ON THE LEVELS OF MAPS AND TOPOLOGICAL REALIZATION OF OBJECTS IN A TRIANGULATED CATEGORY

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Abstract. The level of a module over a differential graded algebra measures the number of steps required to build the module in an appropriate triangulated category. Based on this notion, we introduce a new homotopy invariant of spaces over a fixed space, called the level of a map. Moreover we provide a method to compute the invariant for spaces over a $\mathbb{K}$-formal space. This enables us to determine the level of the total space of a bundle over the 4-dimensional sphere with the aid of Auslander-Reiten theory for spaces due to Jørgensen. We also discuss the problem of realizing an indecomposable object in the derived category of the sphere by the singular cochain complex of a space. The Hopf invariant provides a criterion for the realization.

1. Introduction

Categorical representation theory yields suitable tools for studying certain problems in finite group theory, algebraic geometry and algebraic topology. For example, the Auslander-Reiten quiver of a triangulated category is an interesting combinatorial invariant; see [15], [16], [18], [19] and [35]. The singular (co)chain complex functor is a necessary ingredient in developing algebraic model theory for topological spaces; see [1], [3], [10], [14] and [29]. We will here advertise the idea that this functor, combined with tools from categorical representation theory of the kind just mentioned, is likely to provide new insights into the relationship between algebra and topology. To this end, we introduce and study a homotopy invariant that we call the level of a map.

The notion of levels of objects in a triangulated category was originally introduced by Avramov, Buchweitz, Iyengar and Miller in [2]. Roughly speaking, the level of an object $M$ in a triangulated category $\mathcal{T}$ counts the number of steps required to build $M$ out of a fixed object via triangles in $\mathcal{T}$.

Let $X$ be a space and $\mathcal{TOP}_X$ the category of spaces over $X$. The singular cochain complex functor $C^\ast (\cdot ; \mathbb{K})$ with coefficients in a field $\mathbb{K}$ gives rise to a contravariant functor from $\mathcal{TOP}_X$ to the derived category $D(C^\ast (X; \mathbb{K}))$ of DG (that is, differential graded) modules over the DG algebra $C^\ast (X; \mathbb{K})$. Observe that $D(C^\ast (X; \mathbb{K}))$ is a triangulated category with shift functor $\Sigma; (\Sigma M)^n = M^{n+1}$. We then define the level of a space $Y$ over $X$ to be the level of the DG $C^\ast (X; \mathbb{K})$-module $C^\ast (Y; \mathbb{K})$; see Section 2 for the exact definition.

In the rest of this section, we survey our main results.

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After showing that the level of a space is a weak homotopy invariant on \( \text{TOP}_X \), we give a reduction theorem (Theorem 2.5) for computing the level of a pullback of \( \mathbb{K} \)-formal spaces. An explicit calculation using this theorem tells us that a ‘nice’ space such as the total space \( E \) of a bundle over the sphere \( S^d \) is of low level; see Propositions 2.6 and 2.7. This means that the object \( C^*(E; \mathbb{K}) \) in \( D(C^*(S^d; \mathbb{K})) \) is built out of indecomposable objects of low level in the full subcategory of compact objects \( D'(C^*(S^d; \mathbb{K})) \). These indecomposable objects, which we call molecules of \( C^*(E; \mathbb{K}) \), are visualized with black vertices in the Auslander-Reiten quiver of \( D'(C^*(S^d; \mathbb{K})) \) as drawn below.

Here only the component of the quiver containing \( Z_0 = C^*(S^d; \mathbb{K}) \) is illustrated. Thus one has a new algebraic aspect of a topological object. For more details of the Auslander-Reiten quiver of a space, we refer the reader to Theorem 2.13, which is a remarkable result due to Jørgensen.

The level of a map \( Y \to B \) provides a lower bound on the number of spherical fibrations required to construct \( Y \) from \( B \); see Proposition 2.8 and Theorem 2.9. A topological description of the level is here given. Moreover, Theorem 2.9 and Proposition 3.5 imply that there exists at least one molecule in each row of the Auslander-Reiten quiver of \( D'(C^*(S^d; \mathbb{Q})) \) which is a summand of \( C^*(X; \mathbb{Q}) \) for some space \( X \) over \( S^d \).

Intriguing properties of the notion level are investigated in followups to this article [26] [27]. In particular, we show in [26] that the dual, chain-type level of a map \( f : X \to Y \) provides an upper bound on the Lusternik-Schnirelmann category of \( X \), at least over \( \mathbb{Q} \). In [27] we explain that cochain-type and chain-type levels are related by a sort of Eckmann-Hilton duality.

We deal with the problem of realizing a vertex (molecule) in an Auslander-Reiten quiver by the singular cochain complex of a space. It turns out that almost all molecules which appear in the quiver over the sphere are not realized by finite CW complexes. In fact Theorem 2.18 states that, in the Auslander-Reiten quiver mentioned above, only the arrow

\[
\begin{align*}
Z_0 & \quad \longrightarrow \quad \Sigma^{-(d-1)} Z_1 \\
\end{align*}
\]

is realizable. Proposition 2.17 asserts that a map \( \phi : S^d \to S^{2d-1} \) realizes the arrow if and only if the Hopf invariant of \( \phi \) is non-trivial. This gives a new topological perspective on the Auslander-Reiten quiver.

Statements of all our results can be found in Section 2, while the proofs are in sections 3 through 7.
2. Results

We fix some terminology. Throughout this article differential graded objects are written in the cohomological notation; that is, the differential increases degree by 1. We say that a graded vector space $M$ is locally finite if $M^i$ is of finite dimension for any $i$. Moreover $M$ is said to be non-negative if $M^i = 0$ for $i < 0$. A DG algebra $A$ over a field $K$ is simply-connected if it is non-negative and satisfies the condition that $H^0(A) = K$ and $H^1(A) = 0$. We refer to a morphism between DG $A$-modules as a quasi-isomorphism if it induces an isomorphism on the homology. Note that unspecified DG $A$-modules are right DG $A$-module. Unless otherwise explicitly stated, it is assumed that a space has the homotopy type of a CW complex whose cohomology with coefficients in the underlying field is locally finite. Observe that the cochain algebra $C^*(X; K)$ of a simply-connected space $X$ is simply-connected.

The goal of this section is to state our results in more detail.

Let $\mathcal{T}$ be a triangulated category. To introduce the notion of the level, we first recall from [2] the definition of the thickening of $\mathcal{T}$. For a given object $C$ in $\mathcal{T}$, we define the $0$th thickening by $\text{thick}_0^\mathcal{T}(C) = \{0\}$ and $\text{thick}_1^\mathcal{T}(C)$ by the smallest strict full subcategory which contains $C$ and is closed under taking finite coproducts, retracts and all shifts. Moreover for $n > 1$ define inductively the $n$th thickening $\text{thick}_n^\mathcal{T}(C)$ to be the smallest strict full subcategory of $\mathcal{T}$ which is closed under retracts and contains objects $M$ admitting an exact triangle

$$M_1 \to M \to M_2 \to \Sigma M_1$$

in $\mathcal{T}$ for which $M_1$ and $M_2$ are in $\text{thick}_{n-1}^\mathcal{T}(C)$ and $\text{thick}_n^\mathcal{T}(C)$, respectively.

By definition, a full subcategory $C$ of $\mathcal{T}$ is thick if it is additive, closed under retracts, and every exact triangle in $\mathcal{T}$ with two vertices in $C$ has its third vertex in $C$. As mentioned in [2, 2.2.4], the thickenings provide a filtration of the smallest thick subcategory $\text{thick}_\mathcal{T}(C)$ of $\mathcal{T}$ containing the object $C$:

$$\{0\} = \text{thick}_0^\mathcal{T}(C) \subset \cdots \subset \text{thick}_n^\mathcal{T}(C) \subset \cdots \subset \bigcup_{n \geq 0} \text{thick}_n^\mathcal{T}(C) = \text{thick}_\mathcal{T}(C).$$

For an object $M$ in $\mathcal{T}$, we define a numerical invariant $\text{level}^\mathcal{T}_C(M)$, which is called the $C$-level of $M$, by

$$\text{level}^\mathcal{T}_C(M) := \inf\{n \in \mathbb{N} \cup \{0\} \mid M \in \text{thick}_n^\mathcal{T}(C)\}.$$ 

It is worth noting that $\text{level}^\mathcal{T}_C(M)$ is finite if and only if $M$ is finitely built from $C$ in the sense of Dwyer, Greenlees and Iyenger [6, 3.15]; see also [5].

Let $A$ be a DG algebra over a field $K$. Let $D(A)$ be the derived category of DG $A$-modules, namely the localization of the homotopy category $H(A)$ of DG $A$-modules with respect to quasi-isomorphisms; see [21] and [23, PART III]. Observe that $D(A)$ is a triangulated category with the shift functor $\Sigma$ defined by $(\Sigma M)^n = M^{n+1}$ and that a triangle in $D(A)$ comes from a cofibre sequence of the form $M \xrightarrow{f} N \to C_f \to \Sigma M$ in the homotopy category $H(A)$. Here $C_f$ denotes the mapping cone of $f$.

In what follows, for any object $M$ in $D(A)$, we may write $\text{level}_{D(A)}(M)$ for the $A$-level $\text{level}^\mathcal{T}_{D(A)}(M)$ of $M$.

Let $X$ be a simply-connected space and $\mathcal{TOP}_X$ the category of connected spaces over $X$; that is, objects are maps to the space $X$ and morphisms from $\alpha : Y \to X$ to $\beta : Z \to X$ are maps $f : Y \to Z$ such that $\beta f = \alpha$. For an object $\alpha : Y \to X$ in $\mathcal{TOP}_X$, the singular cochain complex $C^*(Y; K)$ is considered a DG module over
the DG algebra $C^*(X; \mathbb{K})$ via the morphism of DG algebras induced by $\alpha$. We may write $C^*(Y; \mathbb{K})^\alpha$ for this DG-module. Thus we have a contravariant functor

$$C^* (\cdot; \mathbb{K}) : \mathcal{TOP}_X \to D(C^*(X; \mathbb{K})).$$

**Definition 2.1.** Let $\alpha : Y \to X$ be an object in $\mathcal{TOP}_X$. The level of the map $\alpha$, denoted $\text{level}_X(Y)_\mathbb{K}$, is the $C^*(Y; \mathbb{K})$-level of $C^*(X; \mathbb{K})^\alpha$ in the triangulated category $D(C^*(X; \mathbb{K}))$, namely $\text{level}_X(Y)_\mathbb{K}(C^*(X; \mathbb{K})^\alpha)$. When there is no danger of confusion, we will write $\text{level}_X(Y)_\mathbb{K}$ in place of $\text{level}_X(Y)_\mathbb{K}$. Note that, in [26], we call the level of a map $\alpha : Y \to X$ the cochain type level of the space $Y$ and write $\text{level}_{D(C^*(X; \mathbb{K}))}(Y)$ for $\text{level}_X(Y)_\mathbb{K}$.

A straightforward argument shows that the level is a weak homotopy invariant on $\mathcal{TOP}_X$.

**Proposition 2.2.** Let $\alpha : Y \to X$ and $\beta : Z \to X$ be objects in $\mathcal{TOP}_X$. If there exists a weak homotopy equivalence $f : Y \to Z$ such that $\alpha \simeq \beta \circ f$, then

$$\text{level}_X(Z) = \text{level}_X(Y).$$

**Proof.** Let $H : Y \times I \to X$ be a homotopy from $\alpha$ to $\beta \circ f$ and $\varepsilon_i : Y \to Y \times I$ the inclusion defined by $\varepsilon_i(y) = (y, i)$ for $i = 0, 1$. We consider $C^*(Y \times I; \mathbb{K})$ a DG $C^*(X; \mathbb{K})$-module via the induced map $H^* : C^*(Y; \mathbb{K}) \to C^*(Y \times I; \mathbb{K})$. Moreover $C^*(Y; \mathbb{K})$ is endowed with a DG $C^*(X; \mathbb{K})$-module structure via the map $(H \circ \varepsilon_i)^* : C^*(X; \mathbb{K}) \to C^*(Y; \mathbb{K})$ for each $i = 0, 1$. Then there exists a sequence of quasi-isomorphisms of DG $C^*(X; \mathbb{K})$-modules

$$C^*(Z; \mathbb{K})^\beta \xrightarrow{\varepsilon_i \circ \varepsilon_1^*} C^*(Y; \mathbb{K})^{H \circ \varepsilon_1} \xrightarrow{\varepsilon_i^*} C^*(Y \times I; \mathbb{K})^H \xrightarrow{\varepsilon_i^*} C^*(Y; \mathbb{K})^{H \circ \varepsilon_0} = C^*(Y; \mathbb{K})^\alpha.$$ 

Thus we have the result. \qed

It is natural to ask what aspect of topological spaces is captured by the notion of level. To begin to answer this question, it is helpful to compute the level of various interesting maps. As an aid to computation we provide a reduction theorem for levels of certain maps of $\mathbb{K}$-formal spaces.

Let $m_X : TV_X \xrightarrow{\simeq} C^*(X; \mathbb{K})$ be a minimal TV-model for a simply-connected space in the sense of Halperin and Lemaire [14]; that is, $TV_X$ is a DG algebra whose underlying $\mathbb{K}$-algebra is the tensor algebra generated by a graded vector space $V_X$ and, for any element $v \in V_X$, the image of $v$ by the differential is decomposable; see also Appendix.

Recall that a space $X$ is $\mathbb{K}$-formal if it is simply-connected and there exists a sequence of quasi-isomorphisms of DG algebras

$$H^*(X; \mathbb{K}) \xleftarrow{\phi_X} TV_X \xrightarrow{m_X} C^*(X; \mathbb{K}),$$

where $m_X : TV_X \to C^*(X; \mathbb{K})$ denotes a minimal TV-model for $X$. Observe that spheres if $d > 1$, then the sphere $S^d$ is $\mathbb{K}$-formal, for any field $\mathbb{K}$ [7][33]. Moreover a simply-connected space whose cohomology with coefficients in $\mathbb{K}$ is a polynomial algebra generated by elements of even degree is $\mathbb{K}$-formal [31, Section 7].

**Definition 2.3.** Let $q : E \to B$ and $f : X \to B$ be maps between $\mathbb{K}$-formal spaces. The pair $(q, f)$ is relatively $\mathbb{K}$-formalizable if there exists a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{f} & X \\
\downarrow q & & \downarrow f \\
B & \xrightarrow{1} & B
\end{array}
$$

in which $q$ is a weak homotopy equivalence and $f$ is a homotopy equivalence.
up to homotopy of DG algebras

\[
\begin{array}{ccc}
H^*(E; K) & \xrightarrow{\phi_E} & TV_E \\ & \xrightarrow{m_E} & C^*(E; K) \\
H^*(B; K) & \xrightarrow{\phi_B} & TV_B \\ & \xrightarrow{m_B} & C^*(B; K) \\
H^*(X; K) & \xrightarrow{\phi_X} & TV_X \\ & \xrightarrow{m_X} & C^*(X; K),
\end{array}
\]

in which horizontal arrows are quasi-isomorphisms.

In general, for given quasi-isomorphisms \( \phi_E, m_E, \phi_B \) and \( m_B \) as in Definition 2.3, there exist DG algebra maps \( \tilde{q}_1 \) and \( \tilde{q}_2 \) which make the right upper square and left one homotopy commutative, respectively. However, in general, one cannot choose a map \( \tilde{q} \) which makes upper two squares homotopy commutative simultaneously even if the maps \( \phi_E, m_E, \phi_B \) and \( m_B \) are replaced by other quasi-isomorphisms; see Remark 6.3.

The following proposition, which is deduced from the proof of [25, Theorem 1.1], gives examples of relatively \( K \)-formalizable pairs of maps.

**Proposition 2.4.** A pair of maps \((q; f)\) with a common target is relatively \( K \)-formalizable if each of the maps satisfies either of the two conditions below on a map \( \tilde{q} \):

(i) \( H^*(S; K) \) and \( H^*(T; K) \) are polynomial algebras with at most countably many generators in which the operation \( Sq_1 \) vanishes when the characteristic of the field \( K \) is 2. Here \( Sq_1 x = Sq^{n-1} x \) for \( x \) of degree \( n \); see [31, 4.9].

(ii) \( \tilde{H}^i(S; K) = 0 \) for any \( i \) with \( \dim \tilde{H}^i(\Omega T; K) - \dim(QH^*(T; K))^i \neq 0 \).

Let \( q : E \to B \) be a fibration over a space \( B \) and \( f : X \to B \) a map. Let \( F \) denote the pullback diagram

\[
\begin{array}{ccc}
E \times_B X & \xrightarrow{q} & E \\
\downarrow & & \downarrow q \\
X & \xrightarrow{f} & B.
\end{array}
\]

Our main theorem on the computation of the level of a space is stated as follows.

**Theorem 2.5.** Suppose that the spaces \( X, B \) and \( E \) in the diagram \( F \) are \( K \)-formal and the pair \((q, f)\) is relatively \( K \)-formalizable. Then

\[
\text{level}_X(E \times_B X) = \text{level}_{\text{D}(H^*(X; K))}(H^*(E; K) \otimes_{H^*(B; K)} L_{H^*(B; K)} H^*(X; K)).
\]

As Example 4.3 illustrates, the condition that \( X, B \) and \( E \) in \( F \) are \( K \)-formal is not sufficient. We refer the reader to Section 3 for the definition of the left derived functor \( \otimes^L \).

By virtue of Theorem 2.5 and Proposition 2.4, we have

**Proposition 2.6.** Let \( G \) be a simply-connected Lie group and \( G \to E_f \to S^4 \) a \( G \)-bundle with the classifying map \( f : S^3 \to BG \). Suppose that \( H^*(BG; K) \) is a
polynomial algebra on generators of even degree. Then

\[
\text{level}_{S^4}(E_f) = \begin{cases} 
2 & \text{if } H^4(f; \mathbb{K}) \neq 0, \\
1 & \text{otherwise}. 
\end{cases}
\]

**Proposition 2.7.** Let \( G \) be a simply-connected Lie group and \( H \) a maximal rank subgroup. Let \( G/H \to E_g \to S^4 \) be the pullback of the fibration \( G/H \to BH \xrightarrow{\pi} BG \) by a map \( g : S^4 \to BG \). Suppose that \( H^*(BG; \mathbb{K}) \) and \( H^*(BH; \mathbb{K}) \) are polynomial algebras on generators with even degree. Then

\[
\text{level}_{S^4}(E_g) = \begin{cases} 
2 & \text{if } H^4(f; \mathbb{K}) \neq 0, \\
1 & \text{otherwise}. 
\end{cases}
\]

As an introduction of the meaning of the level of a maps \( f \), we show that it provides an lower bound on the number of stages in a factorization

\[
Y = Y_c \xrightarrow{\pi_c} Y_{c-1} \xrightarrow{\pi_{c-1}} \ldots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 \xrightarrow{\pi_0} B
\]
of \( f \), where each \( \pi_i \) is a fibration with an odd sphere as fibre.

**Proposition 2.8.** Suppose that there exists a sequence of fibrations

\[
S^{2n_i+1} \to Y_1 \xrightarrow{\pi_1} B \times (S^{2n_{i+1}}), \quad S^{2m_j+1} \to Y_2 \xrightarrow{\pi_2} Y_1, \ldots,
\]

in which \( B \) is simply-connected and \( n_i, m_j \geq 1 \) for any \( i \) and \( j \). We regard \( Y_c \) as a space over \( B \) via the composite \( \pi_0 \circ \pi_1 \circ \ldots \circ \pi_c \), where \( \pi_0 : B \times (\times_{i=1}^{c} S^{2n_i+1}) \to B \) is the projection onto the first factor. Then

\[
\text{level}_B(Y_c) \leq c + 1.
\]

By using Proposition 2.8 and the homological information of each vertex of the Auslander-Reiten quiver of \( \text{D}(C^*(S^d; \mathbb{K})) \) described in Theorem 2.13, we can construct an object in \( \text{TOP}_{S^d} \) of arbitrary level, provided that \( \mathbb{K} = \mathbb{Q} \).

**Theorem 2.9.** For any integers \( l \geq 1 \) and \( d > 1 \), there exists an object \( P_l \to S^d \) in \( \text{TOP}_{S^d} \) such that

\[
\text{level}_{S^d}(P_l) = l.
\]

The map \( P_l \to S^d \) in the statement above is constructed iteratively by spherical fibrations, as in Proposition 2.8.

Proposition 2.8 also clarifies a link between the level of a rational space \( X \) and the codimension of \( X \) due to Greenlees, Hess and Shamir [13].

**Definition 2.10.** [13, 7.4(i)] A space \( X \) is spherically complete intersection (sci) if it is simply-connected and there exists a sequence of spherical fibrations

\[
S^{m_1} \to X_1 \to KV, \quad S^{m_2} \to X_2 \to Y_1, \ldots, S^{m_c} \to X_c \to X_{c-1}
\]
in which \( X_c = X \) and \( KV \) is a regular space, namely the Eilenberg-MacLane space on a finite dimensional graded vector space \( V \) with \( V^{\text{odd}} = 0 \). The least such integer \( c \) is called the codimension of \( X \), denoted \( \text{codim}(X) \).

The result [13, Lemma 8.1] asserts that the spheres which appear in the definition of a sci space may be taken to be of odd dimension by replacing the regular space \( KV \) by another regular space. Thus if \( X \) is sci, by composing the projections in the fibrations, we have a new fibration \( F \to X \xrightarrow{\pi} KV \) such that

\[
\text{codim}(X) = \dim \pi_*(F) \otimes \mathbb{Q} = \dim \pi^{\text{odd}}_*(X) \otimes \mathbb{Q}.
\]
We call this fibration a standard fibration of $X$. Proposition 2.8 yields immediately the following result.

**Theorem 2.11.** Let $X$ be sci with a standard fibration of the form $F \to X \to KV$. Then one has

$$\text{level}_{KV}(X)_Q \leq \text{codim}(X) + 1.$$  

We next focus on the problem of realizing objects in the triangulated category $D(C^*(S^d, \mathbb{K}))$ as the singular cochain complexes of spaces. To this end, we describe Jørgensen’s result in [18] briefly.

Let $T$ be a triangulated category. An object in $T$ is said to be indecomposable if it is not a coproduct of nontrivial objects. Recall that a triangle $L \overset{u}{\to} M \overset{v}{\to} N \overset{w}{\to} L$ in $T$ is an Auslander-Reiten triangle [15][16] if the following conditions are satisfied:

(i) $L$ and $N$ are indecomposable.
(ii) $w \neq 0$.
(iii) Each morphism $N' \to N$ which is not a retraction factors through $v$.

We say that a morphism $f : M \to N$ in $T$ is irreducible if it is neither a section nor a retraction, but satisfies that in any factorization $f = rs$, either $s$ is a section or $r$ is a retraction.

The category $T$ is said to have Auslander-Reiten triangles if, for each object $N$ whose endomorphism ring is local, there exists an Auslander-Reiten triangle with $N$ as the third term from the left. Recall also that an object $K$ in $T$ is compact if the functor $\text{Hom}_T(K, -)$ preserves coproducts; see [32, Chapter 4].

**Definition 2.12.** The Auslander-Reiten quiver of $T$ has as vertices the isomorphism classes $[M]$ of indecomposable objects. It has one arrow from $[M]$ to $[N]$ when there is an irreducible morphism $M \to N$ and no arrow from $[M]$ to $[N]$ otherwise.

Let $A$ be a locally finite, simply-connected DG algebra over a field $\mathbb{K}$. Assume further that $\dim H^*(A) < \infty$. We denote by $D^c(A)$ the full subcategory of the derived category $D(A)$ consisting of the compact objects. For a DG $A$-module $M$, let $DM$ be the dual $\text{Hom}_K(M, \mathbb{K})$ to $M$.

Put $d := \sup \{i \mid H^i(A) \neq 0\}$. One of the main results in [18] asserts that both $D^c(A)$ and $D^c(A^{op})$ have Auslander-Reiten triangles if and only if there are isomorphisms of graded $H^*A$-modules $H^*(DH^*A) \cong H^*(\Sigma^d H^*A)$ and $(DH^*A)_{H^*A} \cong (\Sigma^d H^*A)_{H^*A}$; that is, $H^*(A)$ is a Poincaré duality algebra. In other words, $A$ is Gorenstein in the sense of Félix, Halperin and Thomas [8]. In this case, the form of the Auslander-Reiten quiver of $D^c(A)$ was determined in [18] and [19].

The key lemma [18, Lemma 8.4] for proving results in [18, Section 8] is obtained by using the rational formality of the spheres. Since the spheres are also $\mathbb{K}$-formal for any field $\mathbb{K}$, the assumption concerning the characteristic of the underlying field is unnecessary for all the results in [18, Section 8]; see [20] and [35]. In particular, we have

**Theorem 2.13.** [18, Theorem 8.13][18, Proposition 8.10] Let $S^d$ be the $d$-dimensional sphere with $d > 1$ and $\mathbb{K}$ an arbitrary field. Then the Auslander-Reiten quiver of

...
the category $\mathrm{D}^c(C^*(S^d; \mathbb{K}))$ consists of $d-1$ components, each isomorphic to the translation quiver $\mathbb{Z}A_{\infty}$; see [15, 5.6]. The component containing $Z_0 \cong C^*(S^d; \mathbb{K})$ is of the form

\[
\begin{array}{ccccccc}
& & & & & & \\
& & Z_3 & & Z_2 & & \\
& & & \cdots & & \cdots & \\
& & & & & & \\
\cdots & & Z_1 & & Z_0 & & \\
& & & & & & \\
& & & \cdots & & \cdots & \\
\end{array}
\]

Moreover, the cohomology of the indecomposable object $\Sigma^{-l}Z_m$ has the form

\[
H^i(\Sigma^{-l}Z_m) \cong \begin{cases} 
\mathbb{K} & \text{for } i = -m(d - 1) + l \text{ and } d + l, \\
0 & \text{otherwise.}
\end{cases}
\]

In what follows, we call an indecomposable object in $\mathrm{D}^c(C^*(X; \mathbb{K}))$ a molecule.

**Remark 2.14.** Let $A$ be a DG algebra with $\dim H(A) < \infty$. Then $\mathrm{D}^c(A)$ is a Krull-Remak-Schmidt category; that is, each object decomposes uniquely into indecomposable objects; see [20, Proposition 2.4].

**Remark 2.15.** The latter half of Theorem 2.13 implies that molecules in $\mathrm{D}^c(C^*(S^d; \mathbb{K}))$ are characterized by their cohomology. Moreover, those objects are also classified by the amplitude of their cohomology of the objects, up to shifts. Here the amplitude of a DG module $M$, denoted $\text{amp}M$, is defined by

\[
\text{amp}M := \sup \{i \in \mathbb{Z} \mid M^i \neq 0\} - \inf \{i \in \mathbb{Z} \mid M^i \neq 0\}.
\]

The cohomology of $\Sigma^{-(d-1)}Z_1$ is isomorphic to $H^*(S^{2d-1}; \mathbb{K})$ as a graded vector space and that there is an irreducible map $\Sigma^{-(d-1)}Z_1 \to Z_0$ that induces $H^*(S^d; \mathbb{K}) = H^*(Z_0) \to H^*(\Sigma^{-(d-1)}Z_1) = H^*(S^{2d-1}; \mathbb{K})$ a morphism of $H^*(S^d; \mathbb{K})$-modules. Thus one might expect that the topological realizability of the morphism in the quiver is related to the Hopf invariant $H : \pi_{2d-1}(S^d) \to \mathbb{Z}$. We define realizability as follows.

**Definition 2.16.** An object $M$ in the category $\mathrm{D}^c(C^*(X; \mathbb{K}))$ is **realizable** by an object $f : Y \to X$ in $\mathcal{T}OP_X$ if $M$ is isomorphic to the cochain complex $C^*(Y; \mathbb{K})$ endowed with the $C^*(X; \mathbb{K})$-module structure via the map $f^* : C^*(X; \mathbb{K}) \to C^*(Y; \mathbb{K})$; that is, $M \cong C^*(Y; \mathbb{K})^f$ in $\mathrm{D}^c(C^*(X; \mathbb{K}))$.

We establish the following proposition.

**Proposition 2.17.** Let $\phi : S^{2d-1} \to S^d$ be a map. The DG module $C^*(S^{2d-1}; \mathbb{K})^\phi$ over $C^*(S^d; \mathbb{K})$ is in $\mathrm{D}^c(C^*(S^d; \mathbb{K}))$ if and only if $H(\phi)_X$ is nonzero, where $H(-)_X$ denotes the composite of the Hopf invariant with the reduction $\mathbb{Z} \to \mathbb{Z} \otimes \mathbb{K}$. In that case, the induced map $\phi^* : C^*(S^d; \mathbb{K}) \to C^*(S^{2d-1}; \mathbb{K})$ coincides with the irreducible map $Z_0 \to \Sigma^{-(d-1)}Z_1$ up to scalar multiple.

Since the 0th cohomology of a space is non-zero and the negative part of the cohomology is zero, only indecomposable objects of the form $\Sigma^{-m(d-1)}Z_m$ ($m \geq 0$) may be realizable; see the beginning of the proof of Theorem 2.18. Observe that
the objects $\Sigma^{-m(d-1)}Z_m$ lie in the line connecting $Z_0$ and $\Sigma^{-(d-1)}Z_1$. However, the following proposition states that most of molecules in $D^*((C^*(X;K))$ are not realizable by finite CW complexes.

**Theorem 2.18.** Suppose that the characteristic of the underlying field is greater than 2 or zero. A molecule of the form $\Sigma^{-i}Z_l$ in $D^*((S^d;K))$ is realizable by a finite CW complex if and only if $i = d - 1$, $l = 1$ and $d$ is even, or $i = 0$ and $l = 0$.

The rest of this paper is organized as follows. Section 3 contains a brief introduction to semifree resolutions. We also recall some results on the levels which we use later on. Section 4 is devoted to proving Theorems 2.5, while Proposition 2.8 and Theorem 2.9 are proved in Section 5. In Section 6, we prove Proposition 2.17 and Theorem 2.18. The explicit computations of levels described in Propositions 2.6 and 2.7 are made in Section 7.

We conclude this section with comments on our work.

**Remark 2.19.** Let $X$ be a simply-connected space whose homology with coefficients in a field $K$ is a Poincare duality algebra. The Auslander-Reiten quiver of $D^*((C^*(X;K))$ then graphically depicts irreducible morphisms and molecules in the full subcategory. Even if a molecule in $D^*((C^*(X;K))$ is not realizable, it may be needed to construct $C^*(Y;K)$ for a space $Y$ over $X$ as a $C^*(X;K)$-module. In fact, it follows from the proofs of Propositions 2.6 and 2.7 that some molecules are retracts of the $C^*(S^d;K)$-modules $C^*(E_f;K)$ and $C^*(E_g;K)$, even though they are not realizable; see also Example 7.3.

**Remark 2.20.** A CW complex $Z$ is built out disks, which are called cells, by iterated attachment of them. It is well-known that the dual to the cellular chain complex of a CW complex $Z$ is quasi-isomorphic to the singular cochain complex $C^*(Z;K)$. Thus $C^*(Z;K)$ is also regarded as ‘a set of cells’ and hence it seems a creature in some sense. When we describe images by the functor $C^*(-;K)$ in terms of representation theory, we may need objects in $D^*((C^*(X;K))$ which are not necessarily realizable. Therefore one might regard such an object as structurally smaller than a cell. This is the reason why we give indecomposable objects in $D^*((C^*(X;K))$ the name ‘molecules’.

### 3. Semifree resolutions and the levels

We begin by recalling the definition of the semifree resolution.

Let $A$ be a DG algebra over $K$.

**Definition 3.1.** [2, 4.1][8][11, §6] A **semifree filtration** of a DG $A$-module $M$ is a family $\{F^n\}_{n\in\mathbb{Z}}$ of DG submodules of $M$ satisfying the condition: $F^{-1} = 0$, $F^n \subset F^{n+1}$, $\bigcup_{n\geq 0} F^n = M$ and $F^n/F^{n-1}$ is isomorphic to a direct sum of shifts of $A$. A DG $A$-module $M$ admitting a semifree filtration is called semifree. We say that the filtration $\{F^n\}_{n\in\mathbb{Z}}$ has class at most $l$ if $F^l = M$ for some integer $l$. Moreover $\{F^n\}_{n\in\mathbb{Z}}$ is called finite if each quotient is finitely generated.

Let $M$ be a DG $A$-module. We say that a quasi-isomorphism of $A$-modules $F \to M$ is a semifree resolution of $M$ if $F$ is semifree. For example, the bar resolution $B(M; A; A)$ of $M$ is $A$-semifree, and its canonical augmentation $\varepsilon : B(M; A; A) \to M$ is therefore a semifree resolution of $M$.

Let $N$ be a left DG $A$-module. We observe that the left derived functor $- \otimes_A^L N$ is defined by $M \otimes_A^L N := F \otimes_A N$ for any right DG module $M$ over $A$, where
$F \xrightarrow{\sim} M$ is a semifree resolution of $M$. We see that by definition $H^*(M \otimes^A_\mathbb{K} N)$ is exactly $\text{Tor}_A(M, N)$.

The following result is useful for computing the $A$-level of an object in $D(A)$.

**Theorem 3.2.** [2, Theorem 4.2] Let $M$ be a DG module over a DG algebra $A$ and $l$ a non-negative integer. Then $\text{level}^A_{D(A)}(M) \leq l$ if and only if $M$ is a retract in $D(A)$ of some DG module admitting a finite semifree filtration of class at most $l - 1$.

In order to study Auslander-Reiten triangles, in [19], Jørgensen introduced the function $\varphi : D(A) \to \mathbb{Z} \cup \{\infty\}$ defined by

$$\varphi(M) := \dim H^*(M \otimes^A_\mathbb{K} \mathbb{K}).$$

This yields a criterion for a given object in $D(A)$ to be compact.

**Proposition 3.3.** [2, Theorem 4.8][12, Proposition 2.3][21, Theorem 5.3] Let $A$ be a simply-connected DG algebra. An object $M$ in $D(A)$ is compact if and only if $\varphi(M) < \infty$. In that case $\text{level}^A_{D(A)}(M) < \infty$.

In particular, for a map $\phi : Y \to X$ from a connected space $Y$ to a simply-connected space $X$, if the total dimension of the cohomology of the homotopy fibre of the map $\phi$ is finite, then $C^\ast(Y; \mathbb{K})$ is in $D^\ast(C^\ast(X; \mathbb{K}))$ and hence $\text{level}_X(Y) < \infty$.

**Remark 3.4.** Let $F_\phi$ be the homotopy fibre of a map $\phi : Y \to X$. The latter half of Proposition 3.3 follows from the fact that $H^\ast(F_\phi; \mathbb{K}) \cong \text{Tor}_{C^\ast(X; \mathbb{K})}(C^\ast(Y; \mathbb{K}), \mathbb{K}) \cong H^\ast(C^\ast(Y; \mathbb{K}) \otimes_{C^\ast(X; \mathbb{K})}^l \mathbb{K})$ as a graded vector space; see [36][11, Theorem 7.5].

We conclude this section with a result due to Schmidt, about the levels of molecules in $D^\ast(C^\ast(S^d; \mathbb{K}))$, which is used in the proof of Theorem 2.9.

**Proposition 3.5.** [35, Proposition 6.6] Let $Z_i$ be the molecule in $D^\ast(C^\ast(S^d; \mathbb{K}))$ described in Theorem 2.13. Then $\text{level}_{D(C^\ast(S^d; \mathbb{K}))}(Z_i) = i + 1$.

4. PROOF OF THEOREM 2.5

In what follows, we write $C^\ast(\cdot)$ and $H^\ast(\cdot)$ for $C^\ast(\cdot; \mathbb{K})$ and $H^\ast(\cdot; \mathbb{K})$, respectively if the coefficients are clear from the context.

Let $X$ be a simply-connected formal space and $m_X : TV_X \xrightarrow{\sim} C^\ast(X; \mathbb{K})$ be a minimal model. We then have the following equivalences of triangulated categories; see [23, Proposition 4.2],

$$D(C^\ast(X; \mathbb{K})) \xrightarrow{m_X^*} D(TV_X) \xrightarrow{- \otimes_{TV_X}^L H^\ast(X; \mathbb{K})} D(H^\ast(X; \mathbb{K})), $$

where $m_X^*$ is the pullback functor; that is, for a $C^\ast(X; \mathbb{K})$-module $M$, $m_X^* M$ is defined to be the module $M$ endowed with the $TV_X$-module structure via $m_X$. We denote by $F_X$ the composite of the functors: $F_X = - \otimes_{TV_X}^L H^\ast(X; \mathbb{K}) \circ m_X^*$. Observe that the functor $F_X$ leaves the cohomology of an object unchanged; see [11, Proposition 6.7] for example.

**Lemma 4.1.** Under the same hypothesis as in Theorem 2.5, the differential graded module $F_X(C^\ast(E \times_B X; \mathbb{K}))$ is isomorphic to $H^\ast(E; \mathbb{K}) \otimes_{H^\ast(B; \mathbb{K})}^L H^\ast(X; \mathbb{K})$ in the category $D(H^\ast(X; \mathbb{K}))$. 
Proof. We use the same notation as in Definition 2.3. Let $H : TV_B \wedge I \to C^*(E)$ and $K : TV_B \wedge I \to C^*(X)$ be homotopies from $q^* \circ m_B$ to $m_E \circ \tilde{q}$ and from $f^* \circ m_B$ to $m_E \circ \tilde{f}$, respectively. Here $TV_B \wedge I$ denotes the cylinder object due to Baues and Lemaire [9] in the category of DG algebras; see Appendix. The homotopies $H$ and $K$ make $C^*(E)$ and $C^*(X)$ into a right $TV_B \wedge I$-module and a left $TV_B \wedge I$-module, respectively, so there exists a right $C^*(X)$-module of the form $C^*(E) \otimes_{TV_B \wedge I} C^*(X)$. Then there exists a sequence of quasi-isomorphisms of $TV_X$-modules

$$C^*(E \times_B X) \xrightarrow{EM} C^*(E) \otimes_{C^*(B)} C^*(X) \xrightarrow{1 \otimes m_X^{-1}} C^*(E) \otimes_{TV_B} C^*(X),$$

where $EM$ denotes the Eilenberg-Moore map; see [36, Theorem 3.2]. Therefore we see that $m_X(C^*(E \times_B X))$ is isomorphic to $TV_E \otimes_{TV_B} TV_X$ in $D(TV_X)$. By considering the bar resolution of $TV_E$ as a $TV_B$-module, we see that, as objects in $D(H^*(X))$, $(TV_E \otimes_{TV_B} TV_X) \otimes_{TV_B} H^*(X)$ is isomorphic to $TV_E \otimes_{TV_B} H^*(X)$. Then a sequence of quasi-isomorphisms similar to that above connects $TV_E \otimes_{TV_B} H^*(X)$ with $H^*(E) \otimes_{H^*(B)} H^*(X)$ in $D(H^*(X))$. In fact we have quasi-isomorphisms

$$TV_E \otimes_{TV_B} H^*(X) \xrightarrow{\phi_B \otimes 1} H^*(E) \otimes_{TV_B} H^*(X) \xrightarrow{1 \otimes \alpha_{TV_B}^{-1}} H^*(E) \otimes_{TV_B \wedge I} H^*(X) \xrightarrow{1 \otimes \alpha_{TV_B}^{-1}} H^*(E) \otimes_{H^*(B)} H^*(X).$$

This completes the proof. \hfill \Box

Proof of Theorem 2.5. We see that in $D(H^*(X))$

$$F_X C^*(X) = (m_X C^*(X)) \otimes_{TV_X} H^*(X) = TV_X \otimes_{TV_X} H^*(X) = H^*(X).$$

Then the result [2, Proposition 3.4 (1)] allows us to deduce that $\text{level}_{D(C^*(X,K))}(M) = \text{level}_{D(H^*(X,K))}(F_X M)$ for any object $M$ in $D(C^*(X,K))$. By virtue of Lemma 4.1, we have the result. \hfill \Box

We recall a fundamental property of an object laying in the thickening of $D(A)$. The result follows from the fact that a triangle induces a long exact sequence in homology.

Lemma 4.2. Let $A$ be a DG algebra, $M$ a DG $A$-module and $n$ a positive integer. Suppose that $\dim H(A) < \infty$. Then $\dim H(M) < \infty$ for any object $M \in \text{thick}_{D(A)}^n(A)$.

Example 4.3. Let $\nu : S^7 \to S^4$ be the Hopf map and $E_\nu$ the pullback of $\nu : S^7 \to S^4$ over itself, giving rise to a fibration $S^9 \to E_\nu \to S^7$. We prove now that

\begin{equation}
\text{level}_{S^7}(E_\nu) \neq \text{level}_{D(H^*(S^7,K))}(H^*(S^7;K)) \otimes_{H^*(S^7,K)} H^*(S^7;K).
\end{equation}

Indeed, there is a Koszul resolution of the form

$$(\Gamma[w] \otimes (s^1 x_4) \otimes H^*(S^4;K), \delta) \to K \to 0$$
with $\delta(s^{-1}x_4) = x_4$ and $\delta(u) = s^{-1}x_4 \otimes x_4$, where $x_4$ denotes the generator of $H^*(S^4; \mathbb{K})$, and $\Gamma$ the divided powers algebra functor; see [24, Proposition 1.2].

This gives rise to a semifree resolution

$$H^*(S^7; \mathbb{K}) \otimes \Gamma[w] \otimes \wedge(s^{-1}x_4) \otimes H^*(S^4; \mathbb{K}) \to H^*(S^7; \mathbb{K}) \to 0$$

of $H^*(S^7; \mathbb{K})$ as an $H^*(S^4; \mathbb{K})$-module. Thus we have

$M := H^*(S^7; \mathbb{K}) \otimes \Gamma_{H^*(S^4; \mathbb{K})} H^*(S^7; \mathbb{K}) = (H^*(S^7; \mathbb{K}) \otimes \Gamma[w] \otimes \wedge(s^{-1}x_4) \otimes H^*(S^7; \mathbb{K}), 0)$. Since $\dim H(M) = \infty$, it follows from Lemma 4.2 that $M$ is not in the thickening $\text{thick}_{D_1}^{D_2}(H^*(S^7; \mathbb{K}))(H^*(S^7; \mathbb{K}))$ for any $n \geq 0$. This implies that the right hand side of (4.1) is infinite.

On the other hand, by Proposition 3.3, we see that $\text{level}_{S^d}(E_v) < \infty$ because the dimension of the cohomology of the fibre $S^d$ is finite. We refer the reader to Example 7.2 for the explicit calculation of the level of $E_v$.

5. PROOFS OF PROPOSITION 2.8 AND THEOREM 2.9

In this section, we work in rational homotopy theory and use Sullivan models for spaces and fibrations extensively. For a thorough introduction to these models, we refer the reader to the book [11].

As mentioned in the Introduction, Theorem 2.9 is deduced from Proposition 2.8. The proof of the proposition is given first.

**Proof of Proposition 2.8.** Let $Y_d$ be the space $B \times (\vee_{i=1}^n S^{2m_i+1})$ and $\Lambda V_B$ a minimal model for $B$. Then the Sullivan model for the fibration $S^{2m_i+1} \to Y_d \to Y_{d-1}$ has the form $\wedge V_{d-1} \to \wedge(x_i) \otimes \wedge V_{d-1} = \wedge V_i$, where $\wedge V_0 = \wedge V_B \otimes \wedge(y_0, \ldots, y_b)$ with $d(y_0) = 0$. Since the DG algebras $C^*(B; \mathbb{R})$ and $\Lambda V_B$ are connected with quasi-isomorphisms, it follows from [23, Proposition 4.2] and [2, Lemma 2.3] that $\text{level}_{d}(V_d) \leq \text{level}_{d}(\wedge V_d) \vee \text{level}_{c+1}(\wedge V_c)$. Define a filtration $\{F_i\}_{0 \leq i \leq c}$ of the $\wedge V_d$-module $\wedge V_c$ by

$F_i = \Lambda V_B \otimes \mathbb{R}[\nu_0 \cdots \nu_b, x_1, \ldots, x_i, | \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_i | = 0, 1]$.

It is immediate that $F_i/F_{i-1}$ is a finitely generated free $\wedge V_B$-module for each $i \geq 0$. Then it follows that $\{F_i\}_{0 \leq i \leq c}$ is a finite semifree filtration of class at most $c$. By virtue of Theorem 3.2, we have $\text{level}_{d}(\wedge V_d) \vee \text{level}_{c+1}(\wedge V_c) \leq c + 1$. \hfill \Box

We now establish a weaker version of Theorem 2.9.

**Lemma 5.1.** For any positive integer $l$, there exists an object $P_l \to S^d$ in $\mathcal{P}(S^d)$ such that

$\text{level}_{S^d}(P_l) \geq l$.

**Proof.** In the case where $l = 1$, the sphere $S^d$ is the space we desire. In what follows, we assume that $l \geq 2$. Let $m$ be an integer sufficiently larger than $ld$.

Assume that $d$ is even. We have a minimal model $B = (\wedge(x, \xi), \delta)$ for $S^d$ with $\delta(\xi) = x^2$, where $\deg x = d$. Consider a Koszul-Sullivan extension of the form

$B \to (\wedge(x, \xi, \rho, w_0, \ldots, w_{l-1}), D) := M_{l+1}$

for which the differential $D$ is defined by

$D(x) = x, D(w_0) = 0$ and $D(w_i) = (\rho x - \xi)w_{i-1}$

for $i \geq 1$, where $\deg w_1 = i(2d - 1) + (2m - 1) - i$. Let $\pi : P_{l+1} \to S^d$ be the pullback of the fibration $\{M_{l+1}\} \to |B| = S^d_{Q^l}$, which is the spatial realization of
the extension, by the localizing map $S^d \to S^d_0$; see [11, Proposition 7.9]. Since $M_{l+1}$ is a semifree $B$-module, it follows that $H^*(M_{l+1} \otimes_B \mathbb{Q}) = H^*(M_{l+1} \otimes_B \mathbb{Q}) = H^*(\wedge(p, w_0, w_1, ..., w_{l-1}))$. The cochain complex $M_{l+1} \otimes_B \mathbb{Q}$ is generated by elements with odd degree so that its homology is of finite dimension. It follows from Proposition 3.3 that $C^*(P_{l+1}; \mathbb{Q})$ is in $D^*(C^*(S^d; \mathbb{Q}))$.

By using the manner in [28, Section 7] for computing the homology of a DG algebra (or by the direct calculation), we have elements 1, $\xi$, $w_0$ and $(px - \xi)w_{l-1}$, which form a basis of $H^*(M_{l+1})$ of degree less than or equal to $l(2d - 1) + (2m - 1) - (l - 1)$. Let $Z$ be an indecomposable direct summand (a molecule) of $C^*(P_{l+1}; \mathbb{Q})$ in $D^*(C^*(S^d; \mathbb{Q}))$ containing a cocycle of degree zero; see Remark 2.14. By virtue of Theorem 2.13, we see that $Z = \Sigma^{-k(d-1)}Z_k$ for some $k \geq 0$; see Remark 2.15.

Suppose that $Z$ contains a representative of $w_0$, $(px - \xi)w_{l-1}$ or a cohomology class of degree greater than $l(2d - 1) + (2m - 1) - (l - 1)$. Proposition 3.5 allows us to conclude that $\text{level}_{S^d}(P_{l+1}) \geq l + 1$.

Suppose that $Z$ contains a representative of the element $\xi$. By Theorem 2.13, we see that $Z = \Sigma^{-d}Z_1$. In that case, let $Z'$ be a molecule of $C^*(P_{l+1}; \mathbb{Q})$ containing a representative of $w_0$. Observe that $Z' \neq Z$. If $Z'$ contains a representative of the element $(px - \xi)w_{l-1}$, then $Z' = \Sigma^{-d}(-\xi)Z_{2d-1}$ since $\dim H^*(Z') = 2$ and the amplitude of $Z'$ should be $2ld - 2l + 1$. If $Z'$ contains a representative of the cohomology class of degree greater than $l(2d - 1) + (2m - 1) - (l - 1)$, then $Z' = \Sigma^{-d}(-\xi)Z_{2d-1}$ for some $k \geq 2l - 1$. Proposition 3.5 yields that $\text{level}_{S^d}(P_{l+1}) \geq 2l$.

Suppose that $d$ is odd. We have a Koszul-Sullivan extension of the form

$$(\wedge(x, 0)) \to (\wedge(x, w_0, w_1, ..., w_{l-1}), D) =: N_l$$

for which the differential $D$ is defined by $D(x) = D(w_0) = 0$ and $D(w_i) = xw_{i-1}$ for $i \geq 1$, where deg $x = d$ and deg $w_0 = 2m - 1$. We assume that the integer $m$ is sufficiently larger than $ld$. Observe that deg $w_i = id + (2m - 1) - i$. Let $\pi : P_t \to S^d$ be the pullback of the fibration $|N_l| \to |(\wedge(x), 0)| = S^d_0$ by the localizing map $S^d \to S^d_0$. The same argument as above works again to show that $\text{level}_{S^d}(P_t) \geq l$. This completes the proof.

**Proof of Theorem 2.9.** Let $P_t \to S^d$ be the fibration constructed in the proof of Lemma 5.1. We have a sequence of fibrations

$S^{|\rho|} \to Y_1 \xrightarrow{\pi_1} S^d \times S^{|w_0|}, S^{|w_1|} \to Y_2 \xrightarrow{\pi_2} Y_1, ..., S^{|w_{l-1}|} \to Y_l \xrightarrow{\pi_l} Y_{l-1}$

in which $Y_l = P_{l+1}$ if $d$ is even, where $|w|$ denotes the degree of an element $w$. If $d$ is odd, we have a sequence of fibrations

$S^{|w_1|} \to Y_1 \xrightarrow{\pi_1} S^d \times S^{|w_0|}, S^{|w_2|} \to Y_2 \xrightarrow{\pi_2} Y_1, ..., S^{|w_{l-1}|} \to Y_{l-1} \xrightarrow{\pi_{l-1}} Y_{l-2}$

in which $Y_{l-1} = P_t$. Observe that the integers $|\rho|$ and $|w_i|$ are odd. It follows from Proposition 2.8 that $\text{level}_{S^d}(P_t) \leq l$. By combining the result with Lemma 5.1, the proof is now completed.

**6. Realization of molecules in $D^*(C^*(S^d; \mathbb{R}))$**

We recall briefly the Hopf invariant. Let $\phi : S^{2d-1} \to S^d$ be a map. Choose generators $[x_{2d-1}] \in H^{2d-1}(S^{2d-1}; \mathbb{Z})$ and $[x_d] \in H^d(S^d; \mathbb{Z})$. Let $\rho$ be an element
of \(C^*(S^{2d-1}; \mathbb{Z})\) such that \(\phi^*(x_d) = d\rho\). Since \([x_d]^2 = 0\) in \(H^*(S^d; \mathbb{Z})\), there exists an element \(\xi\) of \(C^*(S^d; \mathbb{Z})\) such that \(d\xi = x_d^2\). We then have a cocycle of the form \(\rho\phi^*(x_d) - \phi^*(\xi)\) in \(C^{2d-1}(S^{2d-1})\). The Hopf invariant \(H(\phi) \in \mathbb{Z}\) is defined by the equality
\[
[\rho\phi^*(x_d) - \phi^*(\xi)] = H(\phi)[x_{2d-1}].
\]

**Remark 6.1.** If \(d\) is odd, then \(H(\phi)\) is always zero.

We prove Proposition 2.17 by using Proposition 3.3. To this end, we need to consider whether the cohomology \(H^*(F_{\phi}; \mathbb{K})\) is of finite dimension, where \(F_{\phi}\) denotes the homotopy fibre of \(\phi : S^{2d-1} \to S^d\). Observe that \(F_{\phi}\) fits into the pullback diagram \(\mathcal{F}'\):

\[
\begin{array}{c}
\Omega S^d \\
\downarrow \\
F_{\phi} \\
\downarrow \\
S^{2d-1} \\
\phi \\
\end{array}
\xrightarrow{\pi} 
\begin{array}{c}
\Omega S^d \\
\downarrow \\
F_{\phi} \\
\downarrow \\
S^{2d-1} \\
\phi \\
\end{array}
\]

Here \(\Omega S^d \to PS^d \xrightarrow{\pi} S^d\) is the path-loop fibration. The pullback diagram gives rise to the Eilenberg-Moore spectral sequence \(\{E_{r}^{*,*}, d_r\}\) converging to \(H^*(F_{\phi}; \mathbb{K})\) with
\[
E_2^{*,*} \cong \text{Tor}^{\Gamma}_{H^*(S^d; \mathbb{K})}(H^*(S^{2d-1}; \mathbb{K}), \mathbb{K}).
\]

The Koszul resolution of \(\mathbb{K}\) as an \(H^*(S^d; \mathbb{K})\)-module allows us to compute the \(E_2\)-term. It turns out that
\[
E_2^{*,*} \cong \begin{cases} \quad [H^*(S^{2d-1}; \mathbb{K}) \otimes (s^{-1}x_d) \otimes \Gamma[\tau] & \text{if } d \text{ is even}, \\ \quad H^*(S^{2d-1}; \mathbb{K}) \otimes \Gamma[s^{-1}x_d] & \text{if } d \text{ is odd}, \end{cases}
\]
where \(\text{bideg } s^{-1}x_d = (-1, d)\) and \(\text{bideg } \tau = (-2, 2d)\); see [37, Lemma 3.1] and also [24, Proposition 1.2].

We relate the Hopf invariant with a differential of the Eilenberg-Moore spectral sequence (EMSS).

Recall that the Eilenberg-Moore map induces an isomorphism from the homology of the bar complex \((B(C^*(S^{2d-1}; \mathbb{K}), C^*(S^d; \mathbb{K}), \delta_1 + \delta_2)\) to \(H^*(F_{\phi}; \mathbb{K})\). Here \(\delta_1\) denotes the part of the differential coming from the multiplication of the algebra and its action on the module, which decreases bar-length, while \(\delta_2\) is induced by the differentials of the algebra and module and does not change bar-length. By the definitions of differentials \(\delta_1\) and \(\delta_2\), we see that
\[
\delta_1([x_d]x_d) = (-1)^d\phi^*[x_d][x_d] + (-1)^d(-1)^{d+1}[x_d^2] = \delta_2((-1)^d\rho[x_d] + 1[\xi]),
\]
\[
\delta_1((-1)^d\rho[x_d] + 1[\xi]) = (-1)^d((-1)^{d-1}\rho\phi^*[x_d] + \phi^*\xi) = -(\rho\phi^*([x_d] - \phi^*\xi)).
\]

It follows from [22, Lemma 2.1] that \(d_2([x_d]x_d) = H(\phi)[x_{2d-1}]\) in the \(E_2\)-term of the EMSS.

We denote by \(\text{Tor}_{H^*(S^d; \mathbb{K})}(H^*(S^{2d-1}; \mathbb{K}), \mathbb{K})_{\text{bar}}\) the torsion product as computed by the bar complex, which is necessarily isomorphic to the torsion product computed by the Koszul resolution.

By the same argument as in [24, Lemma 1.5], we have:
Lemma 6.2. The element \([x_d|x_d]\) in \(\text{Tor}_{\mathcal{H}^*(S^d; F)}(H^*(S^{2d-1}; K), K)_{\text{bar}}\) coincides with the element \(\tau \in \Gamma[\tau]\) up to isomorphism if \(d\) is even and with the element \(\gamma_d(s^{-1}x_d) \in \Gamma[s^{-1}x_d]\) if \(d\) is odd. Thus one has \(d_2(\tau) = H(\phi)_{x^d}2d-1\) if \(d\) is even and \(d_2(\gamma_d(s^{-1}x_d)) = H(\phi)_{x^d}2d-1 = 0\) if \(d\) is odd.

Proof of Proposition 2.17. Let \(\{E_r^{\cdots}, \tilde{d}_r\}\) be the EMSS converging to \(H^*(\Omega S^d; K)\). We see that
\[
\tilde{E}_r^{\cdots} = \begin{cases} \wedge(s^{-1}x_d) \otimes \Gamma[\tau] & \text{if } d \text{ is even,} \\ \Gamma[s^{-1}x_d] & \text{if } d \text{ is odd,} \end{cases}
\]
where \(\text{bideg } s^{-1}x_d = (-1, d)\) and \(\text{bideg } \tau = (-2, 2d)\). The result [9, Theorem III] implies that the EMSS for the fibre square \(\mathcal{F}'\) is a right DG comodule over \(\{E_r^{\cdots}, \tilde{d}_r\}\); that is, there exists a comodule structure \(\Delta : E_r^{\cdots} \rightarrow E_r^{\cdots} \otimes \tilde{E}_r^{\cdots}\) for any \(r\) such that the diagram
\[
\begin{array}{c}
E_r^{\cdots} \otimes \tilde{E}_r^{\cdots} \xrightarrow{\Delta} E_r^{\cdots} \otimes E_r^{\cdots} \\
\downarrow \Delta \quad \quad \quad \downarrow \Delta \\
E_r^{\cdots} \xrightarrow{\tilde{d}_r} E_r^{\cdots}
\end{array}
\]
is commutative. Since the comultiplication of the bar construction induces the comodule structure, it follows that, in our case,
\[
\Delta(x_{2d-1}^{2} \gamma_l(\tau)) = \sum_{0 \leq l \leq i} x_{2d-1}^{2} \gamma_{i-l}(\tau) \otimes \gamma_l(\tau),
\]
where \(\varepsilon = 0\) or 1. For dimensional reasons, we see that \(\tilde{d}_r = 0\) for all \(r\). In fact if \(i > j\), then we have
\[(6.1) \quad \text{t-deg } \gamma_i(\tau) + 1 = 2i(d - 1) + 1 > (2j + 1)(d - 1) = \text{t-deg } s^{-1}x_d \gamma_j(\tau),\]
where \(\text{t-deg}\) denotes the total degree of an element \(\alpha \in \tilde{E}_r^{\cdots, t}\), namely \(\text{t-deg}\alpha = s + t\). This implies that \(\tilde{d}_r(\gamma_i(\tau)) = 0\) even if \(d\) is even.

Suppose that \(H(\phi)_{\mathbf{K}}\) is nonzero. Then \(d\) is even. The commutativity of the diagram above and Lemma 6.2 together allow us to deduce that \(d_2(\gamma_i(\tau)) = H(\phi)_{x^d}2d-1\gamma_{i-1}(\tau)\), whence \(H^*(\mathcal{F}_{\phi}; K) \cong H^*(S^{2d-1}; K)\). It follows then that the \(C^*(S^d, K)\)-module \(C^*(S^{2d-1}; K)\) is in the category \(\mathcal{D}^*(C^*(S^d; K))\).

We show that the converse holds. Assume that \(C^*(S^{2d-1}; K)\) is a compact object and \(d\) is even. It follows from Proposition 3.3 that \(\dim H^*(\mathcal{F}_{\phi}; K) < \infty\) so that there exists a non-trivial differential in the EMSS \(\{E_r^{\cdots}, \tilde{d}_r\}\). Let \(\gamma_j(\tau) \in E_r^{\cdots}\) be an element with the first non-trivial differential; that is, \(d_s = 0\) for \(s < r\), \(d_r(\gamma_j(\tau)) \neq 0\) and \(d_r(\gamma_i(\tau)) = 0\) for \(i < j\). In view of the inequality (6.1), we can write \(d_r(\gamma_j(\tau)) = ax_{2d-1}^{2} \gamma_k(\tau)\), where \(\alpha \neq 0\). We see that
\[
(d_r \otimes 1 + 1 \otimes \tilde{d}_r)\Delta(\gamma_j(\tau)) = (d_r \otimes 1)
\sum_{0 \leq l \leq j} \gamma_l(\tau) \otimes \gamma_{j-l}(\tau)
\]
\[
= \sum_{0 \leq l \leq j} d_r(\gamma_l(\tau)) \otimes \gamma_{j-l}(\tau) = d_r(\gamma_j(\tau)) \otimes 1.
\]
Consider the commutative diagram mentioned above. We then have
\[(d_r \otimes 1 \pm 1 \otimes \tilde{d}_r) \Delta(\gamma_j(\tau)) = \Delta d_r(\gamma_j(\tau)) = \alpha(x_{2d-1} \otimes \gamma_k(\tau) + \sum_{0 < t \leq k} x_{2d-1} \gamma_t(\tau) \otimes \gamma_{k-t}(\tau)).\]

This amounts to requiring that \( k = 0 \). Thus we have \( d_r(\gamma_j(\tau)) = \alpha x_{2d-1} \). The comparison between the total degrees allows us to deduce that \( j(2(d-1)) + 1 = 2d-1 \) and hence \( j = 1 \). For dimensional reasons, we have \( r = 2 \). Lemma 6.2 yields that \( \alpha = H(\phi)_K \).

In the case where \( d \) is odd, the same argument works well to show the result. It follows from Theorem 2.13 that \( C^*(S^{2d-1}; \mathbb{K}) \cong \Sigma^{-(d-1)}Z_1 \) in \( D^\bullet(C^*(S^d; \mathbb{K})) \); see also Remark 2.15.

We show the latter half of the assertion. By considering the Auslander-Reiten quiver of \( D^\bullet(C^*(S^d; \mathbb{K})) \), we see that there is an irreducible map from \( C^*(S^d; \mathbb{K}) \) to \( C^*(S^{2d-1}; \mathbb{K}) \). Observe that the map is non-trivial.

Suppose that \( \phi^*: C^*(S^d; \mathbb{K}) \to C^*(S^{2d-1}; \mathbb{K}) \) is trivial in \( D(C^*(S^d; \mathbb{K})) \). Then there exists a \( C^* \) \((S^d; \mathbb{K})\)-linear map \( s: C^*(S^d; \mathbb{K}) \to C^*(S^{2d-1}; \mathbb{K}) \) of degree \(-1\) such that \( \phi^* = sd + ds \). We see that \( \phi^*(1) = sd(1) + ds(1) = 0 \) because \( d(1) = 0 \) and \( \deg s = -1 \). This yields that \( \phi^* = 0 \) as a \( C^*(S^d; \mathbb{K})\)-linear map. The definition of the Hopf invariant enables us to conclude that \( H(\phi)_K = 0 \); that is, \( \phi^* \neq 0 \) in \( D(C^*(S^d; \mathbb{K})) \) if \( H(\phi)_K \neq 0 \). Moreover,

\[ \text{Hom}_{D(C^*(S^d; \mathbb{K}))}(C^*(S^d; \mathbb{K}), C^*(S^{2d-1}; \mathbb{K})) = H^0(C^*(S^{2d-1}; \mathbb{K})) = \mathbb{K}. \]

It follows that the map \( \phi : S^{2d-1} \to S^d \) with non-trivial Hopf invariant induces an irreducible map \( \phi^* \) which coincides with the map \( Z_0 \to \Sigma^{-(d-1)}Z_1 \) up to scalar multiple. \( \square \)

**Remark 6.3.** If the pair \((q, f)\) of maps in the fibre square \( F \) described before Theorem 2.5 is relatively \( \mathbb{K} \)-formalizable, then the EMSS sequence with coefficients in \( \mathbb{K} \) for \( F \) collapses at the \( E_2 \)-term; see [25, Proposition 3.2].

Let \( \phi : S^{2d-1} \to S^d \) be a map between spheres and \( F_\phi \) the homotopy fibre of \( \phi \). Then the proof of Proposition 2.17 yields that the EMSS converging to \( H^\bullet(F_\phi; \mathbb{K}) \) does not collapse at the \( E_2 \)-term if \( H(\phi)_K \) is non-zero. Therefore we see that the pair \((\phi, *)\) with the constant map \(* \to S^d \) is not relatively \( \mathbb{K} \)-formalizable if \( H(\phi)_K \neq 0 \), even though \( S^d \) and \( S^{2d-1} \) are \( \mathbb{K} \)-formal. Observe that the map \( \phi \) satisfies neither of the conditions (i) and (ii) in Proposition 2.4.

**Proof of Theorem 2.18.** Recall from Theorem 2.13 the cohomology of the molecule \( \Sigma^{-l}Z_m \) \((m \geq 0)\). Suppose that \( d + l = 0 \). It is immediate that \(-m(d-1) + l < 0 \). Thus if \( \Sigma^{-l}Z_m \) is realizable, then \(-m(d-1) + l = 0 \) so that \( H^\bullet(\Sigma^{-l}Z_m) \cong H^\bullet(\Sigma^{-m(d-1)}Z_m) \cong H^\bullet(S^{(m+1)d-m}; \mathbb{K}) \) as a vector space.

Suppose that \( \Sigma^{-m(d-1)}Z_m \) is realized by a finite CW complex \( X \) with a map \( \phi : X \to S^d \). We then claim that \( m = 0 \) or \( m = 1 \) and \( d \) is even. The \( i \)-th integral cohomology of \( X \) is finitely generated for any \( i \). We see that \( H^\bullet(X) \otimes \mathbb{K} = H^\bullet(X; \mathbb{K}) = H^\bullet(\Sigma^{-m(d-1)}Z_m) = H^\bullet(S^{(m+1)d-m}; \mathbb{K}) \cong \mathbb{K} \oplus \Sigma^{-(m+1)d+m}\mathbb{K} \) and hence the rank of the \((m+1)d-m)\)th integral homology of \( X \) is at most one. It follows that \( H^\bullet(X; \mathbb{Q}) = \mathbb{Q} \oplus \Sigma^{-(m+1)d+m}\mathbb{Q} \) or \( H^\bullet(X; \mathbb{Q}) = \mathbb{Q} \).
Let \( \{ E_r, a_r \} \) be the EMSS converging to \( H^*(F_0; \mathbb{Q}) \). In view of the Koszul resolution of \( K \) as an \( H^*(S^d; K) \)-module, we see that
\[
\bar{E}_2^{*,*} \simeq \begin{cases} 
\wedge (s^{-1} x_d) \otimes \mathbb{Q}[\tau] \otimes H^*(X; \mathbb{Q}) & \text{if } d \text{ is even}, \\
\mathbb{Q}[s^{-1} x_d] \otimes H^*(X; \mathbb{Q}) & \text{if } d \text{ is odd},
\end{cases}
\]
where \( \text{bideg } \tau = (-2, 2d) \) and \( \text{bideg } s^{-1} x_d = (-1, d) \). Therefore, if \( d \) is odd, then the dimension of \( H^*(F_0; \mathbb{Q}) \) is infinite because \( s^{-1} x_d \) is a permanent cycle for dimensional reasons. Suppose that \( d \) is even and \( m > 1 \). Since \( (m + 1)d - m \geq 3d - 2 > 2d - 1 \), it follows that the element \( \tau \) is a permanent cycle and hence \( \dim H^*(F_0; \mathbb{Q}) = \infty \).

The cohomologies \( H^i(X; \mathbb{Z}) \) and \( H^i(\Omega S^d; \mathbb{Z}) \) are finitely generated for any \( i \). By considering the Leray-Serre spectral sequence of the fibration \( \Omega S^d \to F_0 \to X \), we see that \( H^i(F_0; \mathbb{Z}) \) is also finitely generated for any \( i \). This implies that \( \dim H^i(F_0; \mathbb{K}) = \infty \). Thus we conclude from Proposition 3.3 that if \( \Sigma^{-m(d-1)} Z_m \) is realizable, then \( m = 1 \) and \( d \) is even or \( m = 0 \).

In order to complete the proof, it suffices to show that \( \Sigma^{-(d-1)} Z_1 \) is realizable if \( d \) is even. In that case, for the Whitehead product \( [\iota, \iota] : S^{2d-1} \to S^d \) of the identity map \( \iota : S^d \to S^d \), it is well-known that \( H^i([\iota, \iota]) = \pm 2 \); see [30, Chapter 4]. Proposition 2.17 implies that for the irreducible map \( \alpha : Z_0 \to \Sigma^{-(d-1)} Z_1 \), there exists an isomorphism \( \Psi : \Sigma^{-(d-1)} Z_1 \to C^*(S^{2d-1}; \mathbb{K}) \) which fit into the commutative diagram
\[
\begin{array}{ccc}
Z_0 = C^*(S^d; \mathbb{K}) & \xrightarrow{\alpha} & \Sigma^{-(d-1)} Z_1 \\
\downarrow{[\iota, \iota]} & \equiv & \Psi \\
\xrightarrow{\Psi} C^*(S^{2d-1}; \mathbb{K})
\end{array}
\]
in \( D(C^*(S^d; \mathbb{K})) \) up to scalar multiple. Thus we have \( \Psi \alpha = k[\iota, \iota]^* \) for some non-zero element \( k \in \mathbb{K} \). It turns out that the molecule \( \Sigma^{-(d-1)} Z_1 \) is realizable. This completes the proof. \( \square \)

Remark 6.4. There exists an element of Hopf invariant one in \( \tau_{2d-1}(S^d) \) if \( d = 2, 4 \) or \( 8 \). Therefore, the proof of Theorem 2.18 allows us to conclude that the indecomposable element \( \Sigma^{-(d-1)} Z_1 \) is realizable with \( S^{2d-1} \) in \( D^c(C^*(S^d; \mathbb{K})) \) for any field \( \mathbb{K} \) if \( d = 2, 4 \) or \( 8 \).

7. Computational examples

Recall the functor \( F_{st} : D(C^*(S^d; \mathbb{K})) \to D(H^*(S^d; \mathbb{K})) \) described in Section 4, which gives an equivalence between triangulated categories. In order to prove Proposition 2.6, we need a lemma concerning this functor.

Lemma 7.1. Suppose that \( d \) is even. Then, in \( D^c(H^*(S^d; \mathbb{K})) \),
\[
F_{st}(\Sigma^{-(d-1)} Z_1) \cong (\wedge (\tau) \otimes H^*(S^d; \mathbb{K}), d \tau = x_d).
\]

Proof. The functor \( F_{st} \) leaves the cohomology of an object unchanged. Remark 2.15 implies the result. \( \square \)

Proof of Proposition 2.6. By assumption, the cohomology \( H^*(BG; \mathbb{K}) \) is a polynomial algebra generated by elements with even degree, say
\[
H^*(BG; \mathbb{K}) \cong \mathbb{K}[x_1, x_2, \ldots, x_t],
\]
where \( \text{deg} \, x_1 \leq \text{deg} \, x_2 \leq \cdots \leq \text{deg} \, x_l \) and each \( \text{deg} \, x_i \) is even. Since \( G \) is simply-connected, it follows that \( \text{deg} \, x_1 \geq 4 \). Moreover, \( \tilde{H}^i(S^4; \mathbb{K}) \) is nonzero if and only if \( i = 4 \), and \( \dim \tilde{H}^{i-1}(\Omega BG; \mathbb{K}) - \dim(QH^*(BG; \mathbb{K}))^4 = 0 \). Therefore Proposition 2.4 allows us to deduce that the pair \( (f, \pi) \) of maps is relatively \( \mathbb{K} \)-formalizable, where \( \pi : EG \to BG \) denotes the projection of the universal \( G \)-bundle. Theorem 2.5 implies that

\[
\text{level}_{S^4}(E_f) = \text{level}_{D(H^*(S^4; \mathbb{K}))}(\mathbb{K} \otimes_{H^*(BG; \mathbb{K})}^L H^*(S^4; \mathbb{K})) =: L.
\]

Consider the case where \( H^4(f; \mathbb{K}) \neq 0 \). Without loss of generality, we assume that \( H^4(f; \mathbb{K})(x_1) = z_4 \) and \( H^4(f; \mathbb{K})(x_j) = 0 \) for \( j \neq 1 \). Here \( z_4 \) is the generator of the algebra \( H^*(S^4; \mathbb{K}) \) of degree 4. We then have

\[
M := \mathbb{K} \otimes_{H^*(BG; \mathbb{K})}^L H^*(S^4; \mathbb{K}) \cong \wedge((s^{-1}x_2, \ldots, s^{-1}x_l), 0) \otimes (\wedge(s^{-1}x_1) \otimes H^*(S^4; \mathbb{K})) \otimes, \delta,
\]

in \( \text{D}^c(H^*(S^4; \mathbb{K})) \), where \( \delta s^{-1}x_1 = z_4 \). It follows from Lemma 7.1 that \( M \cong \wedge((s^{-1}x_2, \ldots, s^{-1}x_l), 0) \otimes F_{S^4}((\Sigma^{(-4)}Z_1)) \). This fact yields that \( M \) is isomorphic to a coproduct of the molecule \( F_{S^4}((\Sigma^{(-4)}Z_1)) \) and certain shifts as an \( H^*(S^4; \mathbb{K}) \)-module.

The functor \( F_{S^4} \) is exact and gives an equivalence between the triangulated categories \( D(C^*(S^4; \mathbb{K})) \) and \( D(H^*(S^4; \mathbb{K})) \). By [2, Theorem 2.4(6)] and Proposition 3.5, we see that \( L = \text{level}_{D(C^*(S^4; \mathbb{K}))}((\Sigma^{(-4)}Z_1)) = 2 \).

Suppose that \( \tilde{H}^*(f; \mathbb{K}) = 0 \). It follows that \( \mathbb{K} \otimes_{H^*(BG; \mathbb{K})}^L H^*(S^4; \mathbb{K}) \) is isomorphic to the DG module \( \wedge((s^{-1}x_1, s^{-1}x_2), \ldots, s^{-1}x_l) \otimes H^*(S^4; \mathbb{K}) \) with the trivial differential, which is a coproduct of \( H^*(S^4; \mathbb{K}) \) and certain shifts. We conclude that \( L = 1 \).

\[ \square \]

**Proof of Proposition 2.7.** We observe that \( (g, \pi) \) is a relatively \( \mathbb{K} \)-formalizable pair. Indeed the maps \( g \) and \( \pi \) satisfy the conditions (ii) and (i), respectively. Thus Theorem 2.5 yields that the \( C^*(S^4; \mathbb{K}) \)-level of \( C^*(E_g; \mathbb{K}) \) is equal to the \( H^*(S^4; \mathbb{K}) \)-level of \( H^*(BH; \mathbb{K}) \otimes_{H^*(BG; \mathbb{K})}^L H^*(S^4; \mathbb{K}) \). Since \( H \) is a maximal rank subgroup of \( G \), it follows from [4, 6.3 Theorem] that \( H^*(BH; \mathbb{K}) \) is a free \( H^*(BG; \mathbb{K}) \)-module. Therefore \( H^*(BH; \mathbb{K}) \otimes H^*(S^4; \mathbb{K}) \) is isomorphic to a coproduct of shifts of \( H^*(S^4; \mathbb{K}) \). This completes the proof.

\[ \square \]

**Example 7.2.** Let \( E_\nu \to S^7 \) be the fibration described in Remark 6.3, namely the pullback of the Hopf map \( \nu : S^7 \to S^4 \) by itself. We here compute the level of \( E_\nu \).

Consider the commutative diagram

\[
\begin{array}{ccc}
S^3 & \xrightarrow{E_\nu} & S^7 \\
\downarrow pt & \pitchfork & \downarrow \pi \\
S^7 & \xrightarrow{\nu} & S^4
\end{array}
\]

Let \( \{ E_r, d_r \} \) and \( \{ E_r, d_r \} \) be the Eilenberg-Moore spectral sequences for the front square and the back square, respectively. Then the diagram above gives rise to a morphism \( \{ g_r \} : \{ E_r, d_r \} \to \{ E_r, d_r \} \) of the spectral sequences. Observe that \( E_2 \cong H^*(S^7; \mathbb{K}) \otimes \Gamma[w] \otimes (s^{-1}x_4) \) and \( E_2 \cong H^*(S^7; \mathbb{K}) \otimes \Gamma(w) \otimes (s^{-1}x_4) \otimes H^*(S^7; \mathbb{K}) \), where \( \text{bideg} \, w = (-2, 8) \) and \( \text{bideg} \, s^{-1}x_4 = (-1, 4) \). Moreover it follows that
\[ g_2(w) = w, \quad g_2(s^{-1}x_4) = s^{-1}x_4, \quad g_2(x) = x \quad \text{for} \quad x \in H^*(S^7; \mathbb{K}) \otimes 1 \otimes 1 \otimes 1 \quad \text{and} \quad g_2(y) = 0 \quad \text{for} \quad y \in 1 \otimes 1 \otimes 1 \otimes H^*(S^7; \mathbb{K}). \]

By the same argument as in the proof of Proposition 2.17, we see that \( \partial \gamma_i(w) = x \gamma_i - 1(w) \). This implies that \( \partial \gamma_i(w) = x \gamma_i - 1(w) \) and hence \( E_\infty \cong E_3^{*,*} \cong \wedge(s^{-1}x_4) \otimes H^*(S^7; \mathbb{K}) \) as an \( H^*(S^7; \mathbb{K}) \)-module. For dimensional reasons, there is no extension problem. Thus it follows that \( H^*(E_\infty) \cong \wedge(s^{-1}x_4) \otimes H^*(S^7; \mathbb{K}) \) as an \( H^*(S^7; \mathbb{K}) \)-module. We observe that, by Remark 6.3, the pair \((\nu, \nu)\) of maps is not relatively \( \mathbb{K} \)-formalizable.

Define a \( C^*(S^7; \mathbb{K}) \)-module map \( \varphi : \Sigma^{-3}C^*(S^7; \mathbb{K}) \to C^*(E_\infty; \mathbb{K}) \) by \( \varphi(\Sigma^{-3}z) = s^{-1}x_4 \pi^*(z) \), where \( s^{-1}x_4 \) is a representative element of \( s^{-1}x_4 \in H^*(E_\infty; \mathbb{K}) \). We see that the map \( \varphi \oplus \pi^* : \Sigma^{-3}C^*(S^7; \mathbb{K}) \oplus C^*(S^7; \mathbb{K}) \to C^*(E_\infty; \mathbb{K}) \) is a quasi-isomorphism. The fact allows us to conclude that \( \operatorname{level}_{S^7}(E_\infty) = 1 \).

**Example 7.3.** We denote by \( \Sigma^iZA_{\infty} \) the connected component of the Auslander-Reiten quiver containing \( \Sigma^iZ_0 \) in \( D^r(C^*(S^d; \mathbb{K})) \), where \( 0 \leq i \leq d - 2 \).

Let \( G_2 \) be the compact simply-connected simple Lie group of type \( G_2 \). Consider the principal \( G_2 \)-bundle \( G_2 \to X_1 \to S^4 \) with the classifying map \( f : S^4 \to BG_2 \) which represents a generator of \( \pi_4(BG_2) \cong \pi_4(G_2) \cong \mathbb{Z} \). It is well-known that \( H^*(BG_2; \mathbb{F}_2) \cong \mathbb{F}_2[y_4, y_6, y_7] \), where \( \deg y_i = i \). Therefore, it follows from a computation similar to that in the proof of Proposition 2.6 that, in \( D^r(C^*(S^4; \mathbb{F}_2)) \),

\[
C^*(X_1; \mathbb{F}_2) \cong \Sigma^{-3}Z_1 \oplus \mathbb{F}_2[s^{-1}y_6, s^{-1}y_7] \cong \Sigma^{-3}Z_1 \oplus \Sigma^{-3-5}Z_1 \oplus \Sigma^{-3-6}. \]

This yields that \( C^*(X_1; \mathbb{F}_2) \) consists of two molecules \( \Sigma^{-3}Z_1 \) and \( \Sigma^{-3-6}Z_1 \) in \( ZA_{\infty} \) and one molecule \( \Sigma^{-3-5}Z_1 \) in \( \Sigma^2ZA_{\infty} \). We see that \( \operatorname{level}_{X_1}(X_2) = 2 \).

Consider the principal \( SU(4) \)-bundle \( SU(4) \to X_2 \to S^4 \) with the classifying map representing the generator of \( \pi_4(BSU(4)) \cong \mathbb{Z} \). We observe that \( H^*(BSU(4); \mathbb{F}_2) \cong \mathbb{F}_2[c_2, c_3, c_4] \), where \( \deg c_i = 2i \). A computation similar to that above enables us to conclude that

\[
C^*(X_2; \mathbb{F}_2) \cong \Sigma^{-3}Z_1 \oplus \Sigma^{-3-5}Z_1 \oplus \Sigma^{-3-7}Z_1. \]

Observe that the molecules \( \Sigma^{-3}Z_1, \Sigma^{-3-5}Z_1 \) and \( \Sigma^{-3-7}Z_1 \) are in the quivers \( ZA_{\infty}, \Sigma^2ZA_{\infty} \) and \( \Sigma^1ZA_{\infty} \), respectively. This yields that \( \operatorname{level}_{X_4}(X_2) = 2 \).

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8. **Appendix**

We let briefly the \( TV \)-model introduced by Halperin and Lemaire [14].

Let \( TV \) be the tensor algebra \( \sum_{n \geq 0} V^{\otimes n} \) on a graded vector space \( V \) over a field \( \mathbb{K} \) and let \( T^{\otimes k}V \) denote its ideal \( \sum_{n \geq k} V^{\otimes n} \) of the algebra \( TV \), where \( V^{\otimes 0} = \mathbb{K} \).

As usual, we define the degree of the element \( w = v_1 v_2 \cdots v_l \in TV \) by \( \deg w = n_1 + \cdots + n_l \) if \( v_{n_i} \in V^{n_i} \).

Let \( V' \) and \( V'' \) be copies of \( V \). We write \( sv \) for the element of \( \Sigma V \) corresponding to \( v \in V \). The cylinder object \( TV \wedge I = (TV' \oplus V'' \oplus \Sigma V), d \) introduced by Baues
and Lemaire [3, §1] is a DG algebra with differential \( d \) defined by

\[
dv' = (dv)', \quad dv'' = \langle dv \rangle'' \quad \text{and} \quad dsv = v'' - v' - S(dv),
\]

where \( S : TV \to T(V' \oplus V'' \oplus \Sigma V) \) is a map with \( Sv = sv \) for \( v \in V \) and

\[
S(xy) = Sx \cdot y' + (-1)^{\deg x} x' \cdot Sy \quad \text{for} \quad x, y \in TV.
\]

The inclusions \( \varepsilon_0 : TV \to TV \wedge I \) and \( \varepsilon_1 : TV \to TV \wedge I \) are defined by \( \varepsilon_0(v) = v' \) and \( \varepsilon_1(v) = v'' \), respectively.

For DG algebra maps \( \phi' : TV \to A \), we say that \( \phi' \) and \( \phi'' \) are homotopic if the DG algebra map \( (\phi', \phi'') : T(V' \oplus V'') \to A \) extends to a DG algebra map \( \Phi : TV \wedge I \to A \); that is \( \phi' = \Phi \varepsilon_0 \) and \( \phi'' = \Phi \varepsilon_1 \). We refer the reader to [10, Section 3] for the homotopy theory of DG algebras.

A \( TV \)-model for a differential graded algebra \( (A, d_A) \) is a quasi-isomorphism \( (TV, d) \to (A, d_A) \). Moreover the model is called minimal if \( d(V) \subset T^{<2}V \). For any simply-connected space whose cohomology with coefficients in \( K \) is locally finite, there exists a minimal \( TV \)-model \( (TV, d) \) extending to a DG algebra map \( \Phi : TV \wedge I \to A \). The reader is referred to [14] and [33, Introduction] for these facts and more details of \( TV \)-models.

**References**


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