the singular simplex functor $S^D(-)$ assigns a group objects in Diff that in $\mathsf{Sets}^{\Delta^{op}}$. Thus we have

Proposition 4.4. ([10, Proposition 4.30]) *Every diffeological group is fibrant.*

Proposition 4.5. ([10, Proposition 4.28]) *Any diffeological bundle (in the sense in Definition 4.1) with fibrant fibre is fibration.*

Proof. Let $f : X \to Y$ be a diffeological bundle with fibrant fibre F. Consider a commutative diagram

$$\begin{array}{ccc} \Lambda[n]_k & \longrightarrow & S^D(X) \\ \downarrow & & \downarrow{\scriptstyle S^D(f)} \\ \Delta[n] & \xrightarrow[u]{} & S^D(Y) \end{array}$$

in $\mathsf{Sets}^{\Delta^{op}}$. Consider the adjoint to the diagram above and the pullback along $c := ad(u)$, we have commutative diagrams [‡]

$$\begin{array}{ccc} \Lambda^n & \xrightarrow{b} & X \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{R}^n & \xrightarrow[c]{} & Y \end{array} \qquad \begin{array}{cccc} \Lambda^n & & & \\ & \overset{(a,e)}{\searrow} & & \overset{b}{\searrow} \\ & \mathbb{R}^n \times F & \xrightarrow{d} & X \\ {\scriptstyle a}\searrow & {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ & \mathbb{R}^n & \xrightarrow[c]{} & Y \end{array}$$

Here, we apply [25, 8.19], which asserts that the pullback of a diffeological bundle along a *global* plot is a trivial bundle. Let $g : \mathbb{R}^n \to F$ be *arbitrary* smooth map and consider the smooth section $(1, g) : \mathbb{R}^n \to \mathbb{R}^n \times F$. Then, we see that $f \circ \big(d \circ (1, g)\big) = c \circ \pi_1 \circ (1, g) = c$. We want to choose the map g so that the following left-hand side triangle commutes.

$$\begin{array}{ccc} \Lambda^n & \xrightarrow{b} & X \\ {\scriptstyle a}\downarrow & \nearrow_{d\circ(1,g)} & \\ \mathbb{R}^n & & \end{array} \qquad \begin{array}{ccc} \Lambda^n & \xrightarrow{e} & F \\ {\scriptstyle a}\downarrow & \nearrow_{g} & \\ \mathbb{R}^n & & \end{array}$$

[†]We also refer the reader to [38, Section 6, pages 225–229] for the Dress' construction of the spectral sequence.

[‡]$\Lambda^n := |\Lambda[n]_k|_D \cong \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_i = 0 \text{ for some } i\}$

Since F is fibrant, there exists a map g such that $g \circ a = e$. Then, for any $x \in \Lambda^n$, we have $(d \circ (1, g) \circ a)(x) = d((a(x), ga(x)) = d(a(x), e(x)) = b(x)$. □

We give an overview on the Leary–Serre spectral sequence and the Eilenberg–Moore spectral sequence ([33, Theorems 5.4 and 5.5]) in diffeology.

In what follows, we may write $A^*_{DR}(X)$ and $A^*(X)$ for $A^*_{DR}(S^D_\bullet(X)_{\mathrm{sub}})$ and $A^*_{DR}(S^D_\bullet(X))$, respectively.

Remark 4.6. We see that the map $(j^*) : S^D_\bullet(X) \to S^D_\bullet(X)_{\mathrm{sub}}$ induced by inclusion $j : \Delta^n_{\mathrm{sub}} \to \mathbb{A}^n$ gives rise to a natural quasi-isomorphism

$$(j^*)^* : A^*_{DR}(X) \to A^*(X). \tag{4.1}$$

This follows from III) in Section 3.3 and Proposition 3.20. In order to construct the spectral sequences below for pullback diagrams, we need Proposition 4.10. In the proof of the proposition, the compactness of $D(\Delta^n_{\mathrm{sub}})$ is used.

Let $g : (X, \mathcal{D}^X) \to (Y, \mathcal{D}^Y)$ be an induction. Then by definition, the map g is injective and the pullback diffeology $g^*(\mathcal{D}^Y)$ coincides with $\mathcal{D}^X$. The result [25, I. 36] yields that the map $g : X \overset{\cong}{\to} g(X)$ is a diffeomorphism, where $g(X)$ is the diffeological space endowed with subdiffeology.

Theorem 4.7. *Let $\pi : E \to M$ be a smooth map between path-connected diffeological spaces with path-connected fibre L which is* i) *a fibration in the sense of Christensen and Wu or* ii) *the pullback of the evaluation map $(\varepsilon_0, \varepsilon_1) : N^I \to N \times N$ for a connected diffeological space N along an induction $f : M \to N \times N$. Suppose further that in the case* ii) *the cohomology $H(A^*(M))$ is of finite type. Then one has the Leary–Serre spectral sequence $\{{}_{LS}E_r^{*,*}, d_r\}$ converging to $H(A^*(E))$ as an algebra with an isomorphism*

$${}_{LS}E_2^{*,*} \cong H^*(M, \mathcal{H}^*(L))$$

of bigraded algebras, where $H^*(M, \mathcal{H}(L))$ is the cohomology with the local coefficients $\mathcal{H}^*(L) = \{H(A^*(L_c))\}_{c \in S^D_0(M)}$; see, for example, Whitehead's book [45].

Theorem 4.8. *Let $\pi : E \to M$ be the smooth map as in Theorem 4.7 with the same assumption, $\varphi : X \to M$ a smooth map from a connected diffeological space X for which the cohomology $H(A^*(X))$ is of finite type and E_φ the pullback of π along φ. Suppose further that M is simply connected in case of* i) *and N is simply connected in case of* ii)*. Then one has the Eilenberg–Moore spectral sequence $\{{}_{EM}E_r^{*,*}, d_r\}$ converging to $H(A^*(E_\varphi))$ as an algebra with an isomorphism*

$${}_{EM}E_2^{*,*} \cong \mathrm{Tor}^{*,*}_{H(A^*(M))}(H(A^*(X)), H(A^*(E)))$$

of bigraded algebras.

Sketch of the proofs of Theorems 4.7 and 4.8. For the case i), the Leray–Serre spectral sequence and the Eilenberg–Moore spectral sequence are obtained by applying the same argument as in the proofs of [16, 5.1 Theorem and 7.3 Theorem] to the functor $A^*(\ ) := A^*_{DR}(S^D(\ )_\bullet)$. Observe that Dress' construction for the Leary-Serre spectral sequence is applicable to our setting; see [16, 3.3] and [37].

To consider the case ii), we use $A^*_{DR}(\ )$ instead of $A^*(\ )$. For a presheaf $\mathcal{F}$ over a simplicial set K, we define the space $\Gamma(\mathcal{F})$ of global sections of $\mathcal{F}$ by $\Gamma(\mathcal{F}) := \mathrm{Hom}_{\mathsf{Set}^{K^{\mathrm{op}}}}(1, \mathcal{F})$, where 1 denotes the terminal object of $\mathsf{Set}^{K^{\mathrm{op}}}$ the category of presheaves. We regard a simplicial set K as a category whose mmorphisms $u : \sigma \to \tau$ are morphisms of simplicial sets with commutative diagrams

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{\ u\ } & \Delta[n] \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & K. & \end{array}$$

Let K be the simplicial set $S^D_\bullet(M)_{\rm sub}$. Define a filtration $G = \{G^p\}_{p\geq 0}$ by $G^p = \Gamma(\sum_{i\geq p}(A^i_{DR})_\bullet \otimes F)$, where $F = \{F_\sigma\}_{\sigma\in K} := \{A^*_{DR}(\underline{P_\sigma})\}_{\sigma\in K}$ is a presheaf over K constructed with the pullbacks of $\pi : E \to M$ by simplices $\sigma : \Delta^n_{\rm sub} \to M$. The filtration gives rise to a spectral sequence $\{E_r^{*,*}, d_r\}$ converging to $H^*(A_{DR}(E))$; see [34, Pages 956–957].

Recall the integration map defined in (3.2). Then it follows from the proof of [18, 14.18 Theorem] that the integration induces a quasi-isomorphism

$$\int : E_1 = \Gamma((A^*_{DR})_\bullet \otimes H(F)) \to C^*(M; \mathcal{H}(L)),$$

where $C^*(M; \mathcal{H}(L))$ denotes the cochain complex of $S^D_\bullet(M)_{\rm sub}$ with the local coefficients induced by the local system F and $H(F)$ is the local system of coefficients defined by $H(F)_\sigma := H(F_\sigma, d)$ for $\sigma \in K$. An important point of the proof is that $(A^*_{DR})_\bullet$ and $(C_{PL})_\bullet$ are extendable; see [18, 12.37 Theorem] and Section 3. Then, the spectral sequence $\{E_r^{*,*}, d_r\}$ gives the one in Theorem 4.7.

As for the Eilenberg–Moore spectral sequence, by virtue of the result [18, 20.6] and Proposition 4.10 below, we have

$$H^*(A_{DR}(E_f)) \cong \mathrm{Tor}_{A^*_{DR}(M)}(A^*_{DR}(X), A^*_{DR}(E))$$

as an algebra; see also [44, Théorème 4.1.1]. As a consequence, the natural quasi-isomorphism $(j^*)^*$ in (4.1) yields the result in Theorem 4.8. □

4.2. **Chen's iterated integrals in diffeology.** We begin by modifying the iterated integrals due to Chen [7, 8] in the diffeological setting. Let N be a diffeological space and N^I the path space $C^\infty(I, N)$.

Let ω_i be a differential p_i-form in $\Omega^*(N)$ for each $1 \leq i \leq k$ and $\alpha : U \to N^I$ a plot of the diff-space N^I. Let $\rho : \mathbb{R} \to I$ be a cut-off function with $\rho(0) = 0$ and $\rho(1) = 1$. We define a plot $\sharp\alpha$ of N by $\sharp\alpha := ad(\alpha) \circ (1 \times \rho) : U \times \mathbb{R} \to U \times I \to N$. Then, the differential form w_i gives rise to a p_i-form $(\omega_i)_{\sharp\alpha}$ on $U \times \mathbb{R}$. We write $\widetilde{\omega_{i\alpha}}$ for the p_i form $(id_U \times t_i)^*(\omega_i)_{\sharp\alpha}$ on $U \times \mathbb{R}^k$, where

$$t_i : \mathbf{\Delta}^k := \{(x_1, ..., x_k) \in \mathbb{R}^k \mid 0 \leq x_1 \leq \cdots \leq x_k \leq 1\} \to \mathbb{R}$$

denotes the projection in the ith factor. By using integration along the fibre of the trivial fibration $U \times \mathbf{\Delta}^k \to U$, the iterated integral $(\int \omega_1 \cdots \omega_k)_\alpha$ is defined by

$$(\int \omega_1 \cdots \omega_k)_\alpha := \int_{\mathbf{\Delta}^k} \widetilde{\omega_{1\alpha}} \wedge \cdots \wedge \widetilde{\omega_{k\alpha}}.$$

Observe that $(\int \omega_1 \cdots \omega_k)_\alpha$ is of degree $\sum_{1\leq i\leq k}(\deg \omega_i - 1)$.

With a decomposition of the form $\Omega^1(N) = A^1 \oplus d\Omega^0(N)$, we obtain a cochain subalgebra A of $\Omega(N)$ which satisfies the condition that $A^p = \Omega^p(N)$ for $p > 1$ and $A^0 = \mathbb{R}$. The cochain algebra A gives rise to the normalized two-sided bar complex

$B(\Omega(N), A, \Omega(N))$; see [8, §4.1]. Consider the pullback diagram

$$\begin{array}{ccc} E_f & \xrightarrow{\widetilde{f}} & N^I \\ {\scriptstyle \nu = p_f}\downarrow & & \downarrow{\scriptstyle (\varepsilon_0,\varepsilon_1)} \\ M & \xrightarrow[f]{} & N \times N \end{array} \tag{4.2}$$

of $(\varepsilon_0, \varepsilon_1) : N^I \to N \times N$ along a smooth map $f : M \to N \times N$, where ε_i denotes the evaluation map at i. Then we have a chain map

$$\mathsf{It} : \Omega(M) \otimes_{\Omega(N)\otimes\Omega(N)} B(\Omega(N), A, \Omega(N)) \cong \Omega(M) \otimes_f \overline{B}(A) \to \Omega(E_f)$$

defined by $\mathsf{It}(v \otimes [\omega_1|\cdots|\omega_r]) = p_f^* v \wedge \big(\widetilde{f}^* \int \omega_1 \cdots \omega_r\big)$; see [6, Proposition 4.1.2] for the formula $d(\int \omega_1 \cdots \omega_r)$.

Theorem 4.9. *Suppose that, in the pullback diagram (4.2), the diffeological space N is simply connected and f is an induction. Assume further that the factor maps for N and M are quasi-isomorphisms and that the cohomology $H^*(S^D_\bullet(N))$ is of finite type, that is, each vector space $H^i(S^D_\bullet(N))$ is of finite dimension. Then the composite $\alpha \circ \mathsf{It} : \Omega^*(M) \otimes_f \overline{B}(A) \to \Omega(E_f) \to A^*_{DR}(S^D_\bullet(E_f))$ is a quasi-isomorphism of $\Omega^*(M)$-modules.*

Let $j : R := A_{DR}(M) \otimes \wedge V \to A_{DR}(E_f)$ be a Koszul–Sullivan (KS) extensions (relative Sullivan algebras)[§] for the map $\nu^* : A_{DR}(M) \to A_{DR}(E_f)$ induced by the projection $\nu : E_f \to M$. The reader is referred to [18, Chapter 1] for the definition of a KS extension and its fundamental properties. Let P_m denote the fibre over a point $m \in M$. Since the composite of ν and the inclusion $l : P_m \to E_f$ is the constant map at m, it follows that the map $l^* \circ \nu^*$ factors through the augmentation $\varepsilon : A_{DR}(M) \to A_{DR}(\{m\}) = \mathbb{R}$ and then j induces a morphism $k : \wedge V = A_{DR}(\{m\}) \otimes_{A_{DR}(M)} R \to A_{DR}(P_m)$ of cochain algebras.

Proposition 4.10. ([33, Proposition 5.11])[¶] *Suppose that N is simply connected. Then the morphism $k : \wedge V \to A_{DR}(P_m)$ of cochain algebras is a quasi-isomorphism.*

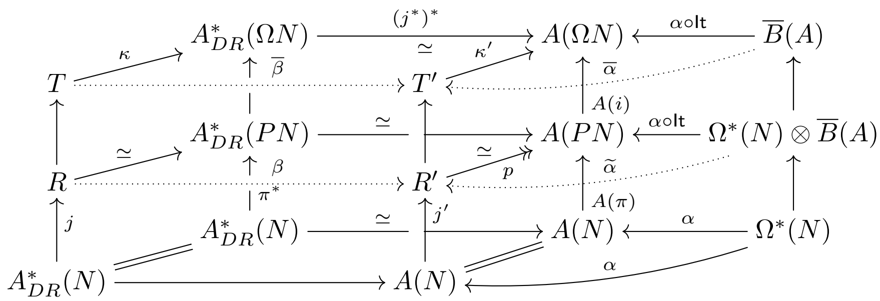
Proof of Theorem 4.9. For a diffeological space X, we recall the quasi-isomorphism $(j^*)^* : A^*_{DR}(X) \to A^*_{DR}(S^D_\bullet(X)) =: A(X)$ in (4.1). Let $\Omega N \to PN \to N$ be the pullback of the evaluation map $(\varepsilon_0, \varepsilon_1) : N^I \to N \times N$ along the induction $s : N \to N \times N$ defined by $s(x) = (*, x)$, where $*$ denotes the base point of N. We

---

[§]By definition, a commutative cochain algebra of the from $(B \otimes \wedge V, d)$ is a *relative Sullivan algebra* if i) $(B \otimes 1, d)$ is a cochain subalgebra and $H^0(B) = \mathbb{Q}$, ii) $1 \otimes V = V = \oplus_{p \geq 1} V^p$ and iii)$V = \cup_{k=0}^{\infty} V(k)$, where $V(0) \subset V(1) \subset \cdots$ is an increasing sequence of graded subspaces with $d : V(0) \to B$ and $d : V(k) \to B \otimes \wedge V(k-1)$ for $k \geq 1$. See [12, Section 14] and [18] for a general theory of relative Sullivan algebras.

[¶]The proof of this proposition is obtained by modifying that of [18, 20.3 Theorem]. The spectral sequence for the case (ii) in Theorem 4.7 is also applied. It is worth mentioning that we elaborate 'smooth objects' in the modification.

have a commutative diagram of solid arrows



in which j and j' are KS extensions of π^* and $A(\pi)$, respectively. Here A in $\overline{B}(A)$ denotes the cochain subalgebra of $\Omega^*(N)$ described in the paragraph before (4.2).

We may assume that the quasi-isomorphism p is a surjection by the surjective trick; see [12, Section 12 (b)]. By applying the Lifting lemma, we have a morphism $\beta : R \to R'$ which makes the two squares commutative. Then we have a morphism $\overline{\beta} : T := \mathbb{R} \otimes_{A_{DR}(N)} R \to T' := \mathbb{R} \otimes_{A(N)} R'$ of cochain algebras. Moreover, the map β is a quasi-isomorphism and hence so is $\overline{\beta}$ by [12, Theorem 6.10 (ii)]†.

Proposition 4.10 implies that κ is a quasi-isomorphism and then so is κ'. Since the bar complex $\Omega^*(N) \otimes \overline{B}(A)$ is a semifree $\Omega^*(N)$-module‡; see [13, Lemma 4.3 (ii)], it follows from the Lifting lemma that there exist a morphism $\widetilde{\alpha} : \Omega^*(N) \otimes \overline{B}(A) \to R'$ of $\Omega^*(N)$-modules and a morphism $\overline{\alpha} : \overline{B}(A) \to T'$ of differential graded modules which fit in the commutative diagram. Observe that $\overline{B}(A) \cong \mathbb{R} \otimes_{\Omega^*(N)} (\Omega^*(N) \otimes \overline{B}(A))$. The complex $\Omega^*(N) \otimes \overline{B}(A)$ is indeed a resolution of $\mathbb{R}$ and the diffeological space PN is smoothly contractible. Then the map $\widetilde{\alpha}$ is a quasi-isomorphism. Since the factor map is a quasi-isomorphism by assumption, it follows from [12, Theorem 6.10(ii)] again that so is $\overline{\alpha}$. We see that $\alpha \circ \mathsf{It} : \overline{B}(A) \to A(\Omega N)$ is a quasi-isomorphism.

We apply the same argument to the pullback $\Omega N \to E_f \to M$ of the evaluation map $(\varepsilon_0, \varepsilon_1) : N^I \to N \times N$ along an induction $f : M \to N \times N$. Then in the diagram above, the bar complex $\Omega^*(N) \otimes \overline{B}(A)$ is also replaced with the complex $\Omega^*(M) \otimes_f \overline{B}(A)$. In order to complete the proof, we use the notion of a semifree module and an 'algebraic spectral sequence argument'§. In view of the quasi-isomorphism $\alpha \circ \mathsf{It} : \overline{B}(A) \to A(\Omega N)$ that we obtain above, the comparison theorem (for the new R' and $\Omega^*(M) \otimes_f B(A)$ enables us to conclude that $\alpha \circ \mathsf{It} : \Omega^*(M) \otimes_f B(A) \to A(E_f)$ is a quasi-isomorphism. □

4.3. **Computational examples.** We recall the diffeological bundle $\mathbb{R} \to \mathbb{T}^2 \to \mathbb{T}_\theta$ in Remark 3.6. Let $f : M \to \mathbb{T}_\theta$ be a smooth map from a diffeological space M. Then we obtain a diffeological bundle $(*) : \mathbb{R} \to M \times_{\mathbb{T}_\theta} \mathbb{T}^2 \stackrel{\pi'}{\to} M$ via the pullback construction along the map f. A diffeological fibre bundle with a diffeological group as the fibre is a fibration; see Proposition 4.5. Then the Leray–Serre spectral

†That asserts that the tensor product preserves quasi-isomorphisms between semifree modules. Observe that KS extension is a semifree module over the base algebra; see [12, Lemma 14.1].

‡A *semifree module* is a version of a Sullivan algebra in an abelian category.

§The filtration $\{F^p\}$ is defined by $F^p := \Omega^{\geq p}(M) \otimes_f \overline{B}$. As for the KS extension $A(M) \to A(M) \otimes T'' \to T''$ in the right-hand side triangles, we use a filtration defined by '$F^p := A(M)^{\geq p} \otimes T''$ in order to construct the spectral sequence.

sequence in Theorem 4.7 for the fibration (*) allows us to deduce that π' gives rise to an isomorphism

$$(\pi')^* : H^*(A^*(M)) \xrightarrow{\cong} H^*(A^*(M \times_{\mathbb{T}_\theta} \mathbb{T}^2)) \tag{4.3}$$

of algebras. Suppose that M is simply connected. Then the comparison of the EMSS's in Theorem 4.8 for LM and $L(M \times_{\mathbb{T}_\theta} \mathbb{T}^2)$ allows us to obtain an algebra isomorphism

$$(L\pi')^* : H^*(A^*(LM)) \xrightarrow{\cong} H^*(A^*(L(M \times_{\mathbb{T}_\theta} \mathbb{T}^2)).$$

Thus if $H^*(A^*(M)) \cong H^*(A^*(S^{2k+1}))$ as an algebra with $k \geq 1$, then we see that

$$H^*(A^*(L(M \times_{\mathbb{T}_\theta} \mathbb{T}^2))) \cong \wedge(\alpha \circ \mathsf{It}((\pi')^*(\omega))) \otimes \mathbb{R}[\alpha \circ \mathsf{It}(1 \otimes (\pi')^*(\omega))] \tag{4.4}$$

as an $H^*(A^*(M))$-algebra, where ω is the volume form of M. In fact, the result follows from Theorem 4.9 and [32, Theorem 2.1 and Corollary 2.2]. Moreover, Corollary 3.13 asserts that we can also determine the singular cohomology algebra of $L(M \times_{\mathbb{T}_\theta} \mathbb{T}^2)$ with coefficients in $\mathbb{R}$.

Remark 4.11. We comment on the isomorphism (4.4). The map $\alpha \circ \mathsf{It}$ in Theorem 4.9 is a quasi-isomorphism of $\Omega^*(M)$-modules. However, in fact, it is a morphism of DGA's, where the bar complex is thought of as an algebra endowed with the shuffle product. This follows from [14, Proposition 4.1]. Thus we have a sequence of morphisms of DGA's except for the last map θ, which is a morphism of algebras on homology,

$$A(LM) \xleftarrow[\simeq]{\alpha\circ\mathsf{It}} \Omega^*(M) \otimes \overline{B}(A) \xleftarrow[\simeq]{v} \wedge V \otimes \overline{B}(\wedge V) \xrightarrow[\simeq]{u} \Gamma \otimes \overline{B}(\Gamma) \xrightarrow[\simeq]{\theta} \Gamma \otimes_{\Gamma\otimes\Gamma} \mathcal{F},$$

where $\wedge V$ denotes a minimal Sullivan model for M and $\Gamma = H^*(M)$. Moreover, the $\Gamma \otimes \Gamma$-module $\mathcal{F}$ is a Koszul resolution of Γ; see [32, Proposition 3.4] for the explicit form and the quasi-isomrphism θ. Thanks the formality of the given diffeological space M, we have the quasi-isomrphisms u and v. Thus, the computation is made with the two steps below. (i) Compute $\mathrm{Tor}_{\Gamma\otimes\Gamma}(\Gamma,\Gamma) = H^*(\Gamma \otimes_{\Gamma\otimes\Gamma} \mathcal{F}, \pm 1 \otimes d)$ as an algebra clarifying generators. (ii) Describe the generators in terms of the bar complex $\Omega^*(M)\otimes\overline{B}(A)$ via θ. The method is applicable to the case where $M = S^{2n}$ and $\mathbb{C}P^n$; see [32, Proposition 3.4] again.

5. Ω^* versus A^*_{DR}

We compare the Souriau–de Rham complex Ω^* with the singular de Rham complexes A^*_{DR} by using the factor map. We refer the reader to [33, Appendix] and [34] for more details of this section.

Let $(X, \mathcal{D}^X)$ be a diffeological space and $\mathcal{G}$ a generating family of $\mathcal{D}^X$ in the sense of [25, 1.65]; see also Section B.2. We may assume that the domain of each plot in $\mathcal{G}$ is a ball in $\mathbb{R}^N$ for some N. Then we define the *nebula* $\mathcal{N}_X$ of X associated with $\mathcal{G}$ by

$$\mathcal{N}_X := \coprod_{\varphi\in\mathcal{G}} \big(\{\varphi\} \times \mathrm{dom}(\varphi)\big)$$

with sum diffeology, where $\mathrm{dom}(\varphi)$ denotes the domain of the plot φ. It is readily seen that the evaluation map $ev : \mathcal{N}_X \to X$ defined by $ev(\varphi, r) = \varphi(r)$ is smooth.

The *gauge monoid* M is a submonoid of the monoid of endomorphisms on the nebula $\mathcal{N}_X$ defined by

$$\mathsf{M} := \{f \in C^\infty(\mathcal{N}_X, \mathcal{N}_X) \mid ev \circ f = ev \text{ and } \sharp\,\mathrm{Supp}\, f < \infty\},$$

where $\mathrm{Supp}\, f := \{\varphi \in \mathcal{G} \mid f|_{\{\varphi\}\times \mathrm{dom}(\varphi)} \neq 1_{\{\varphi\}\times\mathrm{dom}(\varphi)}\}$. Then the original de Rham complex $\Omega^*(\mathcal{N}_X)$ is a left $\mathbb{K}\mathsf{M}^{\mathrm{op}}$-module whose actions are defined by f^* induced by an endomorphism $f \in \mathcal{N}_X$. Moreover, the complex $\Omega^*(\mathcal{N}_X)$ is regarded as a two sided $\mathbb{K}\mathsf{M}^{\mathrm{op}}$-module for which the right module structure is trivial. Then we have the Hochschild complex $C^{*,*} = \{C^{p,q}, \delta, d_\Omega\}_{p,q\geq 0}$ with

$$C^{p,q} = \mathrm{Hom}_{\mathbb{K}\mathsf{M}^{\mathrm{op}}\otimes\mathbb{K}\mathsf{M}}(\mathbb{K}\mathsf{M}^{\mathrm{op}} \otimes (\mathbb{K}\mathsf{M}^{\mathrm{op}})^{\otimes p} \otimes \mathbb{K}\mathsf{M}, \Omega^q(\mathcal{N}_X)) \cong \mathrm{map}(\mathsf{M}^p, \Omega^q(\mathcal{N}_X)),$$

where the horizontal map δ is the Hochschild differential and the vertical map d_Ω is induced by the de Rham differential on $\Omega^*(\mathcal{N}_X)$; see [27, Subsection 12] for more details. The total complex Tot $C^{*,*}$ has the horizontal filtration $F^* = \{F^j\}_{j\geq 0}$ defined by $F^j = \oplus_{q\geq j} C^{*,q}$. Then the filtration gives rise to a first quadrant spectral sequence $\{{}_\Omega E_r^{*,*}, d_r\}$ converging to $H^*(\mathrm{Tot}\ C^{*,*})$ with

$${}_\Omega E_2^{p,q} \cong H^q(HH^p(\mathbb{K}\mathsf{M}^{\mathrm{op}}, \Omega^*(\mathcal{N}_X)), d_\Omega),$$

which is called the Čech–de Rham spectral sequence [27].

The result [27, Proposition in Subsection 9] implies that for a diffeological space X, the evaluation map induces an isomorphism $ev^* : \Omega^*(X) \xrightarrow{\cong} \Omega^*(\mathcal{N}_X)^{\mathsf{M}}$ of cochain algebras, where $\Omega^*(\mathcal{N}_X)^{\mathsf{M}}$ denotes the invariant subcomplex of $\Omega^*(\mathcal{N}_X)$. Since the kernel of the map $\delta : C^{0,q} \to C^{1,q}$ is nothing but the complex $\Omega^*(\mathcal{N}_X)^{\mathsf{M}}$, it follows that the edge homomorphism

$$edge_1 := H(ev^*) : H^*(\Omega^*(X)) \xrightarrow{\cong} {}_\Omega E_2^{0,*}$$

is an isomorphism.

We consider the vertical filtration of the total complex Tot $C^{*,*}$ and the spectral sequence $\{{}^\delta E_r^{*,*}, d_r\}$ associated with the filtration. Then the Poincaré lemma for the original de Rham complex yields that ${}^\delta E_r^{p,q} = 0$ for $q > 0$ and hence the target of $\{{}^\delta E_r^{*,*}, d_r\}$ is the Hochschild cohomology (the Čech cohomology) $\check{H}(X) := HH^*(\mathbb{K}\mathsf{M}, \mathrm{map}(\mathcal{G}, \mathbb{K}))$; see [27, Sections III and IV]. Moreover, we see that the edge homomorphism

$$edge_2 := H(inc_*) : \check{H}(X) \xrightarrow{\cong} H^*(\mathrm{Tot}\ C^{*,*})$$

is an isomorphism, where inc_* is the map induced by the inclusion $\mathrm{map}(\mathcal{G}, \mathbb{K}) \to \Omega^0(\mathcal{N}_X)$ to the constant functions on $\mathcal{N}_X$.

In the constructions of the two spectral sequence above, it is possible to replace the de Rham complex $\Omega^*(X)$ with the singular de Rham complex $A^*(X)$. While the map

$$edge_1 : H^*(A^*(X)) \to {}_A E_2^{0,*}$$

for $A^*(X)$ is merely a morphism of algebras, the map $edge_2$ for $A^*(X)$ is an isomorphism. In fact, Lemma 3.21 and Theorem 3.11 imply that the Poincaré lemma for the singular de Rham complex holds. Since the factor map α gives rise to a natural transformation $\Omega^*(\ ) \to A^*(\ )$, it follows that the map induces a morphism

$$\{f(\alpha)_r\} : \{{}_\Omega E_r^{*,*}, d_r\} \to \{{}_A E_r^{*,*}, d_r\}$$

of spectral sequences. As a consequence, we have a commutative diagram

$$
(5.1)\quad
\begin{array}{ccccccccc}
H^*(\Omega(X)) & \xrightarrow[\cong]{ev^*} & H^*(\Omega(\mathcal{N}_X)^{\mathsf{M}}) & \twoheadrightarrow & {}_{\Omega}E_{\infty}^{0,*} & \rightarrowtail & H^*(\mathrm{Tot}\ C^{*,*}) & \xleftarrow[\cong]{edge_2} & \\
\downarrow{\scriptstyle H(\alpha)} & & \downarrow{\scriptstyle f(\alpha)_2} & & \downarrow{\scriptstyle f(\alpha)_\infty} & & \downarrow{\scriptstyle H^*(\mathrm{Tot}(\alpha))} & & \check{H}^*(X), \\
H^*(A(X)) & \xrightarrow{ev^*} & H^*(A(\mathcal{N}_X)^{\mathsf{M}}) & \twoheadrightarrow & {}_{A}E_{\infty}^{0,*} & \rightarrowtail & H^*(\mathrm{Tot}\ {}'C^{*,*}) & \xleftarrow[\cong]{edge_2} &
\end{array}
$$

where $\mathrm{Tot}(\alpha)$ denotes the cochain map induced by the morphism $C^{*,*} \to {}'C^{*,*}$ of double complexes which $\alpha : A^*(X) \to \Omega^*(X)$ gives. In particular, it follows that $H^q(\mathrm{Tot}(\alpha))$ is an isomorphism for any q. We call the composite of maps in the first line in the diagram (5.1) the *edge homomorphism* of the Čech–de Rham spectral sequence for X. Then the commutative diagram above enables us to deduce the following proposition.

Proposition 5.1. *If the edge homomorphism $H^q(\Omega^*(X)) \to \check{H}^q(X)$ is injective, then the map $H^q(\alpha)$ is injective.*

In the first quadrant spectral sequence $\{{}_{\Omega}E_r^{*,*}, d_r\}$, the differential d_r is of degree $(-r+1, r)$, namely $d_r : {}_{\Omega}E_r^{p,q} \to {}_{\Omega}E_r^{p-r+1,q+r}$. Observe the grading of the filtration defined in [27]. It is readily seen that the sufficient condition in Proposition 5.1 is equivalent to saying that every elements in ${}_{\Omega}E_2^{0,q}$ in the Čech–de Rham spectral sequence is non-exact. By degree reasons, we see that each element in ${}_{\Omega}E_2^{0,1}$ is non-exact. Then Proposition 5.1 gives

Proposition 5.2. *For each diffeological space X, the map $H^1(\alpha) : H^1(\Omega^*(X)) \to H^1(A^*(X))$ induced by the factor map α is injective.*

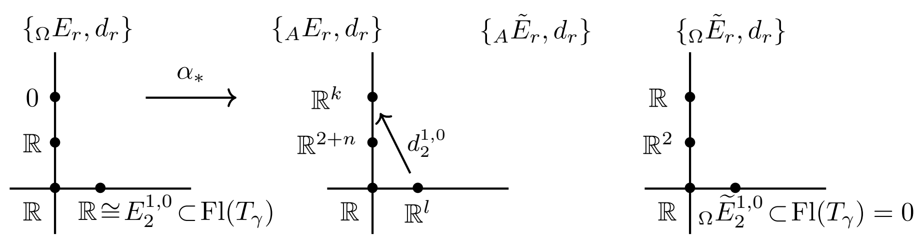
The naturality of the factor map $\alpha : A(X) \to \Omega(X)$ enable us to obtain a commutative diagram of isomorphisms

$$
\begin{array}{ccc}
H^1(\Omega(X)) \oplus {}_{\Omega}E_3^{1,0} & \xrightarrow[\cong]{\Theta} & H^1(A(\mathcal{N}_X)^{\mathsf{M}}) \oplus {}_{A}E_3^{1,0} \\
{\scriptstyle \mathrm{edge}_2,\ \cong}\nwarrow & & \nearrow{\scriptstyle \mathrm{edge}_2,\ \cong} \\
& \check{H}^1(X;\mathbb{R}). &
\end{array}
$$

In fact, by degree reasons, we see that the surjective maps ${}_K E_2^{0,1} \to {}_K E_\infty^{0,1}$ are isomorphisms and ${}_K E_3^{1,0} \cong {}_K E_\infty^{1,0}$ for $K = \Omega$ and A. Thus the map $H^*(\mathrm{Tot}(\alpha))$ yields the homomorphism Θ which fits in the triangle. As a consequence, we see that the map Θ is an isomorphism.

We recall Remark 3.6 which asserts a difference between the Souriau–de Rham cohomology and the singular de Rham cohomology of the irrational torus. For the irrational torus T_γ, we have a diffeological bundle $\mathbb{R} \to T^2 \xrightarrow{\pi} T_\gamma$.

$$
\Omega(T_\gamma) \xrightarrow{\alpha} A(T_\gamma) \xrightarrow[\text{LSSS (Theorem 4.7)}]{\pi^*\ \simeq} A(T^2) \xleftarrow[\simeq]{\alpha} \Omega^*(T^2)
$$



- We have $H^q(\Omega(T_\gamma)) \cong E_2^{0,q}$ and $\text{edge}_1^1 : H^1(A(T^2)) \subset {}_A\tilde{E}_2^{0,1}$. The injectivity of the edge homomorphism follows from the commutativity of the diagram (5.1). The naturarity of π^* in the vertical edge of the E_2-term yields that $\text{edge}_1^1 : H^1(A(T_\gamma)) \to {}_AE_2^{0,1}$ is injective; see [34, Lemma 3.1].
- ${}_\Omega E_r$ and ${}_AE_r$ converge to the Čech cohomology $\check{H}(T_\gamma) \cong H^*(\mathbb{Z} \oplus \mathbb{Z}; \mathbb{R})$. Then $n = 0$, $d_2^{1,0}$ is injective and ${}_AE_\infty^{1,0} = 0$. As a consequence, we have
$$H^1(A(T_\gamma)) \cong H^1(\Omega(T_\gamma)) \oplus {}_\Omega E_2^{1,0}$$

A more refined version of the result above is

Proposition 5.3. ([35, Corollary 2.6]) *There exists an isomorphism $H^*(A(T_\theta)) \cong \wedge(\Theta(t), \Theta(\xi))$ of algebras, where $t \in H^*(\Omega(T_\theta)) \cong \wedge(t)$ is a generator and $\xi \in \mathrm{Fl}^\bullet(T_\theta) \cong \mathbb{R}$ is a flow bundle¶ over T_θ with a connection 1-form, which is a generator of the group $\mathrm{Fl}^\bullet(T_\theta)$.*

6. Remarks –Future perspective and developments–

We conclude this course by describing our (personal) future perspective on diffeology. Recall the diagram (2.1) and consider the right-hand triangle. Recently, Kihara gives the category Diff a model category structure with variants of the singular simplex functor and the realization functor. An important point is that the homotopy category of Diff, which the model structure induces, is equivalent to that of $\mathsf{Sets}^{\Delta^{op}}$. Thus we may develop rational homotopy theory for diffeological spaces via a model structure on the category of CDGAs ‖ and algebraic models models due to Sullivan and Quillen. We address this topic in [35]. Especially, we provide a framework of rational ($\mathbb{R}$-local) homotopy theory in diffeology. In future work, we expect that *differential homotopy theory* ** is developed with the fabric.

Remark 6.1. In general, the realization functor $|{-}|_D : \mathsf{Sets}^{\Delta^{op}} \to \mathsf{Diff}$ does not preserve finite products. In fact, we see that $|\Delta^1|_D \times |\Delta^1|_D \cong \mathbb{R}^2$. On the other hand, the realization $|\Delta^1 \times \Delta^1|_D$ is the pushout of $|\Delta^2|_D \xleftarrow{d^0} |\Delta^1|_D \xrightarrow{d^2} |\Delta^2|_D$ and then it is diffeomorphic to $\Lambda^1 \times \mathbb{R}$; see [10, Proposition 4.9].

The remark above implies that we do not construct, for example, the Brown–Szczarba models for function spaces with the same way as that of topological versions. In order to construct a rational model for a mapping space in Diff by applying the method developed by Brown and Szczarba, we may use the fact that the natural map $|K \times L|_D \to |K|_D \times |L|_D$ is a homotopy equivalence; see the proof of [30, Corollary 4.13]. Here we use the realization functor $|\ |_D$ and the singular simplex functor $S^D(\ )$ due to Kihara [29] in stead of those in Section 2.3.

The proof of [30, Corollary 4.13] also enables us to obtain an equivalence
$$S^D(C^\infty(|K|_D, X)) \simeq \mathrm{Map}(K, S^D(X))$$
of simplicial sets for K in $\mathsf{Sets}^{\Delta^{op}}$ and X in Diff. This fact may be useful in constructing a Brown–Szczarba model [5] for a diffeological mapping space.

¶A principal $\mathbb{R}$-bundle. The term ${}_\Omega E_2^{1,0}$ is isomorphic to a subgroup of the abelian group $\mathrm{Fl}(X)$ of flow bundles over X; see [27, Section 21].

‖We refer the reader to [3, 21] for the model structure.

**This term is due to Iwase; visit the page `https://www2.math.kyushu-u.ac.jp/~iwase/BDHT2/Home.html`

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Appendix A. Stratifolds as diffeological spaces

In order to define a general stratifold, we recall a differential space in the sense of Sikorski [40].

Definition A.1. A *differential space* is a pair $(S, \mathcal{C})$ consisting of a topological space S and an $\mathbb{R}$-subalgebra $\mathcal{C}$ of the $\mathbb{R}$-algebra $C^0(S)$ of continuous real-valued functions on S, which is assumed to be *locally detectable* and C^∞*-closed.*

Local detectability means that $f \in \mathcal{C}$ if and only if for any $x \in S$, there exist an open neighborhood U of x and an element $g \in \mathcal{C}$ such that $f|_U = g|_U$.

C^∞-closedness means that for each $n \geq 1$, each n-tuple $(f_1, ..., f_n)$ of maps in $\mathcal{C}$ and each smooth map $g : \mathbb{R}^n \to \mathbb{R}$, the composite $h : S \to \mathbb{R}$ defined by $h(x) = g(f_1(x), ..., f_n(x))$ belongs to $\mathcal{C}$.

Let $(S, \mathcal{C})$ be a differential space and x an element in S. The vector space consisting of derivations on the $\mathbb{R}$-algebra $\mathcal{C}_x$ of the germs at x is denoted by T_xS, which is called the *tangent space* of the differential space at x; see [31, Chapter 1, section 3].

Definition A.2. An n*-dimensional stratifold* is a differential space $(S, \mathcal{C})$ such that the following four conditions hold:

1. S is a locally compact Hausdorff space with countable basis;
2. for each $x \in S$, the dimension of the tangent space T_xS is less than or equal to n and the *skeleta* $sk_t(S) := \{x \in S \mid \dim T_xS \leq t\}$ are closed in S;
3. for each $x \in S$ and open neighborhood U of x in S, there exists a *bump function* at x subordinate to U; that is, a non-negative function $\phi \in \mathcal{C}$ such that $\phi(x) \neq 0$ and such that the support $\operatorname{supp}\phi := \overline{\{p \in S \mid f(p) \neq 0\}}$ is contained in U;
4. the *strata* $S^t := sk_t(S) - sk_{t-1}(S)$ are t-dimensional smooth manifolds such that each restriction along $i : S^t \hookrightarrow S$ induces an isomorphism of stalks $i^* : \mathcal{C}_x \xrightarrow{\cong} C^\infty(S^t)_x$ for each $x \in S^t$.

Let $(S_1, \mathcal{C}_1)$ and $(S_2, \mathcal{C}_2)$ be stratifolds and $h : S_1 \to S_2$ a continuous map. We call the map h, denoted $h : (S_1, \mathcal{C}_1) \to (S_2, \mathcal{C}_2)$, a morphism of stratifolds if $\phi \circ h \in \mathcal{C}_1$ for every $\phi \in \mathcal{C}_2$.

A parametrized stratifold is constructed from a manifold attaching another manifold with compact boundary. More precisely, let $(S, \mathcal{C})$ be an n-dimensional stratifold. Let W be an s-dimensional manifold with compact boundary ∂W endowed with a collar $c : \partial W \times [0, \varepsilon) \to W$. Suppose that $s > n$. Let $f : \partial W \to S$ be a morphism of stratifolds. Observe that for a manifold M without boundary, we can regard M as the stratifold $j(M) := (M, \mathcal{C}_M)$ defined by $\mathcal{C}_M = \{\phi : M \to \mathbb{R} \mid \phi : \text{smooth}\}$; see [31]. We consider the adjunction topological space $S' := S \cup_f W$. Define the subalgebra of $C^0(S')$ by

$$\mathcal{C}' = \left\{ g : S' \to \mathbb{R} \;\middle|\; \begin{array}{l} g_{|S} \in \mathcal{C}, g_{|W\backslash\partial W} \text{ is smooth and for some positive real} \\ \text{number } \delta < \varepsilon, gc(w,t) = gf(w) \text{ for } w \in \partial W \text{ and } t < \delta \end{array} \right\}.$$

Then the pair $(S', \mathcal{C}')$ is a stratifold; see [31, Example 9] for more details. A stratifold constructed by attaching inductively manifolds with such a way is called a *parametrized stratifold* (p-stratifold for short).

Let Stfd be the category of stratifolds. We recall from [1] that a functor $k : \mathsf{Stfd} \to \mathsf{Diff}$ is defined by $k(S, \mathcal{C}) = (S, \mathcal{D}_{\mathcal{C}})$ and $k(f) = f$ for a morphism $f : S \to S'$ of stratifolds, where

$$\mathcal{D}_{\mathcal{C}} := \left\{ u : U \to S \;\middle|\; \begin{array}{l} U : \text{open in } \mathbb{R}^q, q \geq 0, \\ \phi \circ u \in C^\infty(U) \text{ for any } \phi \in \mathcal{C} \end{array} \right\}.$$

Observe that a plot in $\mathcal{D}_{\mathcal{C}}$ is a set map. We see that the functor k is faithful, but not full in general. It is worth mentioning that the fully faithful embedding $m : \mathsf{Mfd} \to \mathsf{Diff}$ from the category Mfd of manifolds in Assertion 2.8 is indeed an embedding j followed by k; that is, we have a sequence of functors

$$m : \mathsf{Mfd} \xrightarrow[\text{fully faithful}]{j} \mathsf{Stfd} \xrightarrow{\quad k \quad} \mathsf{Diff}.$$

Here the functor j is defined by assignment $M \mapsto j(M)$. We refer the reader to [1, Section 5] for the details of the functors.

Lemma A.3. *Let $(S, \mathcal{C})$ be a stratifold. An open subset of the underlying topological space S is a D-open subset of the diffeological space $k(S, \mathcal{C})$.*

Proof. Let u be an element in $\mathcal{D}_{\mathcal{C}}$ with domain U. Then $u : U \to k(S, \mathcal{C})$ is a smooth map in the sense of diffeology. In fact, for any plot $p : V \to U$ of the diffeology U and for any $\phi \in \mathcal{C}$, we see that $\phi \circ u_*(p) = (\phi \circ u) \circ p$ is in $C^\infty(V)$ and hence $u_*(p)$ is in $\mathcal{D}_{\mathcal{C}}$. Since U is a manifold, it follows from [1, Proposition 5.1] that u is a morphism in Stfd. In particular, the plot u is continuous. It turns out that, by definition, each open set of S is D-open. □

Remark A.4. While we do not know whether the functor k is full, a characterization of a morphism of diff-spaces which stems from a morphism of stratifolds is described in [1, Proposition 5.1].

Appendix B. Other topics related to our objects in diffeology

This section gives comments on some topics in diffeology that we do not deals with in the body of this note.

B.1. **The D-topology.** We review results in [9, 25] concerning the D-topology of a diffeological spaces. Let X be a diffeological space and Y a quotient *set* of X. Then the result [25, 2.12] yields that the D-topology of the quotiemt diffeology on Y coincides with the quotient topology of the D-topology.

As for a subset A of a diffeological space X, we have two topologies of A:

1) $\tau_1(A)$: the D-topology of the sub-diffeology on A and
2) $\tau_2(A)$: the sub-topology of the D-topology on X.

We see that $\tau_1(A)$ is finer than $\tau_2(A)$; that is, $\tau_2(A) \subseteq \tau_1(A)$. Let A be a subset of $\mathbb{R}$. Then $\tau_1(A)$ is discrete if and only if A is totally disconnected under the sub-topology of $\mathbb{R}$; see [9, Example 3.15]. This yields that $\tau_1(\mathbb{Q})$ is the discrete topology and then the topology is strictly finer than $\tau_2(A)$. We observe that the D-topology of $\mathbb{R}$ is the usual topology; see Assertion 2.9.

The results [9, Lemmas 3.16, 3.17 and 3.20] describe sufficient conditions for the two topologies to coincide with each other. In particular, if A is a D-open subset of a diffeological space X, then $\tau_2(A) = \tau_1(A)$.

Lemma B.1. ([9, Lemma 3.16]) *Let A be a convex set of $\mathbb{R}^n$. Then $\tau_1(A) = \tau_2(A)$. In particular, $\tau_1(\Delta^n_{\text{sub}}) = \tau_2(\Delta^n_{\text{sub}})$.*

Remark B.2. The functors C and D give rise to an equivalence between appropriate full subcategories of Diff and Top. To describe this more precisely, we recall that the unit and counit induce isomorphisms $\eta_{CX} : CX \xrightarrow{\cong} CDCX$ and $\varepsilon_{DY} : DCDY \xrightarrow{\cong} DY$; see [9, Propositios 3.3] and also [41].

Let C-Diff be the full subcategory of Diff consisting of objects isomorphic to diffeological spaces in the image of C and Δ-Top the full subcategory of Diff consisting of objects isomorphic to topological spaces in the image of D. We observe that the objects in the image of D are exactly the Δ-generated topological spaces; see [9, Proposition 3.10]. The result [36, Lemma II. 6.4] implies that the functors are restricted to equivalences between C-Diff and Δ-Top. The result [41, Proposition 2.2] asserts that the category NG of numerical generated topological spaces in [41] is nothing but the category Δ-Top. It is worth mentioning that

(1) the category Δ-Top is complete, cocomplete and cartesian closed; see [41, Proposition 2.4, Corollary 3.8],
(2) all CW-complexes are included in Δ-Top; see [41, Corollary 3.4], and
(3) the counit $DCX \xrightarrow{\simeq} X$ is a weak equivalence for any topological space X; see [41, Proposition 4.4].

The result [41, Proposition 2.4] enables us to conclude that, while the colomit in Δ-Top coincides with that in Top, the product $X \times^{\Delta} Y$ in Δ-Top is given by $X \times^{\Delta} Y = DC(X \times Y)$, where $X \times Y$ in the right-hand side denotes the usual product in Top.

Remark B.3. Let X be in the category C-Diff. Since the unit $\eta_X : X \to CDX$ is an isomorphism, it follows that the map $\mathsf{Diff}(\Delta^n_{\text{sub}}, X) \to \mathsf{Diff}(\Delta^n_{\text{sub}}, CDX)$ induced by the unit is bijective and hence so is the composite $\mathsf{Diff}(\Delta^n_{\text{sub}}, X) \to \mathsf{Top}(\Delta^n, DX)$ of the maps in (4.1). Thus the composite gives rise to an isomorphism $H(C^*(S^D_\bullet(X)_{\text{sub}})) \cong H^*(DX, \mathbb{R})$ for each object X in C-Diff, where $H^*(\text{-}, \mathbb{R})$ denotes the singular cohomology with coefficients in $\mathbb{R}$. In particular, we have an isomorphism $H(C^*(S^D_\bullet(CZ)_{\text{sub}})) \cong H^*(Z, \mathbb{R})$ for a CW-complex Z.

B.2. **A generating family of a diffeology.** Let $\mathcal{F}$ be a family of parametrizations of a set X; that is, a set of maps $U \to X$ from open subsets of $\mathbb{R}^n$ for some $n \geq 0$. The diffeology $\langle \mathcal{F} \rangle$ *generated* by $\mathcal{F}$ is defined by the set of parametraizations $P : U \to X$ for each of which there exist an open neighborhood V_r for all $r \in U$ such that $P|_{V_r}$ is constant or there exists a parametrization $Q : W \to X$ in $\mathcal{F}$, and a smooth map $\varphi : V_r \to W$ such that $P|_{V_r} = Q \circ \varphi$.

$$\begin{array}{ccccc} V_r & \xrightarrow{\subset} & U & \xrightarrow{P} & X \\ {\scriptstyle\varphi}\downarrow & & & \nearrow_{Q \in \mathcal{F}} & \\ W & & & & \end{array}$$

Let $(X, \mathcal{D}^X)$ be a diffeological space. A family of parametrizations $\mathcal{F}$ is a *generating family* of $\mathcal{D}^X$ if $\langle \mathcal{F} \rangle = \mathcal{D}^X$. We see that a chart $\{(U_j, \psi_j\}_{j \in J}$ of a manifold M gives a generating family $\mathcal{F} = \{\psi_j^{-1}\}_{j \in J}$ of the standard diffeology of M.

Following Iglesias-Zemmour [24], we introduce the notion of the *dimension* of a diffeological space X which is defined with a generating families of the diffeology of X. For a domain (open subset) U of $\mathbb{R}^n$, we define the dimension of the domain $\dim U$ by $\dim U = n$.

Definition B.4. Let $(X, \mathcal{D}^X)$ be a diffeological space and $\mathrm{GF}(X)$ the set of generating family of the diffeology of $\mathcal{D}^X$. Then the dimension $\dim X$ of X is defined by

$$\dim(X) := \inf_{\mathcal{F}\in \mathrm{GF}(X)} \dim(\mathcal{F}),$$

where $\dim(\mathcal{F}) = \sup_{(P:U\to X)\in\mathcal{F}} \dim U$. If the diffeology of X has no finite dimensional generating family, the dimension of X will be said to be infinity.

The topic on the dimension is discussed in [25, 1.77–1.83]. In particular, the dimension of a manifold as a diffeological space coincides with the usual one[†].

As for a subject concerning the Souriau–de Rham complex $\Omega^*(X)$ of a diff-space X and the dimension of X, we see that $\Omega^p(X) = 0$ if $p > \dim(X)$. In fact, by the definition of the generating family, each plot factors through locally a domain of dimension less than p under the assumption. Observe that $\Omega^p(U) = 0$ for the domain U.

B.3. **Homotopy sets and homotopy groups in Diff.** The homotopy set in Diff is defined by the same way as that in Top. Moreover, by *cut-off functions*, we define the smooth homootpy groups for a diffeological space; see [25, Chapter 5], [10, §3.1] and [29, Theorem 1.4]. For a diffeological bundle; see Definition 4.1, we have the homotopy exact sequence; see [25, §8.21].

Since the functor $S^D(\ )$ is the right adjoint, it preserves the products. Then we have

Lemma B.5. ([10, Lemma 4.10]) *The functor* $S^D : \mathsf{Diff} \to \mathsf{Sets}^{\Delta^{op}}$ *sends smoothly homotopic maps to simplicially homotopic maps.*

The functor $D : \mathsf{Diff} \to \mathsf{Top}$ is the left adjoint, it preserves the product $- \times \mathbb{R}$. This follows from Lemma 2.10. In fact, $D(\mathbb{R}) = \mathbb{R}$ is a locally compact Hausdorff space. Then D sends smoothly homotopic maps to topologically homotopic maps.

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[†] To see this, we use an argument of tangent spaces and the fact that each plot factors through locally the inverse of some chart; see Assertion 2.9.

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