SIMPLICIAL COCHAIN ALGEBRAS FOR DIFFEEOLOGICAL SPACES

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Abstract. We introduce a de Rham complex endowed with an integration map into the singular cochain complex which gives the de Rham theorem for every diffeological space. The theorem allows us to conclude that the Chen complex for a simply-connected manifold is quasi-isomorphic to the new de Rham complex of the free loop space of the manifold with an appropriate diffeology. This result is generalized from a diffeological point of view. In consequence, the de Rham complex behaves as a relevant codomain of Chen’s iterated integrals. Furthermore, the process of the generalization yields the Leray–Serre spectral sequence and the Eilenberg–Moore spectral sequence in the diffeological setting. The spectral sequences enable us to obtain computational examples of the new de Rham cohomology algebras for diffeological spaces containing the irrational torus and its related loop space.

1. Introduction

Diffeological spaces have been introduced by Souriau in the early 1980s [47]. The notion generalizes that of a manifold. More precisely, the category $\text{Mfd}$ of finite dimensional manifolds embeds into $\text{Diff}$ the category of diffeological spaces, which is complete, cocomplete and cartesian closed. As an advantage, we can very naturally define a function space of manifolds in $\text{Diff}$ so that the evaluation map is smooth without arguments on infinite dimensional manifolds; see [33] and also [15, Section 4]. It is worth to mention the existence of adjoint functors between $\text{Diff}$ and $\text{Top}$ the category of topological spaces; see Appendix B for a brief summary of the functors. Thanks to reflective properties of the adjoint functors, the full subcategory of $\Delta$-generated (numerically generated or arc-generated) topological spaces, which contains all CW-complexes, also embeds into $\text{Diff}$; see Remark 6.6. Thus in the category $\text{Diff}$, it is possible to deal with simultaneously such topological spaces and manifolds without forgetting the smooth structure.

The category $\text{Diff}$ is indeed equivalent to the category of concrete sheaves on a concrete site; see [3]. Moreover, Watts and Wolbert [53] have shown that $\text{Diff}$ is closely related to the category of differentiable stacks with adjoint functors between them. As Baez and Hoffnung have mentioned in [3, Introduction], we can use the larger category $\text{Diff}$ for abstract constructions and the smaller one $\text{Mfd}$ for theorems that rely on good control over local structure. In this article, we intend to focus on and develop cohomological methods for considering local and global nature of diffeological spaces in future work; see the end of this section and the diagram consisting of categories and functors related to $\text{Diff}$ in Appendix B.
The de Rham complex for a diffeological space is introduced in [47] and its applications, for example to moment maps, are extensively studied in [32, 34]. Moreover, differential forms on diffeological spaces are effectively used in the study of bundles in Diff; see [41, 52] and also [13, Future work]. Let \((X, \mathcal{D}^X)\) be a diffeological space and \(\Omega^*(X)\) the de Rham complex due to Souriau [47]. It is in fact regarded as a counterpart of the de Rham complex for developing Chen’s iterated integrals [10] in diffeology. In the de Rham calculus for diffeological spaces, Iglesias-Zemmour [34] has introduced an integration, which is a cochain map, 
\[
\int^{IZ} : \Omega^*(X) \rightarrow C^*_\text{cube}(X)
\]
and investigated its properties, where \(C^*_\text{cube}(X)\) denotes the normalized cubic cochain complex whose \(p\)-simplexes are smooth maps from \(\mathbb{R}^p\) to \(X\). However, the de Rham theorem in diffeology, which asserts that such an appropriate integration map induces an isomorphism of cohomology algebras, has not yet been established. Indeed, the de Rham theorem described with \(\int^{IZ}\) does not hold for the irrational torus; see [31, Section 8] and Remark 2.10.

In this article, the de Rham theory in Diff is formulated in the context of simplicial objects. In particular, a new de Rham complex is introduced and a morphism from the original de Rham complex mentioned above to the new one is discussed together with morphisms between other related cochain algebras; see Theorem 2.4 that is the main result in this article. In consequence, the theorem allows one to deduce that the de Rham theorem holds for every diffeological space in our setting; see Corollary 2.5. We deduce that the de Rham complex introduced in this article is quasi-isomorphic to the original one if the given diffeological space stems from a smooth CW-complex [30], a manifold or a parametrized stratifold in the sense of Kreck [38]. As a corollary, we also see that the integration map \(\int^{IZ}\) induces a morphism of algebras on the cohomology; see Corollary 2.6.

The Chen iterated integral map [10] is deeply related to our de Rham complex. Let \(M\) be a simply-connected manifold and \(LM\) the free loop space consisting of smooth loops endowed with Chen space structure; see [10, 12]. Let \(\text{So}\) denote the functor from the category of Chen spaces to Diff introduced by Stacey [48]. Then it follows that the Chen complex which is a cochain subalgebra of the de Rham complex of \(LM\) in the sense of Chen is quasi-isomorphic to our de Rham complex of \(\text{So}(LM)\); see Proposition 2.7. Furthermore, behavior of the Chen iterated integral map in diffeology is described in Theorems 2.8 and 5.2. It seems that the de Rham complex we introduce is a correct target (codomain) of Chen’s iterated integrals; see Remark 2.9. These results are also obtained by adaptation of Theorem 2.4.

The proof of Theorem 2.4 relies on the extendability of simplicial cochain algebras in the real and rational de Rham theory in [20, 26, 49, 54]. Moreover, an argument due to Kihara in [37] with the method of acyclic models [17, 6] is applied to our setting. The latter half of the theorem follows from the usual argument with the Mayer–Vietoris sequence; see [30, 27] for applications of the sequence in diffeology. Properties of the \(D\)-topology for diffeological spaces, which are studied in [15], are also used through this article. Thus the classical result, well-known methods in algebraic topology and recent results in diffeology as well serve mainly in the proofs of our assertions. Then, it seems that no new idea for the study of diffeology is given in this article. However, we would like to emphasize that an advantage of
this work is to give plenty of simplicial objects for homology of diffeological spaces; see Table 1 in Section 5 and the comment that follows. Indeed, there is a suitable choice of a simplicial set and a simplicial cochain algebra with which one deduces the de Rham theorem of the diffeological spaces as mentioned above. It is worth to mention that the de Rham complex, which we choose, definitely concerns the simplicial argument and cohomology developed in \([28, 14, 36, 37, 25]\).

Other choice of simplicial sets for given diffeological spaces enables us to construct the Leray–Serre spectral sequence (LSSS) and the Eilenberg–Moore spectral sequence (EMSS) for an appropriate \textit{fibration} in \(\text{Diff}\); see Theorems 5.4 and 5.5. By elaborate replacement of pullbacks with homotopy pullbacks for considering smooth lifts of fibrations, we obtain the spectral sequences and then Theorem 5.2, which explains a diffeological version of Chen’s isomorphism induced by iterated integrals. This is another highlight in this manuscript; see Remark 5.3. By applying Theorem 5.2, the LSSS and the EMSS to the irrational torus and its related diffeological spaces, we have computational examples; see Remark 2.10. One of the computations shows that appropriate forms on a manifold determine completely the new de Rham cohomology of the loop space of a diffeological space via Chen’s iterated integral map and the map \(\alpha\) connecting the original de Rham complex with the new one in Theorem 2.4. We mention here that in \([31]\), the Čech-de Rham spectral sequence converging to the Čech cohomology of a diffeological space is introduced. Observe that the original de Rham cohomology appears in the vertical edge of the \(E_2\)-term of the spectral sequence; see also \([35]\). This result is recalled in 6.3 Appendix C to discuss the injectivity of \(H(\alpha)\) induced by \(\alpha\) above on the de Rham cohomology; see Proposition 6.10.

In future work, it is expected that the local systems in the sense of Halperin \([26]\) which we use in the proof of Theorem 5.2 develop rational homotopy theory for non-simply connected diffeological spaces and Sullivan diffeological spaces; see \([23, 21]\). Moreover, the new de Rham complex may produce the argument on 1-minimal models as in \([11, 8]\) in diffeology. We also anticipate that our de Rham complex yields an invariant for diffeological stacks, for example, via a spectral sequence obtained by a bisimplicial set; see \([4, 44]\).

2. The main results

We begin by recalling the definitions of a diffeological space and the de Rham complex due to Souriau. A good reference for the subjects is the book \([33]\). We refer the reader to \([10, \S 1.2]\) and \([3, \S 2]\) for Chen spaces and \([48]\) for the comparison between diffeological spaces and Chen spaces.

**Definition 2.1.** For a set \(X\), a set \(\mathcal{D}^X\) of functions \(U \to X\) for each open set \(U\) in \(\mathbb{R}^n\) and for each \(n \in \mathbb{N}\) is a \textit{diffeology} of \(X\) if the following three conditions hold:

1. (Covering) Every constant map \(U \to X\) for all open set \(U \subset \mathbb{R}^n\) is in \(\mathcal{D}^X\);
2. (Compatibility) If \(U \to X\) is in \(\mathcal{D}^X\), then for any smooth map \(V \to U\) from an open set \(V \subset \mathbb{R}^m\), the composite \(V \to U \to X\) is also in \(\mathcal{D}^X\);
3. (Locality) If \(U = \bigcup U_i\) is an open cover and \(U \to X\) is a map such that each restriction \(U_i \to X\) is in \(\mathcal{D}^X\), then the map \(U \to X\) is in \(\mathcal{D}^X\).

We call a pair \((X, \mathcal{D}^X)\) consisting a set and a diffeology a \textit{diffeological space}. An element of a diffeology \(\mathcal{D}^X\) is called a \textit{plot}. Let \((X, \mathcal{D}^X)\) be a diffeological space and
A a subset of $X$. The sub-diffeology $\mathcal{D}^A$ on $A$ is defined by the initial diffeology for the inclusion $i : A \to X$; that is, $p \in \mathcal{D}^A$ if and only if $i \circ p \in \mathcal{D}^X$.

**Definition 2.2.** Let $(X, \mathcal{D}^X)$ and $(Y, \mathcal{D}^Y)$ be diffeological spaces. A map $X \to Y$ is smooth if for any plot $p \in \mathcal{D}^X$, the composite $f \circ p$ is in $\mathcal{D}^Y$.

All diffeological spaces and smooth maps form a category $\text{Diff}$. Let $(X, \mathcal{D}^X)$ be a diffeological space. We say that a subset $A$ of Euclidean spaces and smooth maps between them. We can regard the usual de Rham complex of $X$ with the sense of Souriau [47]. For an open set $U$, the cochain algebra consisting of $\Omega^p(U)$ is the category consisting of open sets of Euclidian spaces and smooth maps between them. We can regard $\mathcal{D}^X$ as the category of sets. Let $\text{Open}$ denote the category consisting of open sets and smooth maps between them. We can regard $\mathcal{D}^N$ as functors from $\text{Open}^{op}$ to $\text{Sets}$ the category of sets. A $p$-form is a natural transformation from $\mathcal{D}^X$ to $\mathcal{D}^N$. Then the de Rham complex $\Omega^*(X)$ is the direct sum of

$$\Omega^p(X) := \left\{ \left[ \text{Open}^{op} \xrightarrow{\mathcal{D}^X} \text{Sets} \right] | \omega \text{ is a natural transformation} \right\}$$

with the cochain algebra structure defined by that of $\Lambda^*(U)$ pointwise. We mention that the interpretation above of the de Rham complex appears in [42] and [30]. Observe that $\Omega^*(M)$ is isomorphic to the usual de Rham complex if $M$ is a manifold; see the comment after Definition 3.1. In what follows, we may use the terminology a differential graded algebra (for short, abbreviated DGA) for a cochain algebra.

**Remark 2.3.** The category $\text{Open}$ is regarded as a concrete site endowed with coverages consisting of open covers; see [3, Lemma 4.14]. Then the result [3, Proposition 4.15] yields that the functor $\mathcal{D}^X$ is a concrete sheaf on $\text{Open}$. On the other hand, the functor $\Lambda^p$ is a sheaf for each $p \geq 0$ but not concrete in general. It is readily seen that $\Lambda^p$ is a concrete sheaf if and only if $p = 0$.

In order to describe our main theorem, we recall appropriate simplicial sets and simplicial cochain algebras. Let $\mathbb{A}^n := \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 1\}$ be the affine space equipped with the sub-diffeology of $\mathbb{R}^{n+1}$. Let $\Delta^n_{\text{sub}}$ denote the sub-diffeo- logical space, whose underlying set is the standard $n$-simplex $\Delta^n$, equipped with the sub-diffeology of the affine space $\mathbb{A}^n$. Let $(\tilde{\Delta}^n_{\text{DR}})^{\bullet}$ be the simplicial cochain algebra defined by $(\tilde{\Delta}^n_{\text{DR}})^{n} := \Omega^*(\mathbb{A}^n)$ for each $n \geq 0$. We denote by $(\tilde{\Delta}^n_{\text{DR}})^{\bullet}$ the sub-simplicial cochain algebra of $\Omega^*(\Delta^n_{\text{sub}})$ consisting of elements in the image of the map $j^* : \Omega^*(\mathbb{A}^n) \to \Omega^*(\Delta^n_{\text{sub}})$ induced by the inclusion $j : \Delta^n_{\text{sub}} \to \mathbb{A}^n$.

In what follows, let $\Delta$ be the category which has posets $[n] := \{0, 1, ..., n\}$ with $k < k + 1$ for $n \geq 0$ as objects and non-decreasing maps $[n] \to [m]$ for $n, m \geq 0$ as morphisms. Let $K$ be a simplicial set, namely a contravariant functor from $\Delta$ to the category $\text{Sets}$ of sets.

For a diffeological space $(X, \mathcal{D}^X)$, let $S^D_\bullet(X)$ be the simplicial set defined by

$$S^D_\bullet(X) := \{\{\sigma : \mathbb{A}^n \to X | \sigma \text{ is a $C^\infty$-map}\}_{n \geq 0}.$$
We mention that $S^D_\bullet(-)$ gives the smooth singular functor defined in [14]. Moreover, let $S^\infty_\bullet(X)$ denote the sub-simplicial set of

$$S^D_\bullet(X)_{\text{sub}} := \{\{\sigma : \Delta^n_{\text{sub}} \to X \mid \sigma \text{ is a } C^\infty\text{-map}\}\}_{n \geq 0}$$

consisting of the elements which are restrictions of $C^\infty$-maps from $\Delta^n$ to $X$; see [28] for the study of the simplicial set $S^D_\bullet(X)_{\text{sub}}$ in diffeology.

We denote by $C^\ast(K)$ the cochain complex of maps from $K_p$ to $\mathbb{R}$ on degree $p$. The simplicial structure gives $C^\ast(K)$ the cochain algebra structure with the standard manner; that is, the cup product is defined on $C^\ast(K)$; see [20, 10 (d)] for example. We also recall the simplicial cochain algebra $(C^p_{PL})_\bullet := C^\ast(\Delta^\bullet)$, where $\Delta[n] = \text{hom}_\Delta(\cdot, [n])$ denotes the standard simplicial set.

For a simplicial cochain algebra $A_\bullet$, we denote by $A(K)$ the cochain algebra

$$\text{Sets}^\Delta_\omega(K, A_\bullet) := \left\{ \Delta^{op} \xymatrix{ & K \ar[ld]_f \ar[rd]_\omega & \cr A_\bullet \ar[rr]^-\omega \ar[rru]^-\Delta^{op} & & \text{Sets} } \quad \omega \text{ is a natural transformation} \right\}$$

whose cochain algebra structure is defined by that of $A_\bullet$. Observe that, for a simplicial set $K$, the map $\nu : C^p_{PL}(K) \to C^p(K)$ defined by $\nu(\gamma)(\sigma) = \gamma(\sigma)(\text{id}_{[p]})$ for $\sigma \in K_p$ gives rise to a natural isomorphism $C^p_{PL}(K) \cong C^\ast(K)$ of cochain algebras; see [20, Lemma 10.11]. In particular, we have a cochain algebra of the form $A_{DR}^\ast(S^D_\bullet(X))$ for a diffeological space $X$. This is a diffeological variant of Sullivan’s polynomial simplicial form construction for a topological space; see [49].

Moreover, we define a map $\alpha : \Omega^\ast(X) \to A_{DR}^\ast(S^D_\bullet(X))$ of cochain algebras by $\alpha(\omega)(\sigma) = \sigma^\ast(\omega)$ and define $\alpha' : \Omega(X) \to A_{DR}^\ast(S^\infty_\bullet(X))$ as well. The naturality and appropriate properties of the maps $\alpha$ and $\alpha'$ will be discussed in Section 4.

One of the aims of this article is to relate cochain algebras induced by simplicial objects mentioned above to one another. The following is the main theorem which describes such a relationship.

**Theorem 2.4.** (cf. [19], [30, Theorem 9.7], [25, Théorèmes 2.2.11, 2.2.14, 2.2.18])

For a diffeological space $(X, D^X)$, one has a homotopy commutative diagram

$$\begin{array}{ccc}
C^\ast(S^D_\bullet(X)_{\text{sub}}) & \xrightarrow{\varphi} & (C^p_{PL} \otimes A_{DR}^\ast)(S^D_\bullet(X)) \\
\downarrow \cong \quad \quad \downarrow \varphi \quad \quad \downarrow \text{mult } \alpha(1 \otimes f) \\
C^\ast(S^D_\bullet(X)_{\text{sub}}) & \xrightarrow{\psi} & A_{DR}^\ast(S^D_\bullet(X)) \\
\downarrow \cong \quad \quad \downarrow \text{"integration" } f \\
C^\ast(\text{cube}(X)) & \xrightarrow{\alpha} & \Omega^\ast(X)
\end{array}$$

in which $\varphi$ and $\psi$ are quasi-isomorphisms of cochain algebras and the integration map $f$ is a morphism of cochain complexes. Here mult denotes the multiplication on the cochain algebra $C^\ast(S^D_\bullet(X)_{\text{sub}})$. Moreover, if $(X, D^X)$ stems from a smooth CW-complex in the sense of Iwase and Izumida [30] or a parametrized stratifolds; see Appendix B, then $\alpha$ is a quasi-isomorphism.

The homology $H(C^\ast(S^D_\bullet(X)_{\text{sub}}))$ has been introduced in [28] in which tangent spaces of diffeological spaces are also discussed. We observe that the latter half of Theorem 2.4 gives a partial answer of [28, Problème D]. As it turns out, the de Rham theorem holds for diffeological spaces.
Corollary 2.5. For every diffeological space $(X, \mathcal{D}^X)$, the integration map

$$
\int : A^*_{DR}(S^{\mathcal{D}}_*(X)) \to C^*(S^{\mathcal{D}}_*(X)_{\text{sub}})
$$

in Theorem 2.4 induces an isomorphism of algebras on the cohomology.

In Theorem 2.4, the map $l$ induces an isomorphism of algebras in the cohomology. This follows from the result [43, Theorem 8.2] for example. Then we have an obstruction for $\int^I_{\mathcal{D}}$ to give an isomorphism on the cohomology.

Corollary 2.6. (i) The functor $H^*(S^{\mathcal{D}}_*(\_))$ from the category $\text{Diff}$ to the category of graded algebras is homotopy invariant and hence so is the de Rham cohomology functor $H^*(A^*_{DR}(S^{\mathcal{D}}_*(\_)))$.

(ii) The integration map $\int^I_{\mathcal{D}} : \Omega^*(X) \to C^*_{\text{cube}}(X)$ induces a morphism of algebras on the cohomology.

(iii) The integration map $\int^I_{\mathcal{D}}$ induces an isomorphism of algebras on the cohomology if and only if so does the morphism $\alpha$ in Theorem 2.4.

We observe that in general, the morphism $\alpha$ does not induce an isomorphism on the cohomology. This is clarified in Remark 2.10 below.

In [30], Iwase and Izumida have proved the de Rham theorem for a smooth CW-complex by using cubic de Rham cohomology, which admits the Mayer–Vietoris sequence for every diffeological space. Let $X$ be a diffeological space. Then the excision axiom for the homology of $X$ holds with respect to the $D$-topology for $X$ and hence so does the homology $H(A^*_{DR}(S^{\mathcal{D}}_*(X)))$; see Section 4. Thus we have the Mayer–Vietoris exact sequence for $H(A^*_{DR}(S^{\mathcal{D}}_*(X)))$ with respect to a $D$-open cover.

We describe two applications of the integration map in Theorem 2.4 which are related to Chen’s iterated integrals [10, 12]. Let $M$ be a manifold and $M^I$ the Chen space of smooth free paths. Then we have the pullback

\begin{equation}
\begin{array}{ccc}
LM & \xrightarrow{\Delta} & M^I \\
\ev \downarrow & & \downarrow (\varepsilon_0, \varepsilon_1) \\
M & \xrightarrow{\Delta} & \times \times M
\end{array}
\end{equation}

of the free path fibration $(\varepsilon_0, \varepsilon_1) : M^I \to \times \times M$ along the diagonal map $\Delta : M \to \times M$ in the category $\text{ChenSp}$ of Chen spaces, where $\varepsilon_i$ is the evaluation map at $i$; see [10, Section 1.2] and [3, Section 2]. We also recall Chen’s iterated integral $\Omega^*_{\text{Chen}} : \Omega^*(M) \otimes B(\Omega^*(M)) \to \Omega^*(LM)_{\text{Chen}}$, which is a morphism of DG-modules; see [12, (2.1)] and [10, THEOREM 4.2.1], where $\Omega^*(X)_{\text{Chen}}$ denotes the de Rham complex for a Chen space $X$ and the source complex is the bar construction; see [9, Section 2] and [42, Section 1.1]. More precisely, let $\omega_i$ be a differential $p_i$-form in $\Omega^*(M)$ for each $1 \leq i \leq k$ and $\rho : U \to M^I$ a plot of the Chen space $M^I$. We define $\bar{\omega}_\rho$ by $\bar{\omega}_\rho := (id_U \times t_1)^* \rho^* \omega_i$, where $\rho : U \times I \to M$ is the adjoint to $\rho$ and $t_i : \Delta^k := \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid 0 \leq t_1 \leq \cdots \leq t_k \leq 1\} \to I$ denotes the projection in the $i$th factor. By using the integration along the fibre of the trivial fibration $U \times \Delta^k \to U$, the iterated integral $(\int \omega_1 \cdots \omega_k)\rho$ is defined by

$$
\left( \int \omega_1 \cdots \omega_k \right)\rho := \int_{\Delta^k} \bar{\omega}_\rho \wedge \cdots \wedge \bar{\omega}_{k\rho}.
$$
Then by definition, Chen’s iterated integral $\text{It}$ has the form

\begin{equation}
\text{It}(\omega_0[\omega_1|\cdots|\omega_k]) = ev^*(\omega_0) \wedge \Delta^*(\int \omega_1 \cdots \omega_k).
\end{equation}

Let $\text{So} : \text{ChenSp} \to \text{Diff}$ be the functor introduced by Stacey [48] for which the underlying set is the same as that of the Chen space and $p : U \to X$ is a plot in $\text{So}X$ if and only if $p : U \to X$ is smooth in $\text{ChenSp}$. Then we shall define a morphism $\beta : \Omega^*(X)_{\text{Chen}} \to \Omega^*(\text{So}X)$ of DGA’s for each Chen space $X$ in Section 4.2. We choose a DG subalgebra $A$ of $\Omega^*(M)$ which satisfies the condition that $A^p = \Omega^p(M)$ for $p > 1$, $A^0 = \mathbb{R}$ and $A^1 \cap d\Omega^p(M) = 0$. Let $\text{Chen}(M)$ be the image of the restriction $\Omega^*(M) \otimes \overline{B}(A) \to \Omega(LM)_{\text{Chen}}$ of $\text{It}$ mentioned above. Here $\Omega^*(M) \otimes \overline{B}(A)$ denotes the reduced bar complex. By applying Theorem 2.4, we have

**Proposition 2.7.** Let $M$ be a simply-connected manifold. Then the composite \[
\alpha \circ \beta \circ \text{It} : \Omega^*(M) \otimes \overline{B}(A) \to A^*_{\text{DR}}(S^D_\bullet(\text{So}(LM)))
\]
is a quasi-isomorphism of DG-modules over $\Omega^*(M)$. Moreover, a restriction of the form $(\alpha \circ \beta)_{\text{Chen}(M)} : \text{Chen}(M) \to A^*_{\text{DR}}(S^D_\bullet(\text{So}(LM)))$ is a quasi-isomorphism of DGA’s.

While the proof we give in the article heavily relies on the results in [10, 12], it may be possible to prove the latter half of Proposition 2.7 with relative Sullivan models for fibrations [26, 20]. In fact, it is realized in a dиффологический framework; see Remark 2.9 below.

We can consider the same diagram as (2.1) in Diff in which $M$ is a general dиффологический space and $M^I$ is the dиффологический space endowed with the functional dиффология. The pullback is denoted by $ev : L_{\text{free}}M \to M$. Modifying the definition of Chen’s iterated integral in Diff, we have a morphism $\text{It} : \Omega^*(M) \otimes \overline{B}(A) \to \Omega^*(L_{\text{free}}M)$ of differential graded $\Omega^*(M)$-modules; see Section 5 for more details. The integration map in Theorem 2.4 and careful treatment of a local system in the sense of Halperin [26] with respect to the evaluation map $M^I \to M \times M$ in (2.1) enable us to deduce the following pivotal theorem.

**Theorem 2.8.** Let $M$ be a simply-connected dиффологический space. Then the composite \[
\alpha \circ \text{It} : \Omega^*(M) \otimes \overline{B}(A) \to \Omega^*(L_{\text{free}}M) \to A^*_{\text{DR}}(S^D_\bullet(L_{\text{free}}M))
\]
is a quasi-isomorphism of $\Omega^*(M)$-modules.

**Remark 2.9.** We observe that the image $\text{Chen}(M)$ of Chen’s iterated integral map is a DG subalgebra of $\Omega(LM)_{\text{Chen}}$ which is a prototype of the original de Rham complex of a dиффологического space; see [10, 21]. The subalgebra is isomorphic to the bar complex $\Omega^*(M) \otimes \overline{B}(A)$ mentioned above as a cochain complex; see [10, Theorem 4.2.1]. Moreover, the result [12, Theorem 0.1] asserts that $\text{Chen}(M)$ is quasi-isomorphic to the singular cochain complex $C^*(LM_{\text{top}})$ if $M$ is a manifold, where $LM_{\text{top}}$ denotes the function space of continuous maps form $S^1$ to $M$ with compact-open topology. However, it seems that the relationship on the cohomology between $\text{Chen}(M)$ and the DGA $\Omega^*(LM)_{\text{Chen}}$ itself is obscure.

On the other hand, Proposition 2.7, Theorem 2.8 and its generalization Theorem 5.2 below reveal that the new de Rham complex functor $A^*_{\text{DR}}(S^D_\bullet(\overline{B}(A)))$ gives rise to an appropriate target of Chen’s iterated integral $\text{It}$. In fact, every map containing $\text{It}$ as a component in the assertions is a quasi-isomorphism.
In Remark 2.10 below, we give computational examples of the de Rham cohomology algebras of diffeological spaces containing the irrational torus and its related ones. As a consequence, we see that the free loop cohomology for a diffeological space associated with the irrational torus is generated by Chen’s iterated integrals.

Remark 2.10. Let $\gamma$ be an irrational number, namely $\gamma \in \mathbb{R} \setminus \mathbb{Q}$. Consider the two dimensional torus $T^2 := \{(e^{2\pi i x}, e^{2\pi i y}) \mid (x, y) \in \mathbb{R}^2\}$, which is a Lie group, and the subgroup $S_\gamma := \{(e^{2\pi i t}, e^{2\pi i \gamma t}) \mid t \in \mathbb{R}\}$ of $T^2$. Then the irrational torus $T_\gamma$ is defined by the quotient $T^2/S_\gamma$ with quotient diffeology. Since $S_\gamma$ is a dense subgroup, it follows that the topology of the homogeneous space $T^2/S_\gamma$ is trivial and hence it is contractible.

In the category $\text{Diff}$, we have a principal diffeological fibre bundle of the form $S_\gamma \to T^2 \xrightarrow{S_\gamma} T_\gamma$; see [33, 8.15]. For a smooth map $f : M \to T_\gamma$ from a diffeological space $M$, we obtain a principal diffeological bundle $(**): S_\gamma \to M \times_{T_\gamma} T^2 \xrightarrow{f} M$ via the pullback construction along the map $f$. For a diffeological space $X$, we may write $A^*(X)$ for the cochain algebra $A^*_{\text{Diff}}(S^2(X))$. A diffeological fibre bundle with a diffeological group as the fibre is a fibration in the sense of Christensen and Wu; see [14, Propositions 4.28 and 4.30]. Then the Leray–Serre spectral sequence in Theorem 5.4 below for the fibration $(**)$ enables us to deduce that $\pi'$ gives rise to an isomorphism

$$\tag{2.3} (\pi')^* : H^*(A^*(M)) \xrightarrow{\cong} H^*(A^*(M \times_{T_\gamma} T^2))$$

of algebras. We observe that the local system $\mathcal{H}^*(S_\gamma)$ is simple. In fact, the fibre $S_\gamma$ is diffeomorphic to $(\mathbb{R}, +)$ as a Lie group and hence it is contractible. In particular, we have isomorphisms

$$\wedge (dt_1, dt_2) \cong H^1_{\text{dRham}}(T^2) \xrightarrow{pr_2^*} H^1(A^*(T^2)) \xrightarrow{(\pi')^*} H^1(A^*(T_\gamma)).$$

Here $H^1_{\text{dRham}}(T^2)$ is the usual de Rham cohomology of the manifold $T^2$ and $dt_i$ denotes the image $(pr_i)^* (d\theta)$ of the volume form $d\theta \in H^1(T)$ of the one dimensional torus $T$ by the map $(pr_i)^*$ induced by the projection $pr_i$ on ith factor. We recall the diffeomorphism $\psi : \mathbb{R}/(\mathbb{Z} + \gamma \mathbb{Z}) \to T_\gamma$ defined by $\psi(t) = (0, e^{2\pi i t})$ in [33, Exercise 31, 3]). Then the isomorphism $\rho^* : \Omega^*(\mathbb{R}/(\mathbb{Z} + \gamma \mathbb{Z})) \xrightarrow{\cong} (\Lambda^*(\mathbb{R}), d = 0)$ induced by the subduction $p : \mathbb{R} \to \mathbb{R}/(\mathbb{Z} + \gamma \mathbb{Z})$ in [33, Exercise 119] fits into the commutative diagram

$$\begin{array}{ccc}
\Omega^*(T) & \xrightarrow{\rho^*} & \Omega^*(\mathbb{R}) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\Omega^*(T^2) & \xrightarrow{\psi^*} & \Omega^*(\mathbb{R}/(\mathbb{Z} + \gamma \mathbb{Z})),
\end{array}$$

where $\rho : \mathbb{R} \to T$ is the projection and $\psi^*$ is the inclusion in the second factor. We may assume that $\rho^*(d\theta) = dt$ for the constant differential form $d\theta \in \Lambda^1(\mathbb{R})$. Then, it follows that $\pi^*(\psi^*)^{-1}(\rho^*)^{-1}dt = dt_2$. In fact, we write $a_1 dt_1 + a_2 dt_2$ for $\pi^*(\psi^*)^{-1}(\rho^*)^{-1}dt$. The commutativity of the diagram above implies that $a_2 = 1$. We see that $(a_1 dt_1 + a_2 dt_2)^2 = (\pi^*(\psi^*)^{-1}(\rho^*)^{-1}dt)^2 = 0$ and hence $a_1 = 0$. 

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Moreover, the naturality of the morphism $\alpha$ in Theorem 2.4 gives a commutative diagram

$$
\begin{array}{ccc}
A^*(T^2) & \overset{\alpha}{\leftarrow} & \Omega^*(T^2) \\
\pi^* \downarrow & & \downarrow \pi^* \\
A^*(T) & \overset{\alpha}{\leftarrow} & \Omega^*(T).
\end{array}
$$

Thus we see that the morphism $H(\alpha)$ of algebras from the original de Rham cohomology of $T$ to the new one is a non-surjective monomorphism; see 6.3 Appendix C for a more general consideration concerning the injectivity of $H(\alpha)$.

We recall the isomorphism $(\pi')^*$ in (2.3). Suppose that $M$ is simply connected. Then the comparison of the EMSS’s in Theorem 5.5 for $L_{\text{free}}M$ and $L_{\text{free}}(M \times T, T^2)$ allows us to obtain an algebra isomorphism

$$(L\pi')^* : H^*(A^*(L_{\text{free}}M)) \xrightarrow{\cong} L_{\text{free}}(M \times T, T^2).$$

Thus if $H^*(A^*(M)) \cong H^*(A^*(S^{2k+1}))$ as an algebra with $k \geq 1$, then we see that

$$H^*(A^*(L_{\text{free}}(M \times T, T^2))) \cong \wedge(\alpha \circ \text{It}((\pi')^*(\omega))) \otimes \mathbb{R}[(\alpha \circ \text{It}(1 \otimes (\pi')^*(\omega)))]$$

as an $H^*(A^*(M))$-algebra, where $\omega$ is the volume form of $M$. In fact, the result follows from Theorem 2.8 and [39, Theorem 2.1 and Corollary 2.2]. Observe that we require no $D$-topological condition on $M$ which is needed for the classification theorem of bundles over a diffeological space; see [41, 13].

The rest of this article is organized as follows. In order to prove Theorem 2.4, the extendability of the simplicial cochain algebra $A^*_{\text{DR}}$ and its variants is verified in Section 3.1. Section 3.2 explains the integration map and the map $\alpha$ in the main theorem. Theorem 2.4, Corollaries 2.5, 2.6 and Proposition 2.7 are proved in Section 4. In Section 5, after recalling Chen’s iterated integrals in a diffeological point of view, we prove Theorem 2.8 as a corollary of a more general result (Theorem 5.2). In Appendix A, the acyclic model theorem for cochain complexes is recalled. Appendix B brieﬂy summarizes the notion of a stratifold due to Kreck [38] and functors between categories related to our subjects in this article. In Appendix C, we discuss the injectivity of $H(\alpha)$ induced by the map $\alpha$ in Theorem 2.4.

3. Preliminaries

3.1. Extendability of the simplicial cochain algebra $A^*_{\text{DR}}$. We begin with the definition of the extendability of a simplicial object. The notion plays an important role in the proof of the main theorem.

Definition 3.1. A simplicial object $A$ in a category $C$ is extendable if for any $n$, every subset set $I \subset \{0, 1, \ldots, n\}$ and any elements $\Phi_i \in A_{n-1}$ for $i \in I$ which satisfy the condition that $\partial_i \Phi_j = \partial_{i-1} \Phi_i$ for $i < j$, there exists an element $\Phi \in A_n$ such that $\Phi_i = \partial_i \Phi$ for $i \in I$.

Let $M$ be a manifold and $\Omega^*_{\text{deRham}}(M)$ the usual de Rham complex of $M$. We recall the tautological map $\theta : \Omega^*_{\text{deRham}}(M) \to \Omega^*(M)$ defined by $\theta(\omega) = \{p^*\omega\}_{p \in \mathcal{P}_M}$. Observe that $\theta$ is isomorphism; see [29, Section 2]. With this in mind, we prove the following lemma due to Emoto [18]. Though the proof indeed uses the same strategy as in [26, 13.8 Proposition] and [20, Lemma 10.7 (iii)], we introduce it for the reader.
Lemma 3.2. The simplicial differential graded algebra $(\Lambda^*_\mathbb{R})_n$ is extendable.

Proof. Let $\mathcal{I}$ be a subset of $\{0, 1, \ldots, n\}$ and $\Phi_i$ an element in $(\Lambda^*_\mathbb{R})_{n-1}$ for $i \in \mathcal{I}$. We assume that $\partial_i \Phi_j = \partial_{j-1} \Phi_i$ for $i < j$. We define inductively elements $\Phi_r \in (\Lambda^*_\mathbb{R})_n$ for $-1 \leq r \leq n$ which satisfy the condition that (*): $\partial_i \Psi_r = \Phi_i$ if $i \in \mathcal{I}$ and $i \leq r$. Put $\Psi_{-1} = 0$ and suppose that $\Psi_{r-1}$ is given with (*). Define a smooth map $\varphi : \mathbb{A}^n - \{v_r\} \to \mathbb{A}^{n-1}$ by $\varphi(t_0, t_1, \ldots, t_n) = (\frac{t_i}{v_r}, \ldots, \frac{t_i}{v_r}, \frac{t_{i+1}}{v_r}, \ldots, \frac{t_{n-1}}{v_r})$, where $v_r$ denotes the $r$th vertex. The map $\varphi$ induces a morphism $\varphi^* : \Omega^*(\mathbb{A}^{n-1}) \to \Omega^*(\mathbb{A}^n - \{v_r\})$ of cochain algebras. For an element $u$ in $(\Lambda^*_\mathbb{R})_{n-1}$, we write $u'$ for an element in $\Omega^*(\mathbb{A}^{n-1})$ with $j^*(u') = u$. If $r$ is not in $\mathcal{I}$, we define $\Psi_r$ by $\partial_r \Psi_r$. In the case where $i \in \mathcal{I}$, we consider the element $\Phi_i - \partial_r \Psi_r$ in $\Omega^*(\mathbb{A}^n)$.

Define $\Psi \in (\Lambda^*_\mathbb{R})_n$ by

$$\Psi := j^*((\rho \circ k_r) \ast \varphi^*(\Phi_r' - \partial_r \Psi_{r-1}));$$

where $k_r : \mathbb{A}^n \to \mathbb{A}$ is the projection in the $r$th factor and $\rho$ is a cut-off function with $\rho(0) = 1$ and $\rho(1) = 0$. We observe that $(\rho \circ k_r)$ is in $\Omega^0_D(\mathbb{A}^n)$ and that the action of $(\rho \circ k_r)$ on $\Omega^r_D(\mathbb{A}^n - \{v_r\})$ defined by the pointwise multiplication gives rise to a linear map $(\rho \circ k_r) \ast : \Omega^*(\mathbb{A}^n - \{v_r\}) \to \Omega^*(\mathbb{A}^n)$. Moreover, we see that the map $(\rho \circ k_r) \ast -$ fits in the commutative diagram

\[
\begin{array}{ccc}
\Omega^*(\mathbb{A}^{n-1}) & \xrightarrow{\varphi^*} & \Omega^*(\mathbb{A}^n - \{v_r\}) \\
\downarrow{id} & & \downarrow{(\rho \circ k_r) \ast} \\
\Omega^*(\mathbb{A}^{n-2}) & \xrightarrow{\varphi^*} & \Omega^*(\mathbb{A}^n - \{v_{r-1}\}).
\end{array}
\]

for $i < r$. Since $\partial_i(\Phi_r - \partial_r \Psi_{r-1}) = \partial_{r-1}(\Phi_i - \partial_r \Psi_{r-1}) = 0$ by assumption for $i < r$, it follows from the commutative diagram above that for $i < r$,

$$\partial_i \Phi = \partial_r j^*((\rho \circ k_r) \ast \varphi^*(\Phi_r' - \partial_r \Psi_{r-1})))
= j^*((\rho \circ k_{r-1}) \ast \partial_{r-1} \varphi^*(\Phi_r' - \partial_r \Psi_{r-1})))
= (\rho \circ k_{r-1}) \ast j^* \varphi^* \partial_{r-1} \varphi^*(\Phi_r' - \partial_r \Psi_{r-1})
= (\rho \circ k_{r-1}) \ast \varphi^* \partial_{r-1} \varphi^*(\Phi_r' - \partial_r \Psi_{r-1}).$$

The third and fifth equalities follow from the commutativity of the diagram

\[
\begin{array}{ccc}
\Omega^*(\mathbb{A}^{n-2}) & \xrightarrow{\varphi^*} & \Omega^*(\mathbb{A}^{n-1} - \{v_{r-1}\}) \\
\downarrow{j^*} & & \downarrow{(\rho \circ k_{r-1}) \ast} \\
\operatorname{Im} j^* & \xrightarrow{\varphi^*} & \operatorname{Im} j^* \subset \Omega^*(\mathbb{A}^{n-1} - \{v_{r-1}\}).
\end{array}
\]

Since $\partial_i(\rho \circ k_r) = 1$ and $\varphi \circ \partial^r = \text{id}_{\mathbb{A}^n}$, it follows that the diagram

\[
\begin{array}{ccc}
\Omega^*(\mathbb{A}^{n-1}) & \xrightarrow{\varphi^*} & \Omega^*(\mathbb{A}^n - \{v_{r-1}\}) \\
\downarrow{id} & & \downarrow{(\rho \circ k_r) \ast} \\
\Omega^*(\mathbb{A}^{n-1}) & \xrightarrow{\partial_r} & \Omega^*(\mathbb{A}^n - \{v_{r-1}\}).
\end{array}
\]
is commutative. Thus we have $\partial_j \Psi = \Phi_r - \partial_r \Psi_{r-1}$. It turns out that $\partial_j (\Psi + \Psi_{r-1}) = \Phi_j$ for $j \in \mathcal{I}$ and $j \leq r$. This completes the proof. □

We verify that the Poincaré lemma holds for $(\widetilde{A}_{DR})_n$.

Lemma 3.3. One has $H^k((\widetilde{A}_{DR})_n) = \mathbb{R}$ for any $n \geq 0$.

Proof. We first remark that the chain homotopy operator defined in [33, 6.83] is natural with respect to smooth maps. A smooth contraction map $\Delta^n_{sub} \to \Delta^n_{sub}$ extends to one on the affine space $\mathbb{A}^n$. Moreover, we have a smooth homotopy between the contraction and the identity map on $\mathbb{A}^n$ whose restriction is such a homotopy on $\Delta^n_{sub}$. The homotopy gives rise to chain homotopy maps $K$ and $K'$ which fit into a commutative diagram

$$
\begin{array}{c}
\Omega^* (\mathbb{A}^n) \\
\downarrow K \\
\Omega^{* -1} (\mathbb{A}^n)
\end{array}
\begin{array}{c}
\longrightarrow j^* \Omega^* (\Delta^n_{sub}) \\
\downarrow K' \\
\longrightarrow j^* \Omega^{*-1} (\Delta^n_{sub})
\end{array}
$$

Observe that in the construction of the chain homotopies, we use the chain homotopy operator mentioned above. Thus $K'$ is restricted to $(\widetilde{A}_{DR})_n$. In consequence, we have the result. □

Thanks to the extendability and the Poincaré lemma for the simplicial cochain algebra $(\widetilde{A}_{DR})_\bullet$, the same argument as in the proof of [20, Theorem 10.9], which gives quasi-isomorphisms between $C^*(X; \mathbb{Q})$ and the rational de Rham complex $A_{PL}(X)$ for a space $X$, does work well in our setting. In fact in the proof, replacing the simplicial complex $(A_{PL})_\bullet$ of polynomial differential forms with $(\widetilde{A}_{DR})_\bullet$, we have

Proposition 3.4. Let $K$ be a simplicial set. Then one has a sequence of quasi-isomorphisms

$$
C^*(K) \xrightarrow{\cong} C_{PL}^*(K) \xrightarrow{\cong} (C_{PL} \otimes \widetilde{A}_{DR})^*(K) \xleftarrow{\cong} \widetilde{A}_{DR}^*(K),
$$

where $\varphi$ and $\psi$ are defined by $\varphi(\gamma) = \gamma \otimes 1$ and $\psi(\omega) = 1 \otimes \omega$, respectively.

Let $(\Omega_{\text{deRham}}^\bullet)_\bullet$ be the simplicial cochain algebra defined by $(\Omega_{\text{deRham}}^*)_{\text{deRham}} (\mathbb{A}^n)$ in degree $n$. The affine space $\mathbb{A}^n$ is a manifold diffeomorphic to $\mathbb{R}^n$ with the projection $\pi: \mathbb{A}^n \to \mathbb{R}^n$ defined by $\pi(x_0, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. Observe that the sub-diffeology of $\mathbb{A}^n$ described in Section 2 coincides with the diffeology comes from the structure of the manifold $\mathbb{A}^n$ mentioned above. We see that the extendability is satisfied and that the Poincaré lemma holds for $(\Omega_{\text{deRham}}^\bullet)_\bullet$ and hence so does $(\widetilde{A}_{DR})_\bullet$. In fact, these results follow from the same arguments as in the proofs of Lemmas 3.2 and 3.3. Therefore, Proposition 3.4 is also valid after replacing $(\widetilde{A}_{DR})_\bullet$ with the simplicial cochain algebra $(A_{DR})_\bullet$. Thus the result [20, Proposition 10.5] enables us to deduce the following corollary.

Corollary 3.5. For a simplicial set $K$, one has a sequence of quasi-isomorphisms

$$
A_{PL}^*(K) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\cong} \Omega_{\text{deRham}}^*(K) \xrightarrow{\theta} A_{DR}^*(K) \xrightarrow{\cong} (A_{DR}^\bullet) \xrightarrow{(j)^*} (\widetilde{A}_{DR})_\bullet(K),
$$

where $l$ denotes the map induced by the inclusion $(A_{PL})_\bullet \to (\Omega_{\text{deRham}}^\bullet)_\bullet$, and $\theta$ is the isomorphism mentioned after Definition 3.1.
3.2. The map \( \alpha \) and an integration map. In this subsection, for a map \( \tau : [n] \to [m] \) in \( \Delta \), we use the same notation for the affine maps \( \mathbb{A}^n \to \mathbb{A}^m \) and \( \Delta^n \to \Delta^m \) induced by the non-decreasing map as \( \tau \). We recall the map \( \alpha' : \Omega^*(X) \to \text{Sets}^{\Delta^p}((S^\infty(X), A_{DR}) = \overline{A}_{DR}(S^\infty(X)) \) defined by \( \alpha'(\omega)(\sigma) = \sigma^*(\omega) \) for \( \sigma \in S^\infty(X) \). Let \( j : \Delta_{\text{sub}}^n \to \mathbb{A}^l \) be the inclusion. By definition, we see that \( \sigma = \tilde{\sigma} \circ j \) for some smooth map \( \tilde{\sigma} : \mathbb{A}^l \to X \). Then we see that \( \sigma^*(\omega) = j^*(\tilde{\sigma}^*\omega) \) and hence \( \alpha' \) is well defined. Moreover, the standard calculation allows us to conclude that \( \alpha' \) is a morphism of cochain algebras. Observe that \( \alpha' \) is natural with respect to diffeological spaces. In fact, for a morphism \( Y \to X \) in Diffe, we have \((f \circ \alpha'(\omega))(\sigma_Y) = \alpha'(\omega)(f \circ \sigma_Y) = (f \circ \sigma_Y)^*\omega = \sigma_Y^*(f^*\omega) = \alpha'(f^*\omega)(\sigma_Y)\), where \( \omega \in \Omega^*(X) \) and \( \sigma_Y \in S^\infty(Y) \). Thus, the map \( \alpha' \) gives rise to a natural transformation \( \alpha' : \Omega^*(\cdot) \to A_{DR}(X) = (S^\infty(\cdot)) \) and \( \alpha' \). We also define a natural transformation map \( \alpha : \Omega^*(\cdot) \to \overline{A}_{DR}(S^\infty(\cdot)) \) by the same way as that for \( \alpha' \). The natural transformation gives the map \( \alpha \) described in the Section 2.

We here define an integration map \( \int_{\Delta^p} : (A_{DR})_{\bullet} \to (C^\infty_{PL})_{\bullet} = C^\infty(\Delta_{\bullet}) \) by
\[
(\int \gamma)(\sigma) = \int_{\Delta^p} \sigma^* \gamma
\]
for \( \gamma \in (A_{DR})_{\bullet} \), where \( \sigma : \Delta^p \to \Delta^n \) is the affine map induced by \( \sigma : [p] \to [n] \). Since the affine map \( \sigma \) is extended to an affine map from \( \mathbb{A}^p \) to \( \mathbb{A}^n \), it follows that \( \sigma^* \gamma \) is in \( (A_{DR})_{\bullet} \). Stokes’ theorem enables us to conclude that the map \( \int \) is a cochain map. In fact, let \( \sigma \) be an element in \( \Delta[n]_p \) and \( \gamma' \) a form in \( (A_{DR})_{\bullet} \) with \( \sigma^* (\gamma') = j^* \theta(\eta') \) for some \( \eta' \in \Omega^p_{DR}(\mathbb{A}^p). \) Then we have
\[
(\int d\gamma' )(\sigma) = \int_{\Delta^p} \sigma^*(d\gamma') = \int_{\Delta^p} d(\sigma^* \gamma') = \int_{\Delta^p} d(\eta')
\]
\[
= \int_{\Delta^p} \eta' = \sum_i (-1)^i \int_{\Delta^{p-1}} d_i \eta' = \sum_i (-1)^i \int_{\Delta^{p-1}} d_i \sigma^* \gamma'
\]
\[
= \sum_i (-1)^i \int_{\Delta^{p-1}} (\sigma \circ d_i)^* \gamma' = (d(\int \gamma'))(\sigma).
\]

The fourth and fifth equalities follows from Stokes’ theorem for a manifold; see [7, V. Sections 4 and 5] for example. We show that the integration is a morphism of simplicial sets. Let \( \sigma : [p] \to [m] \) and \( \tau : [m] \to [n] \) be a map in \( \Delta \). For a \( \gamma \in (A_{DR})_{\bullet} \), we take a differential form \( \eta \) in \( \Omega^p_{DR}(\mathbb{A}^p) \) with \( (\tau \circ \sigma)^* \gamma = j^* \theta(\eta) \). Then it follows that
\[
\tau^*(\int \gamma)(\sigma) = (\int \gamma)(\tau \circ \sigma) = \int_{\Delta^p} \eta = \int_{\Delta^p} \sigma^* (\tau^* \gamma) = (\int \tau^* \gamma)(\sigma).
\]
The fact that \( \sigma^* (\tau^* \gamma) = j^* \theta(\eta) \) yields the third equality. In consequence, we see that \( \int \) is a morphism of simplicial differential graded modules.
Let 1 be the unit of $\widetilde{A_{DR}}^\bullet$, which is in $\text{Diff}(A, \mathbb{R}) = \Omega_{DR}^1(A^n) = (\widetilde{A_{DR}})^n$. Then it follows that $f \cdot 1 = 1$ in $(C_{PL}^n)^n$ for $n \geq 0$. This yields the commutative diagram

$$
(C_{PL}^\bullet \xrightarrow{\varphi} (C_{PL} \otimes \widetilde{A_{DR}})^\bullet) \xrightarrow{\psi} \widetilde{A_{DR}}^\bullet
$$

The argument above in this subsection does work well for the simplicial cochain algebra $A_{DR}^\bullet$. In consequence, in the diagram above, the commutativity remains valid even if $\widetilde{A_{DR}}^\bullet$ is replaced with the simplicial cochain algebra $A_{DR}^\bullet$; see [20, Remark, page 130] for the same triangles as above for the polynomial de Rham complex $A_P^\bullet$.

4. Proofs

4.1. Proofs of the main theorem and corollaries. We may write $H^D(X)$ for the homology of $ZS^D_\bullet(X)_{sub}$, which is the chain complex with coefficients in $\mathbb{Z}$ induced by the simplicial set $S^D_\bullet(X)_{sub}$.

The homotopy axiom for the homology $H^D(X)$ is now discussed. Let $f$ and $g$ be smooth maps from $X$ to $Y$ which are homotopic smoothly in the sense of Iglesias-Zemmour [33]. Then the homomorphisms $f_*$ and $g_*$ induced on the homology coincide with each other: $f_* = g_* : H_\ast(X) \to H_\ast(Y)$. The construction of the chain homotopy is almost verbatim repetition of the usual one on the singular chain. Observe that the proof uses the fact that $\Delta^n_{sub} \times \mathbb{R} \cong (\Delta^n \times \mathbb{R})_{sub}$ as a diffeological space and the following lemma; see [20, 1.10 Theorem] for example and the sequence (4.1) below.

**Lemma 4.1.** If $X$ is a convex set of $\mathbb{R}^k$ with sub-diffeology, then the $n$th homology $H_n(S^D_\bullet(X))$ is trivial for $n > 0$. The same assertion is valid for the functors $ZS^D_\bullet(-)_{sub}$ and $ZS^\infty_\bullet(-)$.

**Proof.** For a smooth simplicial $\sigma$ in $S^D_n(X)$ and a point $v \in X$, we define a cone $K_v(\sigma)$ by

$$K_v(\sigma)(t_0, \ldots, t_{n+1}) = \begin{cases} \rho(1 - t_0)\sigma(t_0, \ldots, t_{n+1}) + \tau(1 - t_0)v & \text{for } t_0 \neq 1 \\ v & \text{for } t_0 = 1 \end{cases}$$

where $\rho$ is a cut-off function with $\rho(0) = 0$, $\rho(1) = 1$ and $\tau$ is the smooth function defined by $\tau = 1 - \rho$. We see that $K_v(\sigma)$ is in $S^D_{n+1}(X)$. By extending $K_v$ linearly, we have a homomorphism $K_v : ZS^D_n(X) \to ZS^D_{n+1}(X)$. This gives a homotopy between the identity and the zero map; see the proof of [50, 1.8 Lemma]. The same argument as above does work in $S^D_n(X)_{sub}$.

As for the functor $S^\infty_\bullet(-)$, we can define a cone $K_v : ZS^\infty_n(X) \to ZS^\infty_{n+1}(X)$ with an extension $\bar{\sigma} : A^n \to X$ for $\sigma : \Delta^n_{sub} \to X$. This gives a homotopy between the identity and the zero map. \qed

In order to apply the method of acyclic models for proving the main theorem, we need the following result.

**Lemma 4.2.** Let $\mathcal{M}$ be the set of convex sets of $\mathbb{R}^k$ for $k \geq 0$. Then the three functors $ZS^D_n(-)$, $ZS^D_\bullet(-)_{sub}$ and $ZS^\infty_\bullet(-)$ are representable for $\mathcal{M}$ in the sense of Eilenberg–Mac Lane for each $n$. 

Theorems Ia and Ib] with Lemmas 4.1 and 4.2. Then the map category of chain complexes. This follows from the method of acyclic models [17, j inverse induced by (Cj II) The restriction map (I) The method of acyclic models [17, Section 8] implies that there exists a chain are given.

The considerations in I), II), III) and respectively, it follows from the same argument as above that the functors Z^n(D^(-)) and Z^n(S^(-) sub are representable for M. This completes the proof.

We consider the excision axiom for the homology of S^P_0(X) sub. Kihara’s consideration in the proof of [37, Proposition 3.1] enables us to regard the chain complex Z^n(S^P(X) sub as a subcomplex of the singular chain complex C_\Sigma(DX), where D : Diff \to Top denotes the functor mentioned in Appendix B. In fact, [15, Lemma 3.16] implies that D(\Delta^n sub) is the simplex \Delta^n with the standard topology. We observe that for the diffeology \mathbb{R}^n with smooth plots, D(\mathbb{R}^n) is the Euclidean space. Thus for a diffeology X, the unit id : X \to CDX yields the sequence of inclusions (4.1) Diff(\Delta^n sub, X) \to Diff(\Delta^n sub, CDX) \cong Top(D(\Delta^n sub), DX) = Top(\Delta^n, DX).

Observe that S^P_0(X) sub = Diff(\Delta^n sub, X). Then we can prove the excision axiom by applying the barycentric subdivision argument. Indeed, the subdivision map Sd : S^P_0(X) sub \to S^P_0(X) sub is defined by restricting the usual one for the singular chain complex, which is chain homotopic to the identity. It turns out that the relative homology H^P(D(X, A) satisfies the excision axiom for the D-topology; that is, the inclusion i : (X - U, A - U) \to (X, U) induces an isomorphism on the relative homology if the closure of U is contained in the interior of A with respect to the D-topology of X; see [7, IV, Section 17] for example. Thus we also see that the (co)homology of S^P_0(X) sub has the Mayer–Vietoris exact sequence.

More observations concerning cochain complexes in the diagram in Theorem 2.4 are given.

I) The method of acyclic models [17, Section 8] implies that there exists a chain homotopy equivalence \tilde{I} : Z^n(S^P_0(X) sub \cong C_{cube^+}(X). The dual to \tilde{I} yields a cochain homotopy equivalence \tilde{I} : C_{cube^+}(X) \cong C^*(S^P_0(X) sub), which induces a morphism of algebras on the cohomology; see also [43, Theorem 8.2] for example.

II) The restriction map j^* : Z^n(S^P_0(X) \to Z^n(S^P_0(X) sub) has a homotopy inverse k in the category of chain complexes. This follows from the method of acyclic models [17, Theorems 1a and 1b] with Lemmas 4.1 and 4.2. Then the map k^* : C^*(S^P_0(X)) \to C^*(S^P_0(X) sub)) induces an isomorphism of algebras on the homology. In fact, the inverse induced by (j^*)^* : C^*(S^P_0(X) sub) \to C^*(S^P_0(X)) is a morphism of algebras.

III) The homotopy commutativity of the square in Theorem 2.4 also follows from the method of acyclic models for cochain complexes; see Appendix A.

Proof of the first assertion in Theorem 2.4. The considerations in I), II), III) and the commutative diagram (3.2) allow us to deduce the first part.

Proof of Corollary 2.5. By Theorem 2.4, we see that mult \circ (1 \otimes f) is a quasi-isomorphism. The commutativity of the right triangle implies that the integration map is also a quasi-isomorphism.
Proof of Corollary 2.6. The argument in the beginning of Subsection 4.1 gives (i). The assertion (ii) follows from II) above and the fact that the integration map \( f \) induces a morphism of algebras. The first assertion in Theorem 2.4 yields (iii). □

Proof of the latter half of Theorem 2.4. We first observe that the Poincaré lemma for the cohomology of the de Rham complex in the sense of Souriau and the homotopy axiom hold for diffeological spaces; see [33]. By Corollary 2.5, it suffices to show that the composite \( v := f \circ \alpha \) induces an isomorphism on the cohomology.

In case of a smooth CW-complex \( K \), we can use the Mayer–Vietoris exact sequence argument for proving the result; see [30, 27]. In fact, we have a partition of unity of \( CK \) with respect to \( D \)-topology; see Appendix B for the functor \( C : \text{Top} \to \text{Diff} \).

Suppose that \( (X, D^X) \) is a manifold. Then the usual argument as in [7, V. §9] enables us to deduce that the map \( H(v) \) induced by \( v \) on the cohomology is an isomorphism. By definition, a \( p \)-stratifolds \( (S, C) \) is constructed from manifolds with boundaries via an attaching procedure; see Appendix B. In general, a stratifold admits a partition of unity; see [38]. Moreover, we see that an open set of the underlying topological space \( S \) is a \( D \)-open set of the diffeology \( k(S, C) \); see Lemma 6.5. Thus the Mayer–Vietoris sequence argument does work to show that \( H(v) \) is an isomorphism in case of a \( p \)-stratifolds. □

Remark 4.3. Let \( (S, C) \) be a \( p \)-stratifolds. By virtue of Lemma 6.5, we have the map \( i : D(S) \to S \) is in \( \text{Top} \). Consider the map \( i : H_* (S^D(k(S, C))_\bullet) \to H_* (D(S)) \to H_* (S) \) to the singular homology, which is induced by the map \( i \) and the inclusion mentioned in (4.1); see Appendix B for the functor \( k \). If \( S \) is a manifold, then \( i \) is an isomorphism. This follows from the same argument as in [7, 9.5 Lemma]. Then the usual argument with the Mayer–Vietoris sequence enables us to conclude that \( i \) is an isomorphism for every parametrized stratifold; see Remark 6.7 below for the cases of CW-complexes and more general one.

4.2. Applications of the integration map in the main theorem. In this section, we describe applications of the integration map on \( A^i_{DR}(S^D(X)) \) mentioned in Theorem 2.4. Let \( j^* : S^D_0(X) \to S^\infty_0(X) \) and \( j^* : (A^i_{DR})_0 \to (A^i_{DR})_n \) be the restriction maps induced by the inclusion \( j : \Delta^\infty_{\text{ub}} \to \mathbb{H}^n \). The naturality of the integration map \( f \) in the theorem implies that the map \( \alpha' \) described in Section 2 is an extension of \( \alpha \) on the cohomology.

Proposition 4.4. (i) One has the diagram:

\[
\begin{array}{ccc}
\text{Sets}^\Delta(S^\infty_*(X), (A^\infty_{DR})_\bullet) & \xrightarrow{\alpha'} & A^\infty_{DR}(S^\infty_*(X)) \\
\Omega^*(X) & \xrightarrow{\alpha} & \text{Sets}^\Delta(S^\infty_*(X), (A^i_{DR})_\bullet) \\
& \xrightarrow{(j^*)_*} & A^i_{DR}(S^\infty_*(X)) \\
& \xrightarrow{(j^*)_*} & \text{Sets}^\Delta(S^D_*(X), (A^i_{DR})_\bullet) \\
& \xrightarrow{(j^*)_*} & A^i_{DR}(S^D_*(X))
\end{array}
\]

in which \( (j^*)_* \) and \( (j^*)_* \) are quasi-isomorphisms. Moreover, the diagram is commutative on the cohomology.

(ii) The maps \( \alpha \) and \( \alpha' \) are natural with respect to diffeological spaces; that is, they give rise to natural transformations.
Observe that a natural map from $\Omega^*(X)$ to $\mathcal{A}^\bullet_{DR}(\mathcal{S}^\infty_{\Omega}(X))$ cannot be defined in such a way as to give the map $\alpha$. However, Proposition 3.4 and the commutative diagram (3.2) imply that the integration $\int : (\mathcal{A}^\bullet_{DR}) \to (C^\bullet_{PL})$ in Section 3.2 gives rise to a quasi-isomorphism $\int : \mathcal{A}^\bullet_{DR}(K) \to C^\bullet_{PL}(K) \cong C^\bullet(K)$ of differential graded modules for each simplicial set $K$, which is an isomorphism of algebras on the cohomology. This is a key to proving Proposition 4.4.

\textit{Proof of Proposition 4.4.} (i) We consider the diagram

\[
\begin{array}{c}
\Omega^*(X) \cong \int \circ \phi \circ \int \circ \alpha \\
\downarrow (J^+) \circ \int \\
\int \circ (J^+) \circ \phi \circ \int \circ \alpha \\
\end{array}
\]

The method of acyclic models implies that the restriction $(J^+) : \mathcal{Z}^\bullet_{PL}(X) \to \mathcal{Z}^\infty_{\Omega}(X)$ is a quasi-isomorphism and then so is the map $(J^+)^*$ in the right hand side. It follows that the center triangle and square are commutative by the definition of the integration map; see (3.1). The commutative diagram (3.2) implies that the integration maps are quasi-isomorphisms. Then we see that maps $(J^+)^*$ and $(J^+)^*$ in the left hand side are quasi-isomorphisms. Moreover, the direct calculation indicates that $(J^+)^* \circ \int \circ \alpha = \int \circ \alpha$. Therefore, the left triangle is commutative on the cohomology. This completes the proof.

(ii) It is readily seen that $(\varphi_\ast)^* \circ \alpha = \alpha \circ \varphi^*$ and $(\varphi_\ast)^* \circ \alpha' = \alpha' \circ \varphi^*$ for each smooth map $\varphi : X \to Y$. We have the result. \hfill \square

Let $X$ be a Chen space in the sense of [10, Definition 1.2.1]; see also [48, Definition 2.5]. We here define the map $\beta : \Omega^*(X)_{\text{Chen}} \to \Omega^*(\text{So}X)$ in Proposition 2.7 by

$$\beta(\omega) = \{(\rho^* \omega)_\psi \in \text{Charts}(\rho) \}_{\rho \in D^\infty_{\text{So}X}},$$

where $\text{Charts}(\rho)$ denotes the set of appropriate charts of the domain of $\rho$ and $\rho^* : \Omega^*(X)_{\text{Chen}} \to \Omega^*(U)_{\text{Chen}} \cong \Lambda^*_{\text{deRham}}(U)$ is a map defined by $(\rho^* \omega)_\psi = \omega_{\rho \psi}$ for any chart $\psi : C \to U$ of the domain $U$ of $\rho$. Observe that $\{(\rho^* \omega)_\psi \}_\psi \in \text{Charts}(\rho)$ is an equivalent class in $\Omega^*_{\text{deRham}}(U)$; see [5] for example.

\textbf{Lemma 4.5.} \textit{The map $\beta$ is a well-defined morphism of DGA's.}

\textit{Proof.} The differential graded algebra structure of $\Omega^*(C)_{\text{Chen}}$ is defined by that of the usual de Rham complex $\Lambda^*_{\text{deRham}}(C)$, where $C$ is a convex set of $\mathbb{R}^n$ for some $n \geq 0$. Then we see that $\beta$ is a morphism of DGA's if the map is well defined. In order to show the well-definedness, it suffices to prove that for any $\omega \in \Omega^p(X)_{\text{Chen}}$ and each smooth map $u : V \to U$, the diagram

\[
\begin{array}{ccc}
\mathcal{D}^\infty_{\text{So}X}(U) & \xrightarrow{\beta(\omega)} & \Lambda^p(U) \\
\downarrow u^* & & \downarrow u^* \\
\mathcal{D}^\infty_{\text{So}X}(V) & \xrightarrow{\beta(\omega)} & \Lambda^p(V)
\end{array}
\]

is commutative. This comes from the direct calculation. In fact, it follows that $u^*(\beta(\omega)_\rho) = u^*\{\omega_{\rho \psi}\}_\psi = \{(\psi^{-1} u \varphi)^* \omega_{\rho \psi}\}_\varphi$, where $\varphi : C' \to V$ are appropriate...
charts of $V$. On the other hand, we see that $\beta(\omega)_{a^*} = \beta(\omega)_a = \{((pu)^*\omega)_p\}_p = \{\omega_{p^*u}\}_p$. By definition, the $p$-form $\omega \in \Omega^p(X)_{\text{Chen}}$ satisfies the condition that $(\psi^{-1}u\psi)^*\omega_{\psi^{-1}u} = \omega_{puu}$. We have $u^*(\beta(\omega)_p) = \beta(\omega)_{a^*p}$. \hfill $\square$

Proof of Proposition 2.7. Let $C^*_{\text{cube},I}(LM)$ be the cubic cochain complex whose $p$-simplexes are smooth maps to $LM$ from $I^p$ which is regarded as Chen subspace of $\mathbb{R}^p$. We recall the morphism $\Gamma : \Omega^*(M) \otimes B(\Omega^*(M)) \to C^*_{\text{cube},I}(LM)$ of differential graded modules constructed via the appropriate pairing in [12, (2.2)]. Here we may use the normalized bar complex as $\Omega^*(M) \otimes B(\Omega^*(M))$; see [9, (2.1)]. Then we see that the composite $i \circ \Gamma : \Omega^*(M) \otimes \overline{B}(A) \to \Omega^*(M) \otimes B(\Omega^*(M)) \to C^*_{\text{cube},I}(LM)$ is a quasi-isomorphism; see the proof of [12, Theorem 0.1]. Consider the diagram

\[
\begin{array}{ccc}
A^*_D(S^D(\text{So}(LM))) & \xrightarrow{\alpha} & \Omega^*(\text{So}(LM)) \\
\downarrow{\cong} & & \downarrow{\cong} \\
C^*(S^D(\text{So}(LM))) & \xrightarrow{\beta} & C^*_{\text{cube},I}(LM) \\
\end{array}
\]

in which right triangles are commutative. Here $l'$ is a quasi-isomorphism obtained by the method of acyclic models whose models are consisting of the Chen spaces $\mathbb{A}^n$ and $I^n$ for $n \geq 0$. Observe that $\text{So}(\mathbb{A}^n) = \mathbb{A}^n$ by the standard argument and then the identity map $\mathbb{A}^n \to \text{So}(\mathbb{A}^n)$ is in Diff. Thus the method of acyclic models is applicable to $C^*_{\text{cube},I}(\cdot)$ and also $C^*(S^D(\text{So}(\cdot)))$. We have a quasi-isomorphism $l' : C^*_{\text{cube},I}(X) \to C^*(S^D(\text{So}(X)))$ for each Chen space $X$.

The functor $\Omega^*(\cdot) \otimes B(\Omega^*(\cdot))$ is acyclic for models $\mathbb{A}^n$ with $n \geq 0$. Moreover, the functor $C^*(S^D(\cdot))$ is corepresentative; see Appendix A below. In fact, the result follows from the same argument as that after Theorem 6.2. Then Theorem 6.2 implies that the left square is commutative up to homotopy. The argument of a spectral sequence implies that the inclusion $i$ is quasi-isomorphism. Therefore the map $\Gamma$ is a quasi-isomorphism. We see that the composite $\alpha \circ \beta \circ \Gamma$ in the first row is quasi-isomorphism. We observe that the iterated integral map $I$ to the Chen complex is an isomorphism; see [10, Theorem 4.2.1]. This yields the latter half of the assertions. \hfill $\square$

5. CHEN’S ITERATED INTEGRAL MAP IN DIFELOGIE

We recall iterated integrals due to Chen [10] modifying them in the diffeological setting. Let $N$ be a diffeological space and $\rho : \mathbb{R} \to I$ the cut-off function. Then a $p$-form $u$ on the diffeological space $I \times M$ is called a $\Omega^p(N)$-valued function on $I$ if for any plot $\psi : U \to N$ of $N$, the $p$-form $u_{\rho \times \psi}$ on $\mathbb{R} \times U$ is of the type $\sum a_{i_1 \cdots i_p}(t, \xi) d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_p}$, where $(\xi_1, \ldots, \xi_n)$ denotes the coordinates of $U$ we fix. For such an $\Omega^p(N)$-valued function $u$ on $I$, we define the integration $\int_0^1 u \, dt \in \Omega^p(N)$ by

$$\left(\int_0^1 u \, dt\right)_\psi = \sum (\int_0^1 a_{i_1 \cdots i_p}(t, \xi) \, dt) d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_p}.$$ 

Each $p$-form $u$ has the form $u = dt \wedge ((\partial/\partial t)|u) + u''$, where $(\partial/\partial t)|u$ and $u''$ are an $\Omega^{p-1}(N)$-valued function and an $\Omega^p(N)$-valued function on $I$, respectively. Let $F : I \times N^I \to N^I$ be the homotopy defined by $F(t, \gamma)(s) = \gamma(ts)$. The Poincaré operator $\int_F : \Omega(N^I) \to \Omega(N^I)$ associated with the homotopy $F$ is defined
by \( \int_{F_t} v = \int_0^1 \left((\partial/\partial t) [F^*v]\right) dt \). Moreover, for forms \( \omega_1, ..., \omega_r \) on \( N \), the \textit{iterated integral} \( \int \omega_1 \cdots \omega_r \) is defined by \( \int \omega_1 = \int_{F_t} \varepsilon_t^* \omega_1 \) and

\[
\int \omega_1 \cdots \omega_r = \int_F \left\{ J \left( \int \omega_1 \cdots \omega_{r-1} \right) \wedge \varepsilon_r^* \omega_r \right\},
\]

where \( \varepsilon_i \) denotes the evaluation map at \( i \), \( Ju = (-1)^{\deg u} u \) and \( \int \omega_1 \cdots \omega_r = 1 \) if \( r = 0 \); see [10, Definition 1.5.1]. Observe that the operator is of degree \( -1 \) and then \( \int \omega_1 \cdots \omega_r \) is of degree \( \sum_{1 \leq i \leq r} (\deg \omega_i - 1) \).

With a decomposition of the form \( A^1 \oplus d\Omega^0(N) \), we obtain the DG subalgebra \( A \) of \( \Omega(N) \) which satisfies the condition that \( A^p = \Omega(N) \) for \( p > 1 \) and \( A^0 = \mathbb{R} \). The DGA \( A \) gives rise to the normalized bar complex \( B(\Omega(N), A, \Omega(N)) \); see [10, §4.1]. Consider the pullback diagram

\[ (5.1) \]

\[
\begin{array}{ccc}
E_f & \longrightarrow & N^I \\
\downarrow p_f & & \downarrow (\varepsilon_0, \varepsilon_1) \\
M & \longrightarrow & N \times N
\end{array}
\]

of \( (\varepsilon_0, \varepsilon_1) : N^I \rightarrow N \times N \) along a smooth map \( f : M \rightarrow N \times N \). In what follows, we assume that the cohomology \( H^*(A_{DR}(S^*_D(N))) \) is of finite type. We write \( \overline{B}(A) \) for \( B(\mathbb{R}, A, \mathbb{R}) \). Then we have a map

\[ \text{It} : \Omega(M) \otimes_{\Omega(N) \otimes \Omega(N)} B(\Omega(N), A, \Omega(N)) \cong \Omega(N) \otimes_f \overline{B}(A) \rightarrow \Omega(E_f) \]

defined by \( \text{It}(v \otimes [\omega_1] \cdots [\omega_r]) = p_f^* v \wedge \tilde{f}^* \int \omega_1 \cdots \omega_r \). Observe that the source of \( \text{It} \) gives rise to the differential on \( \Omega(M) \otimes_f \overline{B}(A) \). Since \( \rho(0) = 0 \) and \( \rho(1) = 1 \), for the cut-off function \( \rho \), it follows that the result [10, Lemma 1.4.1] remains valid. Then the formula of iterated integrals with respect to the differential in [10, Proposition 1.5.2] implies that It is a well-defined morphism of differential graded \( \Omega(N) \)-modules.

\textbf{Remark 5.1.} The cut-off function \( \rho \) does not satisfy the formula \( \rho(s)\rho(t) = \rho(st) \) for \( s, t \in \mathbb{R} \) in general. Then we do not have the same assertion as that of [10, Lemma 1.5.1] which allows us to deduce a constructive definition of the iterated integral as in (2.1); see [10, page 840].

\textbf{Theorem 5.2.} Suppose that in the pullback diagram (5.1), the diffeological space \( N \) is simply connected and \( f \) is an induction; that is, the map \( f : M \rightarrow f(M) \) is a diffeomorphism, where \( f(M) \) is a diffeological space endowed with subdiffeology. Assume further that the cohomology \( H^*(S^*_D(M)) \) is of finite type. Then the composite \( \alpha \circ \text{It} : \Omega^* (M) \otimes_f \overline{B}(A) \rightarrow \Omega(E_f) \rightarrow A^*_C (S^*_D(E_f)) \) is a quasi-isomorphism of \( \Omega^* (M) \)-modules.

\textbf{Proof of Theorem 2.8.} The diagonal map \( \Delta : M \rightarrow M \times M \) is an induction. Then the result follows from Theorem 5.2.

\textbf{Remark 5.3.} Theorem 5.2 is regarded as a generalization of the result [12, Theorem 0.1] in which \( M \) and \( N \) are assumed to be manifolds, however \( f \) is a more general smooth map. The theorem due to Chen asserts that the homology of the bar complex \( \Omega^* (M) \otimes_f \overline{B}(A) \) is isomorphic to the cubical cohomology of the Chen space \( E_f \) via the pairing with the iterated integrals and cubic smooth chains; see [12, (2.2)].
The rest of this section is devoted to proving Theorem 5.2. We here recall a local system over a simplicial set and its global sections. Let \( K \) be a simplicial set. We regard \( K \) as a category whose objects are simplicial maps \( \sigma : \Delta[p] \to K \) for \( p \geq 0 \) and whose morphisms \( \alpha : \tau \to \sigma \) are simplicial maps \( \alpha : \Delta[q] \to \Delta[p] \) with \( \tau = \sigma \circ \alpha \), where \( \tau : \Delta[q] \to K \) and \( \sigma : \Delta[p] \to K \). Then a local system \( F \over K \) of differential coefficients is defined to be a contravariant functor from \( K \) to the category \( \text{DGAs} \) of unital differential graded algebras with non-negative grading which satisfies the condition that the map \( F(\alpha) : F_{\sigma} \to F_{\alpha \circ \sigma} \) is a quasi-isomorphism for each \( \alpha \) in the category \( K \); see [26, Definition 12.15]. Observe that such a local system \( F \) is an object of the functor category \( \mathcal{E} := \text{DGAs}^{K^{op}} \). We define the space \( \Gamma(F) \) of global sections of \( F \) by \( \Gamma(F) := \text{Hom}_\mathcal{E}(\mathbb{R}, F) \), where \( \mathbb{R} \) denotes the DGA concentrated in degree zero with trivial differential.

There are at least two kinds of fibrations in the category \( \text{Diff} \). One of them is the fibration \( f : X \to Y \) in the sense of Christensen and Wu [14], namely a smooth map \( f : X \to Y \) which induces a fibration \( S^D(f) : S^D(X) \to S^D(Y) \) of simplicial sets in \( \text{Sets}^{\Delta^{op}} \). An important example of such a fibration is a diffeological bundle in the sense of Iglesias-Zemmour [33, Chapter 8] whose fibre is fibrant; see [14, Proposition 4.24].

Another type concerns mapping spaces with evaluation maps. For example, with the interval \( J = [0, 1] \), the map \( (\varepsilon_0, \varepsilon_1) : N^J \to N \times N \) defined by the evaluation map \( \varepsilon_i \) at \( i \) is a fibration in \( \text{Top} \) if \( N^J \) is endowed with compact open topology; that is, the map \( (\varepsilon_0, \varepsilon_1) \) enjoys the right lifting property with respect to the inclusion \( \Delta^n \to \Delta^n \times \{0\} \to \Delta^n \times I \) for \( n \geq 0 \). However, it seems that a smooth lifting problem is not solved in general for such an evaluation map in \( \text{Diff} \). Then in order to prove Theorem 2.8, it is needed to reformulate the Leray–Serre spectral sequences and algebraic models for path spaces in the diffeological framework.

In what follows, we may write \( A^*_{\text{DR}}(X) \) and \( A^*(X) \) for \( A^*_{\text{DR}}(S^D_{\text{op}}(X)_{\text{sub}}) \) and \( A^*_{\text{DR}}(S^D_{\text{op}}(X)) \), respectively. Observe that the natural map \( (j^*)^* : S^D_{\text{op}}(X) \to S^D_{\text{op}}(X)_{\text{sub}} \) induces a quasi-isomorphism from the inclusion \( j : \Delta^n_{\text{sub}} \to \Delta^n \).

This follows from II) in Section 4.1, Proposition 3.4 and the naturality of the integration map; see the proof of Proposition 4.4. The argument in [24, Sections 5, 6 and 7] due to Grivel enables us to obtain the Leray–Serre spectral sequence with a local system for a fibration and the Eilenberg–Moore spectral sequence for a fibre square.

Theorem 5.4. Let \( \pi : E \to M \) be a smooth map between path-connected diffeological spaces with path-connected fibre \( F \) which is i) a fibration in the sense of Christensen and Wu or ii) the pullback of the evaluation map \( (\varepsilon_0, \varepsilon_1) : N^J \to N \times N \) for a connected diffeological space \( N \) along an induction \( f : M \to N \times N \). Suppose further that in the case ii) the cohomology \( H(A^*(M)) \) is of finite type. Then one has the Leray–Serre spectral sequence \( \{LS^E_{\pi^*}, d_r\} \) converging to \( H(A^*(E)) \) as an algebra with an isomorphism

\[
LS^E_{\pi^*} \cong H^*(M, H^*(F))
\]

of bigraded algebras, where \( H^*(M, H(F)) \) is the cohomology with the local coefficients \( H^*(F) = \{H(A^*(F))\}_{(c) \in S^D_{\text{op}}(M)} \); see Lemma 5.10 below.

Theorem 5.5. Let \( \pi : E \to M \) be the smooth map as in Theorem 5.4 with the same assumption, \( \varphi : X \to M \) a smooth map from a connected diffeological space \( X \) for
which the cohomology \( H(A^*(X)) \) is of finite type and \( E_\varphi \) the pullback of \( \varphi \) along \( \varphi \).
Suppose further that \( M \) is simply connected in case of i) and \( N \) is simply connected in case of ii). Then one has the Eilenberg–Moore spectral sequence \( \{ EM E_n^{*,*}, d_r \} \) converging to \( H(A^*(E_\varphi)) \) as an algebra with an isomorphism

\[
EM E_2^{*,*} \cong \text{Tor}_{H(A^*(M))}^r(H(A^*(X)), H(A^*(E)))
\]
as a bigraded algebra.

**Proofs of Theorems 5.4 and 5.5.** For the case i), the Leray–Serre spectral sequence and the Eilenberg–Moore spectral sequence are obtained by applying the same argument as in the proofs of [24, 5.1 Theorem and 7.3 Theorem] to the functor \( A^*(\ ) := A_{DR}^p(S^D(\ )) \).

We consider the case ii). By replacing \( A^*(\ ) \) with \( A_{DR}^p(\ ) \), Theorem 5.4 follows from the argument of the proof of Proposition 5.11 below. By virtue of the result [26, 20.6] and Proposition 5.11, we have \( H^*(A_{DR}(E_\varphi)) \cong \text{Tor}_{A_{DR}(M)}^r(A_{DR}(X), A_{DR}(E)) \) as an algebra and then Theorem 5.5 follows; see [51, Théorème 4.1.1]. In consequence, the natural quasi-isomorphism \( \lambda(j^r)^* \) in (5.2) yields the results. □

**Remark 5.6.** In Theorems 5.4 and 5.5, we have dealt with fibrations of the type i) and of the type ii). We do not know whether the second fibration is indeed the first one. Therefore, we have considered the two cases separately.

One might regard that an appropriate concept of a smooth relative CW-complex \( i : A \to X \) enables us to obtain a map \( i^* : \text{map}(X,Y) \to \text{map}(A,Y) \) with the homotopy extension property in \( \text{Diff} \) for a diffeological space \( Y \), where \( \text{map}(X,Y) \) and \( \text{map}(A,Y) \) are endowed with functional diffeology. As consequence, we expect the spectral sequences as in Theorems 5.4 and 5.5 for the map \( i^* \). We do not pursue this topic in this manuscript.

The argument in [26, Chapter 19] will be replaced with that in our setting. Recall the standard face and degeneracy maps \( \eta_i : \Delta_{sub}^{p-1} \to \Delta_{sub}^p \) and \( \zeta_j : \Delta_{sub}^{p+1} \to \Delta_{sub}^p \). For \( 0 \leq m \leq p \), let \( \alpha_m : \Delta_{sub}^{p+1} \to \Delta_{sub}^p \times I \) be a smooth map defined by \( \alpha_m(x) = (\zeta_m(x), \sum_{i=m+1}^p x_i) \), where \( x = (x_0, x_1, \ldots, x_{p+1}) \). Observe that the maps \( \alpha_m \) give the standard triangulation of \( \Delta^p \times I \) in the category of topological spaces. For any nondecreasing map \( u : [p] \to [n] \), we denote by the same notation the affine map \( \Delta_{sub}^n : \Delta_{sub}^m \to \Delta_{sub}^n \) defined by \( u(a) \). Such an affine map \( u \) gives a set map \( \text{map}(\Delta_{sub}^{p+1}, \Delta_{sub}^p) \to \text{map}(\Delta_{sub}^{p+1}, \Delta_{sub}^n) \) defined by \( \alpha_m \). It is readily seen that the composite \( \alpha_m \) is a smooth map for each \( m \).

Let \( N \) be a diffeological space and \( f : M \to N \times N \) an induction. Then the pullback \( \nu' : E_f \to M \) of the map \( (\varepsilon_0, \varepsilon_1) : N^I \to N \times N \) along \( i \) is identified with the map

\[
\nu := f^{-1} \circ (\varepsilon_0, \varepsilon_1) : P_M N := \{ \gamma \in N^I \mid (\varepsilon_0, \varepsilon_1)(\gamma) \in f(M) \} \to M.
\]

We consider the homotopy pullback \( \pi : P^h_\sigma \to \Delta_{sub}^n \) of \( \nu \) along the \( n \)-simplex \( \sigma \). By definition, it is given by

\[
P^h_\sigma := \{ (a, \zeta, \gamma) \in \Delta_{sub}^{n+1} \times M^I \times P_M N \mid \sigma(a) = \zeta(0), \nu(\gamma) = \zeta(1) \}
\]

with \( \pi \) the projection. Let \( P^h_\sigma \) be the sub-simplicial set of \( S^p_\sigma(P^h_\sigma) \) consisting of \( p \)-simplexes each of which satisfies the condition that \( \pi \circ \sigma : \Delta^p \to \Delta^n \) is the affine map defined by a nondecreasing map \( [p] \to [n] \). By the same way, we have a sub-simplicial set \( P^D_\sigma \) of \( S^p_\sigma(P^D_\sigma) \), where \( P^D_\sigma \) denotes the pullback of \( \nu \) along
the n-simplex \( \sigma : \Delta^n_{\text{sub}} \to M \). The restriction map of \( \sigma \) to the vertex \( \sigma(n) \) gives rise to the pullback \( P_{\sigma(n)} \) of \( \nu : P_MN \to M \). Then we have the natural inclusion \( j : P_{\sigma(n)} \to P_{\sigma}^b \) defined by \( j(\gamma) = (\sigma(n), C_{\sigma(n)}, \gamma) \), where \( C_{\sigma(n)} \) is the constant map at \( \sigma(n) \). The following lemma is proved modifying the argument of the proof of [26, Lemma 19.9] in the diffeological framework.

**Lemma 5.7.** The homomorphism \( A^*_\text{DR}(P_{\sigma(n)}) \xrightarrow{A^*(j)} A^*_\text{DR}(P_{\sigma}) \) induced by the natural map \( j : S^\bullet(P_{\sigma(n)})_{\text{sub}} \to P_{\sigma}^b \) of simplicial sets is a quasi-isomorphism.

**Proof.** By Theorem 2.4, it suffices to show that the map \( j_* : C_\text{c}((S^\bullet(P_{\sigma(n)}))) \to C_\text{c}(P_{\sigma}^b) \) of chain complexes induced by \( j \) is chain homotopy equivalent. We identify \( M \) with the subspace \( \{ i \} \) of \( N \times N \). For each \( \tau \in (P^b_\sigma)_p \), we have the map \( \pi \circ \tau : \Delta^p \times I \to \Delta^n \) mentioned above. Let \( J \) denote the space \( (I \times \{0, 1\}) \cup \{\{0\} \times I\} \) endowed with the subdiffeology of \( I \times I \). For each point \( z \in I \times I \), join \((2, \frac{1}{2})\) to \( z \) by a straight line, and make the line beyond \( z \) until it meets \( J \) at a point \( z' \). Then one defines a retraction \( r : I \times I \to J \) by \( r(z) = z' \) as a set map. Moreover, by using the projections \( p_i \) from \( P_\sigma^b \) in the ith factor and the adjoint \( ad \) of a map to \( M^I \), we define a map \( \tilde{r} : \Delta^p \times I \to M \) by

\[
\tilde{r} := ((\sigma \circ \pi \circ \tau)_0 \circ (1 \times \rho)) \cup ((\sigma \circ \pi \circ \tau)_1 \circ (1 \times \rho)) \cup (\omega^{-1} \ast \ell \ast \omega)(1 \times r),
\]

where \( \omega(z, 0, s) = \text{ad}(\text{pr}_2\tau)(z, \rho(s)), (\omega^{-1}(z, 0, s) = \text{ad}(\text{pr}_2\tau)(z, 1 - \rho(s)), \ell(z, 0, s) = \text{ad}(\text{pr}_3\tau)(z, \rho(s)) \) and \( \rho \) is a cut-off function. Moreover, \( (\sigma \circ \pi \circ \tau)_i \) denotes the map defined by \( (\sigma \circ \pi \circ \tau)_0 \) on \( \Delta^p \times \{0\} \) for \( i = 0, 1 \). We observe that \( \tilde{r} \) is smooth on \( (\text{Im } \alpha_n) \times I \). In fact, the map is constant in appropriate neighborhoods of the rays from \((2, \frac{1}{2})\) to the points \((0, 0), (0, 1), (0, \frac{1}{2}) \) and \((0, \frac{3}{2}) \) in \( I \times I \). For any plot \( p : U \to \Delta^p \times I \times I \), we talk a point \( r \in U \) and write \( p(r) = (x, (a, b)) \). Then there exists an open neighborhood \( U \) of \((a, b)\) such that \( \tilde{r} \circ p|_A \) is constant or a composite of \((1 \times r) \) and \( (\sigma \circ \pi \circ \tau)_i \) or \( (\omega^{-1} \ast \ell \ast \omega) \), where \( A = p^{-1}(\Delta^p \times U) \). This implies that \( \tilde{r} \circ p|_A \) is locally smooth. It follows that \( \tilde{r} \circ p \) is in \( D^{M^I} \) the diffeology of \( M \) and then \( \tilde{r} \) is smooth.

We define \( \overline{\gamma} : \Delta^p \times I \to \text{Map}(I, M) \) by the adjoint to \( r \). The homotopy \( H_r : (\Delta^p \times I) \times I \to M \) from \( \sigma \circ (\pi \circ \tau) \) to \( \nu \circ \overline{\gamma} \) defined by \( H_r((z, t), s) = (\sigma \circ \pi \circ \tau)(z, (1 - s)t + sp(t)) \) gives a map \( \overline{\gamma} : \Delta^p \times I \to P_\sigma^b \) with

\[
\overline{\gamma}(z, t) = ((\sigma \circ \pi \circ \tau)(z, t), \text{ad}(H_r)(z, t), \overline{\gamma}(z, t)).
\]

Observe that the domain of the map will be restricted to the space \( (\text{Im } \alpha_n) \) when constructing a simplicial homotopy below. We call \( \overline{\gamma} \) the canonical lift with respect to \( r \).

\[
(5.3)
\]

Since \( \pi \circ \tau(z, 1) \) is the constant map at \( v_n \), it follows that \( \overline{\gamma}(-1, 1) \) factors through \( P_{v_n} \). Then we define a simplicial map \( \lambda : P_{v_n}^b \to S^\bullet(P_{\sigma(n)})_{\text{sub}} \) by \( \lambda(\tau) = \overline{\gamma}(-, 1) \).
In order to show that \( j \circ \lambda \) is homotopic to the identity, we define \( h_m : (P^h_\sigma)_{P^h_\sigma} \to (P^h_\sigma)_{P^h_\sigma} \) by \( h_m(\tau) = \overline{\pi}_2 \circ \alpha_m \) for any \( 0 \leq m \leq p \) by using the canonical lift. Since \( \pi(\tau) \cdot \alpha_m(z) \), it follows that \( h_m \) is well defined. Then we see that \( \{h_m\}_{0 \leq m \leq p} \) gives rise to a simplicial homotopy; that is, the maps \( h_m \) satisfy the following equalities

\[
d_j h_m = \begin{cases} 
  f & \text{if } j = m = 0, \\
  h_{m-1}d_j & \text{if } j < m, \\
  d_j h_{m+1} & \text{if } 0 \leq j - 1 = m < p, \\
  h_m d_j & \text{if } 0 \leq m < j - 1 \leq p, \\
  g & \text{if } j - 1 = m = p,
\end{cases}
\]  

(5.4)

where \( f(\tau) = \overline{\pi}(\tau, 1) \) and \( g(\tau) = \overline{\pi}(\tau, 0) \); see [2] for example. The construction of the canonical lift yields that \( d_j h_m(\tau) = \overline{\pi} \circ \alpha_m \circ \eta_j \) and \( h_m d_j(\tau) = \overline{\pi} \circ (\eta_j \times 1) \circ \alpha_m \). Thus each equality mentioned above follows from the the relations between \( \alpha_m \) and \( \eta_m \).

It remains to verify that \( \overline{\pi}(\tau, 0) \) is homotopic to the identity. Let \( \tau \) be an element in \((P^h_\sigma)_{P^h_\sigma}\). We write \( \tau(z) = (\pi \circ \tau(z), \ell(z)) \) and \( \overline{\pi}(z, 0, C_y, \ell(z)) \), where \( z \in \Delta^n \) and \( C_y \) denotes the constant path at \( y = \overline{\pi}(\tau(z), 0) \). Observe that \( \pi \pi \pi(z, 0) = (\pi \circ \tau(z)) \) and \( \ell' \) is nothing but the path \( (\omega^{-1}) \ast \ell \ast \omega \) in (5.2). Thus by using homotopies from \( C_y \) to \( \omega(z) \), from \( (\omega^{-1}) \ast \ell \ast \omega(z,-) \) to \( \ell(z) \) and form the cut-off function \( \rho \) to the identity, we can construct a homotopy \( H_{\tau} : \Delta^n \times I \to P^h_\sigma \)
from \( \overline{\pi}(\tau, 0) \) to \( \tau \) with \( H_{d_j \tau} = H_{\tau} \circ (\eta_j \times 1) \). Define an element \( h_m(\tau) \) in \((P^h_\sigma)_{P^h_\sigma}\) by \( h_m(\tau) = H_{\tau} \circ \alpha_m \). By the direct calculation, we see that \( h_m \) satisfies the equalities in (5.4), where \( f(\tau) = \tau \) and \( g(\tau) = \overline{\pi}(\tau, 0) \).

As for the composite \( \lambda \circ j : S^D_\sigma(P_{\sigma(n)}) \to P^h_\sigma \to S^D_\sigma(P_{\sigma(n)}) \), it is also homotopic to the identity. The same homotopy as \( H_{\tau} \) mentioned above gives such a chain homotopy. This completes the proof.

Remark 5.8. We define a map \( \varphi : M^l \to \text{stPath}_c(M) \) by composing the cut-off function \( \rho \), where the target denotes the stationary path space. Then the map \( \varphi \) is smooth. This follows from the smoothness of the evaluation map. Moreover, it follows from the locality of plots that the concatenation \( \text{stPath}_c(M) \times \text{stPath}_c(M) \to \text{stPath}_c(M) \) is also smooth. By using these facts, we have proved Lemma 5.7.

Lemma 5.9. The map \( S^D_\sigma : P_\sigma \to P^h_\sigma \) defined by the inclusion \( : P_\sigma \to P^h_\sigma \) induces a quasi-isomorphism \( \iota^* : A^*_{DR}(P^h_\sigma) \to A^*_{DR}(P_\sigma) \).

Proof. The inclusion \( \iota \) is given by \( i(a, \gamma) = (a, C_{(a)}, \gamma) \). We define a map \( \mu : P^h_\sigma \to P_\sigma \) by \( \mu(a, \omega, \gamma) = (a, \omega^{-1} \gamma \omega) \), where \( \omega^{-1} \gamma \omega = (\omega^{-1} \circ \rho) \ast (\gamma \circ \rho) \ast (\omega \circ \rho) \). By adjusting the parameters of paths \( \omega \) and \( \gamma \), we can construct smooth homotopies \( H : P^h_\sigma \times I \to P^h_\sigma \) from 1 to \( \iota \circ \mu \) and \( G : P_\sigma \times I \to P_\sigma \) from 1 to \( \mu \circ \iota \) which preserve the first factor. For an \( n \)-simplex \( \tau : \Delta^n \to P^h_\sigma \), define \( h_m(\tau) \) by the composite \( H \circ (\tau \times 1) \circ \alpha_m : \Delta^n \times I \to P^h_\sigma \). Then the same argument as in the proof of Lemma 5.7 yields that the family \( \{h_m\}_{0 \leq m \leq n} \) \( n \geq 0 \) gives a simplicial homotopy on \( S^D_\sigma(P^h_\sigma) \). By using the homotopy \( G \), we have a simplicial homotopy on \( S^D_\sigma(P_\sigma) \). This completes the proof.
Lemmas 5.7 and 5.9 imply that the map $\text{top}_n : [0] \to [n]$ defined by $\text{top}_n(0) = (n)$ induces a quasi-isomorphism $\text{top}_n^* : A^*_\text{DR}(P_\sigma) \to A^*_\text{DR}(P_{\sigma(n)})$. A map $\alpha(\eta) : \Delta^m \to \Delta^m_{\text{sub}}$ defined by a nondecreasing map $\eta : [m] \to [n]$

\[
P_{\alpha(\eta)}^* \xi_{\alpha(\eta)} \xrightarrow{\pi_{\alpha(\eta)}^* \sigma} P_\sigma \xrightarrow{\pi_\sigma} E_f \xrightarrow{\nu} \Delta^m_{\text{sub}} \xrightarrow{\alpha(\eta)} \Delta^m_{\text{sub}} \sigma \to M.
\]

Moreover, the smooth map $\xi_{\alpha(\eta)}$ gives rise to a simplicial map $\xi_{\alpha(\eta)} : P_\sigma^* \to P_\sigma$. By using the quasi-isomorphisms $\text{top}_n^*$ mentioned above, we have

**Lemma 5.10.** The family $F := \{A^*_\text{DR}(P_\sigma)\}_{\sigma \in K}$ of DGA’s gives an extendable local system over $K$ of differential coefficients.

**Proof.** For a nondecreasing map $\eta : [m] \to [n]$, we define a map $\tau : [n] \to [n + 1]$ which sends $\eta(m), \eta(m) + 1, \ldots, n$ to $n + 1$. Since $\tau \circ \text{top}_n = \text{top}_{n+1}$ and $\tau \circ \text{top}_{m+1} = \text{top}_{m+1}$, it follows that $\xi_{\alpha(\eta)}$ induces a quasi-isomorphism $\xi_{\alpha(\eta)}^* : A^*_\text{DR}(P_\sigma) \to A^*_\text{DR}(P_{\tau(\eta)})$. The extendability of the local system follows from Lemma 3.2; see the proof of [26, 19.17 Lemma].

Let $j : R := A^*_\text{DR}(M) \otimes \wedge V \to A^*_\text{DR}(E_f)$ be a KS extension for the map $\nu^* : A^*_\text{DR}(M) \to A^*_\text{DR}(E_f)$ induced by the projection $\nu : E_f \to M$. Let $F_m$ denote the fibre over a point $m \in M$. Since the composite $\nu$ and the inclusion $l : F_m \to E_f$ is the constant map at $m$, it follows that the map $l^* \circ \pi^*$ factors through the augmentation $\varepsilon : A^*_\text{DR}(M) \to A^*_\text{DR}([m]) = \mathbb{R}$ and then $j$ induces a map $k : \wedge V = A^*_\text{DR}([m]) \otimes A^*_\text{DR}(M) R \to A^*_\text{DR}(P_m)$ of DGA’s.

**Proposition 5.11.** Suppose that $N$ is simply connected. Then the morphism $k : \wedge V \to A^*_\text{DR}(P_m)$ of DGA’s is a quasi-isomorphism.

This result follows from [26, 20.3 Theorem]. We here prove Proposition 5.11 by constructing the Leary-Serre spectral sequence and by applying the comparison theorem of spectral sequences.

To this end, we first recall an isomorphism $a : A^*_\text{DR}(E_f) \to \Gamma(F)$ of DGA’s in [26, 19.21 Lemma] defined by $(\alpha \psi)_{\sigma} = a_{\sigma} \psi$, where $a_{\sigma}$ denotes the composite $A^*_\text{DR}(E_f) \xrightarrow{\xi^*} A^*_\text{DR}(P_{\sigma}) \to A^*_\text{DR}(P_\sigma)$.

For the map $P_m \to \{m\}$, Lemma 5.10 enables us to obtain a local system over $L := \{S^D([m])\}_{\sigma \in L}$ of the form $F^* := \{A^*_\text{DR}(P_m)_{\tau}\}_{\tau \in L}$. Observe that the inclusion $i : P_m \to E_f$ induces a morphism $i^* : F \to F^*$ of local systems. Moreover, we have an isomorphism $a : A^*_\text{DR}(P_m) \to \Gamma(F)$ by applying [26, 19.21 Lemma]. Recall the quasi-isomorphism $i_F : \Gamma(F) \to \Gamma((A^*_\text{DR})_n \otimes F)$ which is defined by the inclusion $F_\sigma \to 1 \otimes F_\sigma \subset A^*_\text{DR}((\mathcal{A}^n) \otimes F_\sigma$ for $\sigma \in K_n$; see [26, 13.12 Theorem]. Moreover, the map $\xi_F : \Gamma(A \otimes F) \to \Gamma(F)$ is defined by

$$((\xi_F(a \otimes \Phi))_{\sigma} = a_{\sigma} \cdot \Phi_{\sigma},$$

where $\cdot$ denotes the multiplication on $(A^*_\text{DR})_\bullet$. It is readily seen that $\xi_F$ is a left inverse of $i_F$ and then it is a surjective quasi-isomorphism. We observe that $\xi_F$ is
a morphism of $A_{DR}(M)$-algebras. These maps gives a commutative diagram

\[
\begin{array}{ccc}
A_{DR}(\{m\}) & \cong & R \\
\downarrow & & \downarrow \\
\land V & \xrightarrow{k} & A_{DR}(P_m) \\
\downarrow & & \downarrow \\
A_{DR}(M) \otimes \land V & \xrightarrow{j} & A_{DR}(E_j) \\
\downarrow & & \downarrow \\
A_{DR}(M) & \xrightarrow{i} & A_{DR}(F).
\end{array}
\]

containing the KS extension, where the maps $i_{F'}$ and $\xi_{F'}$ are defined by the same way as $i_F$ and $\xi_F$, respectively. Lifting lemma gives rise to a morphism $\Psi : A_{DR}(M) \otimes \land V \to \Gamma((A_{DR})_\bullet \otimes F)$ of $A_{DR}(M)$-algebras with $\xi_F \circ \Psi = j \circ a$. More precisely, we define $\Psi$ by $\Psi(v) = i_F \circ j \circ a(v)$ for $v \in V$. The commutativity of three squares enables us to deduce that

(5.5) $\quad \quad \quad \quad \quad \quad \quad a \circ k \circ q|_{1 \otimes \land V} = \xi_{F'} \circ \Gamma(1 \otimes i^*) \circ \Psi|_{1 \otimes \land V}.$

Define filtrations $G = \{G^p\}_{p \geq 0}$ of $A_{DR}(M) \otimes \land V$ and $'G = \{'G^p\}_{p \geq 0}$ by $G^p = \sum_{i \geq p} A^i_{DR}(M) \otimes \land V$ and $'G^p = \Gamma(\sum_{i \geq p} (A^i_{DR})_\bullet \otimes F)$, respectively. Since the morphism $\Psi$ of DGA’s over $A_{DR}(M)$ preserves the filtrations, it follows that the map induces a morphism $\{f_r\}_{r \geq 0} : \{E^*_r, d_r\} \to \{'E^*_r, 'd_r\}$ of spectral sequences constructed from the filtrations mentioned above; see [26, (12.43)]. We recall the integration map defined in (3.1). The integration induces a quasi-isomorphism

\[
\int : 'E_1 = \Gamma((A^*_{DR})_\bullet \otimes H(F)) \to C^*(M; H(F)),
\]

where $C^*(M; H(F))$ denotes the cochain complex of $S^D(M)_{sub}$ with the local coefficients induced by the local system $F$. This follows form the same argument as in [26, 14.13].

Remark 5.12. The above argument yields that in the Leray–Serre spectral sequence in Theorem 5.4, one has an isomorphism $\text{LS} E_2 \cong H^*(S^D(M)_{sub}) \otimes H^*(A_{DR}(F_m))$ as an algebra if $M$ is simply connected and the cohomology $H^*(S^D(M)_{sub})$ is of finite type.

Proof of Proposition 5.11. Since $N$ is simply connected, it follows that the local system $F$ on $S^D(M)_{sub}$ is simple. We observe that the action of $\pi_1(M)$ on $F$ is induced by that of $\pi_1(N)$. Then we see that $f_2$ is a morphism of free $H^*(M)$-modules. Therefore if $f^{0,q}_2$ is an isomorphism, then so is $f^{0,q}_2$. It follows form the comparison theorem ([26, 17.17 Theorem]) that $f^{0,q}_2$ is an isomorphism for any $q \geq 0$. The formula (5.5) implies that the isomorphism $f^{0,q}_2$ is nothing but the map

\[
H(\int \circ \xi_{F'} \circ a \circ k) : H(\land V) \to H(F') = H(F_m).
\]

This completes the proof. ☐

We are ready to prove the main theorem in this section.
Proof of Theorem 5.2. For a diffeological space $X$, we recall the quasi-isomorphism $(j^*)^*: A^*_{DR}(X) \to A^*_{DR}(S^\infty(X)) =: A(X)$ in (5.2). Let $\Omega M \to PM \to M$ be the pullback of the evaluation map $(\varepsilon_0, e_1): M^I \to M \times M$ along the induction $s: M \to M \times M$ defined by $s(x) = (x, x)$, where $*$ denotes the base point of $M$. We have a commutative diagram of solid arrows

\[
\begin{array}{ccc}
A^*_{DR}(PM) & \xrightarrow{\cong} & A^*_{DR}(PM) \\
\downarrow \cong & & \downarrow \cong \\
A^*_{DR}(M) & \xrightarrow{\cong} & A^*_{DR}(M) \\
\end{array}
\]

in which $j$ and $j'$ are KS extension of $\pi^*$ and $A(\pi)$, respectively. Here $A$ denotes the DG subalgebra of $\Omega^*(M)$ described in the paragraph before (5.1).

We may assume that the quasi-isomorphism $p$ is a surjection by the surjective trick; see [20, Section 12 (b)]. By applying Lifting lemma, we have a morphism $\beta: R \to R'$ which makes the two squares commutative. Then we have a morphism $\beta: T := \mathbb{R} \otimes_{A_{DR}(M)} R \to T' := \mathbb{R} \otimes_{A(M)} R'$ of DGA's. Moreover, the the map $\beta$ is a quasi-isomorphism and hence so is $\beta'$ by [20, Theorem 6.10]. Proposition 5.11 implies that $k$ is a quasi-isomorphism and then so is $k'$. Since the bar complex $\Omega^*(M) \otimes \overline{B}(A)$ is a semifree $\Omega(M)$-module, it follows from Lifting lemma that there exist a morphism $\tilde{\alpha}: \Omega^*(M) \otimes \overline{B}(A) \to R'$ of $\Omega(M)$-modules and a morphism $\tilde{\alpha}: \overline{B}(A) \to T'$ of differential graded modules which fit in the commutative diagram. Observe that $\Omega^*(M) \otimes \overline{B}(A)$ is indeed a resolution of $\mathbb{R}$ and the diffeological space $PM$ is smoothly contractible. Then the map $\tilde{\alpha}$ is a quasi-isomorphism and hence so is $\tilde{\alpha}$. We see that $\alpha \circ \pi : \Omega^*(M) \otimes \overline{B}(A) \to A(\Omega M)$ is a quasi-isomorphism.

We apply the same argument to the pullback $\Omega N \to E_f \to M$ of the evaluation map $(\varepsilon_0, e_1): N^I \to N \times N$ along an induction $f: M \to N \times N$. Then in the diagram above, the bar complex $\Omega^*(M) \otimes \overline{B}(A)$ is also replaced with the complex $\Omega^*(M) \otimes \overline{B}(A)$. The proof of [26, 7.1 Theorem] enables us to conclude that $\alpha \circ \pi : \Omega^*(M) \otimes \overline{B}(A) \to A(E_f)$ is a quasi-isomorphism. We have the result. \qed

We conclude this section with a table which summarizes simplicial objects used in this manuscript.

<table>
<thead>
<tr>
<th>$(A^<em><em>{DR})</em></em>$</th>
<th>$S^\infty_*(X)$</th>
<th>$S^\infty_*(X)_{sub}$</th>
<th>$S^\infty_*(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>This pair is used in proving the de Rham theorem</td>
<td>We use this pair when constructing the SSes in Theorems 5.4 and 5.5</td>
<td>This is used in describing Proposition 4.4</td>
<td></td>
</tr>
<tr>
<td>$(\hat{A}^<em><em>{DR})</em></em>$</td>
<td>This case (1) is not used in our framework</td>
<td>This case (2) is not used in our framework</td>
<td>We use this pair when constructing $\alpha'$ in Proposition 4.4</td>
</tr>
</tbody>
</table>

**Table 1**

The pair of a simplicial set in the first row and a simplicial cochain algebra in the first column gives a DGA. These DGA’s are quasi-isomorphic to one another. In
fact, the quasi-isomorphisms are induced by the inclusion \( S^\infty_*(X) \rightarrow S^D_*(X) \), the restrictions \( j^*: S^D(X)_* \rightarrow S^\infty(X) \) and \( j^*(A^*_{DR})_* \rightarrow (A^*_{DR})_* \). More precisely, the results for the pairs in the second row follow from Lemmas 4.1, 4.2 and the commutativity of the same right square as in the diagram (4.2) in the proof of Proposition 4.4. We have the result for the pair in each column by considering the commutativity of the same right triangle as in (4.2).

In [36], Kihara has introduced standard simplexes \( \Delta^p_{Ki} \) for \( p \geq 0 \) in Diff whose underlying topological spaces are the standard ones in the category of topological spaces. With the simplexes, it is proved that Diff admits a Quillen model category structure; see [36, Theorem 1.3]. For a diffeological space \( X \), we can consider the complex associated with the singular simplex \( S^D_*(X)_{Ki} \) consisting of smooth maps \( \Delta^p_{Ki} \rightarrow X \), which is quasi-isomorphic to the complex for \( S^D_*(X)_{sub} \); see [37, Remark 3.8]. Then the pairs (3):= \((S^D_*(X)_{Ki}, (A^*_{DR})_*)\) and (4):= \((S^D_*(X)_{Ki}, (A^*_{DR})_*^)\) give rise to the DGA’s which are quasi-isomorphic to the DGA for the pair (2). While it is possible to choose the pairs (1), (2), (3) and (4) when considering the cohomology algebras, we do not use them explicitly in this manuscript. We anticipate that such a pair is relevant in the study of diffeological spaces.

Acknowledgements. The author thanks Akinori Emoto and Norio Iwase for many valuable discussions on the simplicial cochain algebras concerning the de Rham theory for diffeological spaces. The author is also grateful to Hiroshi Kihara for his valuable discussions on the simplicial cochain algebras, which is quasi-isomorphic to the complex for \( S^D_*(X)_{Ki} \), the morphism \( \eta_{Ki} : \mathbb{R} \rightarrow F(X) \) in \( \text{Ch}^*(\mathbb{R}) \). Let \( \mathcal{M} \) be a set of objects in \( \mathcal{C} \), which is called models. A functor \( F \) with unit is acyclic on models \( \mathcal{M} \) if for any \( M \in \mathcal{M} \), there exists a morphism \( \varepsilon_M : F(M) \rightarrow \mathbb{R} \) such that \( \varepsilon_M \circ \eta_M \simeq \text{id} \) and \( \eta_M \circ \varepsilon_M \simeq \text{id} \).

Let \( F : \mathcal{C} \rightarrow \mathbb{R}\text{-Mod} \) be a functor form a category with models \( \mathcal{M} \) to the category of vector spaces over \( \mathbb{R} \). Then we define a contravariant functor \( \hat{F} : \mathcal{C} \rightarrow \mathbb{R}\text{-Mod} \) by

\[
\hat{F}(X) := \prod_{M \in \mathcal{M}} (F(M) \times \mathcal{C}(M, X)) = \prod_{M \in \mathcal{M}, \sigma \in \mathcal{C}(M, X)} (F(M) \times \{\sigma\}),
\]

where for a morphism \( f : X \rightarrow Y \) in \( \mathcal{C} \), the morphism \( \hat{F}(f) : \hat{F}(Y) \rightarrow \hat{F}(X) \) is defined by \( \hat{F}(f) \{m_\sigma, \sigma\} = \{m_{f\tau}, \tau\} \). Moreover, we define a natural transformation

6. Appendix

6.1. Appendix A: The acyclic model theorem for cochain complexes. We recall the acyclic model theorem due to Bousfield and Gugenheim [6].

Definition 6.1. Let \( \mathcal{C} \) be a category and \( \text{Ch}^*(\mathbb{R}) \) the category of cochain complexes over a field \( \mathbb{R} \). A contravariant functor \( F : \mathcal{C} \rightarrow \text{Ch}^*(\mathbb{R}) \) admits a unit if for each object \( X \) in \( \mathcal{C} \), there exists a morphism \( \eta_X : \mathbb{R} \rightarrow F(X) \) in \( \text{Ch}^*(\mathbb{R}) \). Let \( \mathcal{M} \) be a set of objects in \( \mathcal{C} \), which is called models. A functor \( F \) with unit is acyclic on models \( \mathcal{M} \) if for any \( M \in \mathcal{M} \), there exists a morphism \( \varepsilon_M : F(M) \rightarrow \mathbb{R} \) such that \( \varepsilon_M \circ \eta_M \simeq \text{id} \) and \( \eta_M \circ \varepsilon_M \simeq \text{id} \).

Let \( F : \mathcal{C} \rightarrow \mathbb{R}\text{-Mod} \) be a functor form a category with models \( \mathcal{M} \) to the category of vector spaces over \( \mathbb{R} \). Then we define a contravariant functor \( \hat{F} : \mathcal{C} \rightarrow \mathbb{R}\text{-Mod} \) by

\[
\hat{F}(X) := \prod_{M \in \mathcal{M}} (F(M) \times \mathcal{C}(M, X)) = \prod_{M \in \mathcal{M}, \sigma \in \mathcal{C}(M, X)} (F(M) \times \{\sigma\}),
\]
\( \Phi : F \to \hat{F} \) by \( \Phi_X(u) = \{F(x)u, x\} \). We say that \( F \) is corepresentative on the models \( \mathcal{M} \) if there exists a natural transformation \( \Psi : \hat{F} \to F \) such that \( \Psi \circ \Phi = id_F \).

**Theorem 6.2.** [6, 2.4 Proposition] Let \( \mathcal{C} \) be a category with models \( \mathcal{M} \). Let \( K_1 \) and \( K_2 \) be contravariant functors from \( \mathcal{C} \to \text{Ch}^* (\mathbb{R}) \) with units \( \eta : \mathbb{R} \to K_1^0, K_2^0 \). Here \( \mathbb{R} \) denotes the constant functor defined by \( \mathbb{R}(X) = \mathbb{R} \). Suppose that \( K_1 \) is acyclic on models \( \mathcal{M} \) and \( U_k \circ K_2 \) is corepresentative on the models, where \( U_k \) denotes the forgetful functor to \( \mathcal{M} \)-Mod on the degree \( k \). Then there exists a natural transformation \( T : K_1 \to K_2 \) which preserves the unit. Moreover any two such natural transformations are naturally homotopic.

We here consider III) in Section 4.1 more precisely. In the theorem above, we take the category \( \text{Diff} \) as \( \mathcal{C} \) and then put \( K_1 = \Omega^* (-) \) and \( K_2 = C^* (S^D_\infty (-)) \). Let \( \mathcal{M} \) be the subset of objects in \( \mathcal{C} \) consisting of the affine spaces \( \mathbb{A}^n \) for any \( n \geq 0 \). Then the category \( \text{Diff} \) is regarded as a category with models \( \mathcal{M} \). Poincaré lemma for diffeology implies that the functor \( \Omega^* (-) \) is acyclic for \( \mathcal{M} \); see [33, 6.83].

For a non-negative integer \( k \geq 0 \), we define a map

\[
\Psi_X : C^k(S^D_\infty (\mathbb{A}^n)) := \prod_{\mathbb{A}^n \in \mathcal{M}} (C^k(S^D_\infty (\mathbb{A}^n)) \times C^\infty (\mathbb{A}^n, X)) \to C^k(S^D_\infty (\mathbb{A}^n))
\]

by \( \Psi_X (\{m_\sigma, \sigma\})(\tau) = m_\tau (id_{\mathbb{A}^n}) \), where \( \tau \in S^D_\infty (\mathbb{A}^n) \). Then \( \Psi_X \) is a natural transformation. In fact, we see that for a smooth map \( f : X \to Y \) and \( u \in S^D_\infty (X) \),

\[
\Psi_X (C^k(S^D_\infty (f))\{m_\sigma, \sigma\})(u) = \Psi_X \{m_f, \tau\}(u) = m_{fu}(id_{\mathbb{A}^n}) \quad \text{and}
\]

\[
((C^k(S^D_\infty (f)))(\Psi_Y \{m_\sigma, \sigma\})(u)) = \Psi_Y \{m_\sigma, \sigma\}(fu) = m_{fu}(id_{\mathbb{A}^n}).
\]

Since \( \Phi_X (u) = \{C^k(S^D_\infty (\sigma))u, \sigma\} \) for \( u \in C^k(S^D_\infty (\mathbb{A}^n)) \) by definition, it follows that

\[
(\Psi_X \Phi_X (u))(\tau) = \Psi_X \{(C^k(S^D_\infty (\sigma))u, \sigma\})(\tau) = C^k(S^D_\infty (\tau))(u(id_{\mathbb{A}^n}) = u(\tau \circ id_{\mathbb{A}^n}) = u(\tau)
\]

for \( \tau \in S^D_\infty (\mathbb{A}^n) \). Then we have \( \Psi \Phi = id \) and hence \( C^k(S^D_\infty (-)) \) is corepresentative. Theorem 6.2 enables us to deduce the homotopy commutativity of the right square in Theorem 2.4.

**6.2. Appendix B.** We begin with the definition of a differential space in the sense of Sikorski [45] in order to define a stratifold.

**Definition 6.3.** A differential space is a pair \((S, \mathcal{C})\) consisting of a topological space \( S \) and an \( \mathbb{R} \)-subalgebra \( \mathcal{C} \) of the \( \mathbb{R} \)-algebra \( C^0(S) \) of continuous real-valued functions on \( S \), which is supposed to be locally detectable and \( C^\infty \)-closed.

Local detectability means that if \( f \in \mathcal{C} \) if and only if for any \( x \in S \), there exist an open neighborhood \( U \) of \( x \) and an element \( g \in \mathcal{C} \) such that \( f|_U = g|_U \).

\( C^\infty \)-closedness means that for each \( n \geq 1 \), each \( n \)-tuple \((f_1, ..., f_n)\) of maps in \( \mathcal{C} \) and each smooth map \( g : \mathbb{R}^n \to \mathbb{R} \), the composite \( h : S \to \mathbb{R} \) defined by \( h(x) = g(f_1(x), ..., f_n(x)) \) belongs to \( \mathcal{C} \).

Let \((S, \mathcal{C})\) be a differential space and \( x \in S \). The vector space consisting of derivations on the \( \mathbb{R} \)-algebra \( \mathcal{C}_x \) of the germs at \( x \) is denoted by \( T_x S \), which is called the tangent space of the differential space at \( x \); see [38, Chapter 1, section 3].

**Definition 6.4.** A stratifold is a differential space \((S, \mathcal{C})\) such that the following four conditions hold:

- \( \mathcal{C}_x \text{ is a differential space} \)
- \( \text{for any } n \geq 1, \text{each } n \)-tuple \((f_1, ..., f_n)\) of maps in \( \mathcal{C} \) and each smooth map \( g : \mathbb{R}^n \to \mathbb{R} \), the composite \( h : S \to \mathbb{R} \) defined by \( h(x) = g(f_1(x), ..., f_n(x)) \) belongs to \( \mathcal{C} \)
- \( \text{for any } x \in S, \text{the tangent space } T_x S \text{ is a differential space} \)
- \( \text{for any } x \in S, \text{the tangent space } T_x S \text{ is a differential space} \)
Lemma 6.5. Let \( \text{Mfd} \to \text{Stfd} \to \text{Stacks}_{\text{Mfd}} \to \text{Diff} \to \text{Sets}^{\Delta^{op}} \to \text{Top} \) be an open set of the underlying topological space \( S \) is a \( D \)-open set of the diffeological space \( k(S,C) \).

Proof. Let \( u \) be an element in \( D_C \) with domain \( U \). Then \( u : U \to k(S,C) \) is a smooth map in the sense of diffeology. In fact, for any plot \( p : V \to U \) of the diffeology \( U \) and for any \( \phi \in C \), we see that \( \phi \circ u_*(p) = (\phi \circ u) \circ p \) is in \( C^\infty(V) \) and hence \( u_*(p) \) is in \( D_C \). Since \( U \) is a manifold, it follows from [1, Proposition 5.1] that \( u \) is a morphism in \( \text{Stfd} \). In particular, the plot \( u \) is continuous. It turns out that, by definition, each open set of \( S \) is \( D \)-open.

We here summarize categories and functors related to our work.

The category \( \text{Mfd} \) also embeds into the category of differentiable stacks \( \text{Stacks}_{\text{Mfd}} \) with the Yoneda embedding \( \mathcal{Y} \). Moreover, the category \( \text{Stacks}_{\text{Mfd}} \) connects with
Diff using the Grothendieck construction functor $f$ and the coarse moduli space functor $\text{Coarse}$, which are adjoint to each other; see [53] for more details.

The $D$-topology of diffeological spaces gives rise to the functor $D : \text{Diff} \to \text{Top}$. For a topological space $X$, all continuous maps make the diffeology $D^X$ of the underlying set $X$. Thus we have the functor $C$ mentioned in the diagram above; see Remark 6.6 for an important property of the adjoint pair. The realization of a simplicial set in $\text{Diff}$ with affine spaces $A^n$ for $n \geq 0$ gives the realization functor $|\cdot|_D$. The results [14, Propositions 4.14 and 4.15] assert that the functors $S^D \circ C$ and $D \circ |\cdot|_D$ coincide with the usual singular simplex functor and the realization functor up to weak equivalence, respectively. We refer the reader to [46] and [14] for more properties of the adjoint pairs $(D, C)$ and $(|\cdot|_D, S^D)$, respectively.

**Remark 6.6.** The functors $C$ and $D$ give rise to an equivalence between appropriate full subcategories of $\text{Diff}$ and $\text{Top}$. In fact, we see that the unit and counit induce isomorphisms $\eta_{C,X} : CX \xrightarrow{\sim} CDCX$ and $\varepsilon_{DY} : DCDY \xrightarrow{\sim} DY$; see [15, Propositions 3.3] and also [46]. Let $C\text{-Diff}$ be the full subcategory of $\text{Diff}$ consisting of objects isomorphic to diffeological spaces in the image of $C$ and $\Delta \text{-Top}$ the full subcategory of $\text{Diff}$ consisting of objects isomorphic to topological spaces in the image of $D$. We observe that the objects in the image of $D$ are exactly the $\Delta$-generated topological spaces; see [15, Proposition 3.10]. The result [40, Lemma II. 6.4] implies that the functors are restricted to equivalences between $C\text{-Diff}$ and $\Delta \text{-Top}$. It is worth to mention that all CW-complexes are included in $\Delta \text{-Top}$; see [46, Corollary 3.4].

**Remark 6.7.** Let $X$ be in the category $C\text{-Diff}$. We can assume that the units $\eta_X : X \to CDX$ is an isomorphisms. Then it follows that for an object $X$ in $\text{Diff}_{\text{top}}$, the map $\text{Diff}(\Delta^n_{\text{sub}}, X) \to \text{Diff}(\Delta^n_{\text{sub}}, CDX)$ induced by the unit in Section 6 is bijective and hence so is the composite $\text{Diff}(\Delta^n_{\text{sub}}, X) \to \text{Top}(\Delta^n, DX)$. This yields that the functor $C_nD$ is representable for each $n$, where $C_n$ denotes the singular chain functor. Thus by the method of acyclic models, we see that $H(C^*(S^D_n(X)_{\text{sub}})) \cong H^*(DX, \mathbb{R})$ for each object $X$ in $\text{Diff}_{\text{top}}$, where $H^*(\cdot, \mathbb{R})$ denotes the singular cohomology with coefficients in $\mathbb{R}$.

**Remark 6.8.** Let $(S, C)$ be a stratifold. Then it follows from [1, Corollary 5.2] that $S^D_n(k(S, C)) \cong \text{Stfd}(A^\bullet, (S, C))$ as a simplicial set. Corollary 2.5 implies that the de Rham cohomology of $k(S, C)$ is isomorphic to the cohomology of the chain complex induced by $\text{Stfd}(A^\bullet, (S, C))$ as an algebra.

### 6.3. Appendix C

Let $(X, D^X)$ be a diffeological space. We recall the morphism $\alpha : \Omega^*(X) \to A^*(X) := A_{DR}(S^D_\bullet(X))$ of cochain algebras defined in Section 2. In this section, we prove the following proposition.

**Proposition 6.9.** For each diffeological space $X$, the map $H^1(\alpha) : H^1(\Omega^*(X)) \to H^1(A^*(X))$ induced by $\alpha$ is injective.

Moreover, we shall introduce an obstruction for the map $H^*(\alpha)$ to be injective with the Čech–de Rham spectral sequence defined in [31, 35].

**Proof of Proposition 6.9.** We consider the map $C_1(S^D_\bullet(X)) \to C_{\text{cube},1}(X)$ induced by the inverse of the projection $pr_1 : A^1 \to \mathbb{R}^1$ on the first factor, which is a diffeomorphism. Then the map gives a chain map $\tilde{m} : C_\ast(S^D_\bullet(X)) \to C_{\text{cube},\ast}(X)$ for $\ast \leq 1$. Moreover, the method of acyclic models extends $\tilde{m}$ to a chain map $\hat{m} = \{\hat{m}_i\}_{i \geq 0}$ defined on all of the degrees. We write $m$ for the dual to the chain
map $\tilde{m}$. Then by the same argument as in Section 4.1 III) with Theorem 6.2, we have a homotopy commutative diagram

\[
\begin{array}{c}
\text{A}(X) \\
\text{C}^*(S^D(X)) \xrightarrow{m} \text{C}(\text{cube}(X))
\end{array}
\]

where $f'$ denotes the composite of the integration map $\int$ defined in Section 3.2 and the isomorphism $\nu : C^\infty_{PL}(S^D(X)) \xrightarrow{\cong} C^*(S^D(X))$ mentioned in Section 2. Moreover, the definition of $\tilde{m}_1$ enables us to obtain a commutative diagram

\[
\begin{array}{ccc}
H_1(S^D(X)) & \xrightarrow{H(\tilde{m})} & H_1(\text{C}(\text{cube}, X)) \\
\phi & \mapsto & \pi_1(X),
\end{array}
\]

where $\Theta$ and $\Theta'$ denote the Hurewicz maps. Composing the diagram (6.1) on the first cohomology with one obtained by dualizing the diagram above, we have a commutative diagram

\[
\begin{array}{ccc}
H^1(A^*(X)) & \xrightarrow{\text{Hom}(\pi_1(X), \mathbb{R})} & H^1(\Omega^*(X)) \\
\int_{\tilde{m}} & \quad & \int_{\tilde{m}}
\end{array}
\]

in which $\int_{\tilde{m}}$ is nothing but the first de Rham homomorphism. It follows from [33, 6.89] that the de Rham homomorphism is a monomorphism in general. The commutativity yields the result.

The Čech–de Rham spectral sequence defined in [31, 35] for a diffeological space $X$ contains the original de Rham cohomology $H^*(\Omega^*(X))$ in the edge of the $E_2$-term. After briefly describing the spectral sequence and the fact, we shall deduce that the injectivity of the edge homomorphism gives an obstruction for the map $H^*(\alpha)$ to be injective.

Let $(X, D^X)$ be a diffeological space and $\mathcal{G}$ a generating family of $D^X$. We may assume that the domain of each plot in $\mathcal{G}$ is a ball in $\mathbb{R}^N$ for some $N$. Then we define the nebula $\mathcal{N}_X$ of $X$ associated with $\mathcal{G}$ by

\[
\mathcal{N}_X := \bigsqcup_{\varphi \in \mathcal{G}} \{\varphi\} \times \text{dom}(\varphi)
\]

with sum diffeology, where $\text{dom}(\varphi)$ denotes the domain of the plot $\varphi$. It is readily seen that the evaluation map $ev : \mathcal{N}_X \to X$ defined by $ev(\varphi, r) = \varphi(r)$ is smooth. The gauge monoid $M$ is a submonoid of the monoid of endomorphisms on the nebula $\mathcal{N}_X$ defined by

\[
M := \{ f \in C^\infty(\mathcal{N}_X, \mathcal{N}_X) \mid ev \circ f = ev \text{ and } \# \text{Supp} f < \infty \},
\]

where $\text{Supp} f := \{ \varphi \in \mathcal{G} \mid f_{\{\varphi\} \times \text{dom}(\varphi)} \neq 1_{\{\varphi\} \times \text{dom}(\varphi)} \}$. Then the original de Rham complex $\Omega^*(\mathcal{N}_X)$ is a left $M^{op}$-module whose actions are defined by $f^*$ induced by an endomorphism $f \in \mathcal{N}_X$. Moreover, the complex $\Omega^*(\mathcal{N}_X)$ is regarded as a two
sided $M^{op}$-module for which the right module structure is trivial. Then we have the Hochschild complex $C^{*,*} = \{C^p,q, \delta, d_\Omega\}_{p,q \geq 0}$ with

$$C^{p,q} = \text{Hom}_{R^{op} \otimes R^{op}}(R^{op} \otimes (R^{op})^{op} \otimes R, \Omega^q(N_X)) \simeq \text{map}(M^p, \Omega^q(N_X)),$$

where the horizontal map $\delta$ is the Hochschild differential and the vertical map $d_\Omega$ is induced by the de Rham differential on $\Omega^*(N_X)$; see [35, Subsection 8]. The total complex $\text{Tot} C^{*,*}$ has the horizontal filtration $F^* = \{F^j\}_{j \geq 0}$ defined by $F^j = \oplus_{q \geq j} C^{p,q}$. Then the filtration gives rise to a first quadrant spectral sequence $\{d_2^{E^*_r}, d_r\}$ converging to $H^*(\text{Tot} C^{*,*})$ with

$$E_2^{p,q} = H^q(\text{HH}^p(M^{op}, \Omega^*(N_X)), d_\Omega),$$

which is called the Čech–de Rham spectral sequence.

The result [35, Proposition in Subsection 6] implies that for a diffeological space $X$, the evaluation map induces an isomorphism $ev^*: \Omega^*(X) \rightarrow \Omega^*(N_X)^M$ of cochain algebras, where $\Omega^*(N_X)^M$ denotes the invariant subcomplex of $\Omega^*(N_X)$. Since the kernel of the map $\delta: C^{0,q} \rightarrow C^{1,q}$ is nothing but the complex $\Omega^*(N_X)^M$, it follows that the edge homomorphism

$$\text{edge}_1 := H(ev^*): H^*(\Omega^*(X)) \xrightarrow{\cong} dE_2^0$$

is an isomorphism.

We consider the vertical filtration of the total complex $\text{Tot} C^{*,*}$ and the spectral sequence $\{d_2^{E^*_r}, d_r\}$ associated with it. Then the Poincaré lemma for the original de Rham complex yields that $d_2^{E^*_r} = 0$ for $q > 0$ and hence the target of $\{d_2^{E^*_r}, d_r\}$ is the Hochschild cohomology (the Čech cohomology) $\tilde{H}(X) = HH^*(R, \text{map}(G, R))$; see [35, Subsections 9 and 16]. Moreover, we see that the edge homomorphism

$$\text{edge}_2 := H(\text{inc}_e): \tilde{H}(X) \xrightarrow{\cong} H^*(\text{Tot} C^{*,*})$$

is an isomorphism, where $\text{inc}_e$ is the map induced by the inclusion map$(G, R) \rightarrow \Omega^0(N_X)$ to the constant functions.

In the constructions of the two spectral sequence above, it is possible to replace the de Rham complex $\Omega^*(X)$ with the new one $A^*(X)$. While the map $\text{edge}_1: H^*(A^*(X)) \rightarrow dE_2^0$ for $A^*(X)$ is merely a morphism of algebras, the map $\text{edge}_2$ for $A^*(X)$ is an isomorphism. In fact, Lemma 4.1 and Theorem 2.4 imply that the Poincaré lemma for the new de Rham complex holds. Since the map $\alpha$ gives rise to a natural transformation $\Omega^*(\ ) \rightarrow A^*(\ )$; see Proposition 4.4 (ii), it follows that the map induces a morphism $\{f(\alpha)_r\} : \{d_2^{E^*_r}, d_r\} \rightarrow \{d_2^{E^*_r}, d_r\}$ of spectral sequences. In consequence, we have a commutative diagram

$$(6.2) \quad \begin{array}{ccccccccc}
H^*(\Omega^*(X)) & \xrightarrow{\text{edge}_1} & dE_2^0 & \rightarrow & dE_2^0 & \rightarrow & H^*(\text{Tot} C^{*,*}) & \xrightarrow{\text{edge}_2} & \tilde{H}^*(X), \\
H^*(A^*(X)) & \xrightarrow{\text{edge}_1} & dE_2^0 & \rightarrow & dE_2^0 & \rightarrow & H^*(\text{Tot} C^{*,*}) & \xrightarrow{\text{edge}_2} & \tilde{H}^*(X),
\end{array}$$

where $\text{Tot}(\alpha)$ denotes the cochain map induced by the morphism $C^{*,*} \rightarrow C^{*,*}$ of double complexes which $\alpha: A^*(X) \rightarrow \Omega^*(X)$ gives. In particular, it follows that $H^q(\text{Tot}(\alpha))$ is an isomorphism for any $q$. We call the composite of maps in the first line in the diagram (6.2) the edge homomorphism of the Čech–de Rham spectral
sequence for $X$. Then the commutative diagram above enables us to deduce the following proposition.

**Proposition 6.10.** The map $H^q(\alpha)$ is injective if so is the edge homomorphism $H^q(\Omega^*(X)) \to H^q(X)$.

In the first quadrant spectral sequence $\{dE^p,q, d_r\}$, the differential $d_r$ is of degree $(-r+1, r)$, namely $d_r : dE^p,q \to dE^{p+r, q+r}$. Observe the grading of the filtration defined in [35]. It is readily seen that the sufficient condition in Proposition 6.10 is equivalent to state that every elements in $dE^0,2$ in the Čech–de Rham spectral sequence is non-exact. By degree reasons, we see that each element in $dE^0,1$ is non-exact. Then Proposition 6.10 gives another proof of Proposition 6.9.

**Remark 6.11.** Iglesias-Zemmour has introduced a linear map $b_q : H^q(\Omega^*(X)) \to H^q(X)$, which is called the $q$th Čech–de Rham homomorphism, in [35, Subsection 12]. By definition, we see that the map $b_k$ is nothing but the edge homomorphism for $X$ mentioned above.

**References**


