RATIONAL VISIBILITY OF A LIE GROUP IN THE MONOID OF SELF-HOMOTOPY EQUIVALENCES OF A HOMOGENEOUS SPACE

KATSUHIKO KURIBAYASHI

Abstract. Let G be a connected Lie group and M a homogeneous space admitting a left translation by G. Let $\mathrm{aut}_1(M)$ denote the identity component of the monoid of self-homotopy equivalences of M. Then the action of G on M gives rise to a map $\lambda : G \to \mathrm{aut}_1(M)$. The purpose of this article is to investigate the injectivity of the homomorphism which λ induces on the rational homotopy. In particular, *the visible degrees* are determined explicitly for all the cases of simple Lie groups and their associated homogeneous spaces of rank one which are classified by Oniscik. Moreover a function space model description of the Kedra-McDuff μ-classes is given. As a consequence, we see that the rational cohomology of the classifying space of the monoid $\mathrm{aut}_1(M)$ is a polynomial algebra generated by μ-classes if M is a cohomologically symplectic manifold whose rational cohomology is generated by a single element.

1. Introduction

Let $f : X \to Y$ be a map between connected spaces whose fundamental groups are abelian. We say that X is *rationally visible* in Y with respect to the map f if the induced map $f_* \otimes 1 : \pi_i(X) \otimes \mathbb{Q} \to \pi_i(Y) \otimes \mathbb{Q}$ is injective for any $i \geq 1$. Let $\mathrm{aut}_1(M)$ be the identity component of the monoid of self-homotopy equivalences of a space M. Let G be a connected Lie group and M an appropriate homogeneous space M admitting a left translation by G. We then define a map of monoids

$$\lambda_{G,M} : G \to \mathrm{aut}_1(M)$$

by $\lambda_{G,M}(g)(x) = gx$ for $g \in G$ and $x \in M$. In this paper, we investigate the rational visibility of G in $\mathrm{aut}_1(M)$ with respect to the map $\lambda_{G,M}$.

The monoid map $\lambda_{G,M}$ factors through the identity component $\mathrm{Homeo}_1(M)$ of the monoid of homeomorphisms of M as well as the identity component $\mathrm{Diff}_1(M)$ of the space of diffeomorphisms of M. Therefore the rational visibility of G in $\mathrm{aut}_1(M)$ implies that of G in $\mathrm{Homeo}_1(M)$ and $\mathrm{Diff}_1(M)$. We also expect that non-trivial characteristic classes of the classifying spaces $B\mathrm{aut}_1(M)$, $B\mathrm{Homeo}_1(M)$ and $B\mathrm{Diff}_1(M)$ can be obtained through the study of rational visibility. Very little is known about the (rational) homotopy of the groups $\mathrm{Homeo}_1(M)$ and $\mathrm{Diff}_1(M)$ for a general manifold M; see [6] for the calculation of $\pi_i(\mathrm{Diff}_1(S^n)) \otimes \mathbb{Q}$ for i in some

2000 Mathematics Subject Classification: 55P62, 57R19, 57R20, 57T35.

Key words and phrases. Self-homotopy equivalence, homogeneous space, Sullivan model, evaluation map, characteristic class.

This research was partially supported by a Grant-in-Aid for Scientific Research (C)20540070 from Japan Society for the Promotion of Science.

Department of Mathematical Sciences, Faculty of Science, Shinshu University, Matsumoto, Nagano 390-8621, Japan e-mail:kuri@math.shinshu-u.ac.jp

range. Then such implication and expectation inspire us to consider the visibility problems of Lie groups. Furthermore this work is also motivated by various results due to Kedra and McDuff [16], Notbohm and Smith [27], Sasao [30] and Yamaguchi [35] as will be seen below. We refer the reader to papers [9] and [34] for the study of rational homotopy types of $\mathrm{aut}_1(M)$ itself and related function spaces.

In the rest of this section, we state our main results.

Theorem 1.1. *Let G be a simply-connected Lie group and T a torus in G which is not necessarily maximal. Then G is rationally visible in $\mathrm{aut}_1(G/T)$ with respect to the map $\lambda_{G,G/T}$ defined by the left translation of G/T by G.*

Theorem 1.1 is a generalization of the result [27, Proposition 2.4], in which T is assumed to be the maximal torus of G. We mention that the result due to Notbohm and Smith plays an important role in the proof of the uniqueness of fake Lie groups with a maximal torus; see [26, Section 1]. Theorem 1.1 is deduced from Theorem 1.2 below, which gives a tractable criterion for the rational visibility.

In order to describe Theorem 1.2, we fix notations. We write $H^*(X)$ for the cohomology of a space X with coefficients in the rational field $\mathbb{Q}$. Let G be a connected Lie group and U a closed connected subgroup of G. Let $B\iota : BU \to BG$ be the map induced by the inclusion $\iota : U \to G$. We assume that the rational cohomology of BG is a polynomial algebra, say $H^*(BG) \cong \mathbb{Q}[c_1, ..., c_k]$.

Consider the Lannes division functor $(H^*(BU) : H^*(G/U))$ in the category of differential graded algebras (DGA's). Then the functor is isomorphic to a quotient of the free algebra $\wedge(H^*(BU) \otimes H_*(G/U))$; see Remark 2.1. Let $\pi : H^*(BU) \otimes H_*(G/U) \to (H^*(BU) : H^*(G/U))$ denote the composite the projection and the inclusion $H^*(BU) \otimes H_*(G/U) \to \wedge(H^*(BU) \otimes H_*(G/U))$. Observe that $(H^*(BU) : H_*(G/U))$ is isomorphic to $\wedge(QH^*(BU) \otimes H_*(G/U))$ as an algebra, where $QH^*(BU)$ denotes the vector space of indecomposable elements. Under the isomorphism, we can define an algebra map $u : (H^*(BU) : H^*(G/U)) \to \mathbb{Q}$ by $u(h \otimes b_*) = \langle j^*(h), b_* \rangle$, where $j : G/U \to BU$ is the fibre inclusion of the fibration $G/U \stackrel{j}{\to} BU \stackrel{B\iota}{\to} BG$. Moreover let M_u be the ideal of $(H^*(BU) \otimes H_*(G/U))$ generated by the set

$$\{\eta | \deg \eta < 0\} \cup \{\eta - u(\eta) | \deg \eta = 0\}.$$

Theorem 1.2. *With the above notation, assume that for $c_{i_1}, ..., c_{i_s} \in \{c_1, ..., c_k\}$, there are elements $c_{j_1}, ..., c_{j_s} \in H^*(BG)$ and $u_{1*}, ..., u_{s*} \in H^{\geq 1}(G/U)$ such that*

$$\pi((B\iota)^*(c_{i_t}) \otimes 1_*) \equiv \pi((B\iota)^*(c_{j_t}) \otimes u_{t*})$$

for $t = 1, ..., s$ modulo decomposable elements in $(H^*(BG) : H^*(G/U))/M_u$. Then there exists a map $\rho : \times_{j=1}^{s} S^{\deg c_{i_j} - 1} \to G$ such that $\times_{j=1}^{s} S^{\deg c_{i_j} - 1}$ is rationally visible in $\mathrm{aut}_1(G/U)$ with respect to the map $(\lambda_{G,G/U}) \circ \rho$. In particular, if $(B\iota)^*(c_{i_1}), ..., (B\iota)^*(c_{i_s})$ are decomposable elements, then $\pi((B\iota)^*(c_{i_t}) \otimes 1_*) \equiv 0$ in $(H^*(BG) : H^*(G/U))/M_u$ and hence one has the same conclusion.

We have an important corollary.

Corollary 1.3. *There exist elements with infinite order in $\pi_l(\mathrm{Diff}_1(G/U))$ and $\pi_l(\mathrm{Homeo}_1(G/U))$ for $l = \deg c_{i_1} - 1, ..., \deg c_{i_s} - 1$.*

For a Lie group G and a homogeneous space M which admits a left translation by G, put $n(G) := \{i \in \mathbb{N} \mid \pi_i(G) \otimes \mathbb{Q} \neq 0\}$ and define the set $vd(G, M)$ of *visible* *degrees* by

$$vd(G, M) = \{i \in n(G) \mid (\lambda_{G,M})_* : \pi_i(G) \otimes \mathbb{Q} \to \pi_i(\mathrm{aut}_1(M)) \otimes \mathbb{Q} \text{ is injective}\}.$$

Example 1.4. Since $SO(d+1)/SO(d)$ is homeomorphic to the sphere S^d, we can define the maps $\lambda_{SO(d+1),S^d} : SO(d+1) \to \mathrm{aut}_1(S^d)$ by left translations. The Brown and Szczarba model for the function space $\mathrm{aut}_1(S^d)$ allows us to deduce that $\mathrm{aut}_1(S^{2m+1}) \simeq_{\mathbb{Q}} S^{2m+1}$ and $\mathrm{aut}_1(S^{2m}) \simeq_{\mathbb{Q}} S^{4m-1}$; see Example 2.4 below. Therefore $\lambda_{SO(d+1),S^d}$ is not injective on the rational homotopy in general. However it follows that the induced maps

$$(\lambda_{SO(2m+2),S^{2m+1}})_* : \pi_{2m+1}(SO(2m+2)) \otimes \mathbb{Q} \to \pi_{2m+1}(\mathrm{aut}_1(S^{2m+1})) \otimes \mathbb{Q},$$

$$(\lambda_{SO(2m+1),S^{2m}})_* : \pi_{4m-1}(SO(2m+1)) \otimes \mathbb{Q} \to \pi_{4m-1}(\mathrm{aut}_1(S^{2m})) \otimes \mathbb{Q}$$

are injective. In fact it is well known that, as algebras, $H^*(BSO(2m+1)) \cong \mathbb{Q}[p_1, ..., p_m]$ and $H^*(BSO(2m+2)) \cong \mathbb{Q}[p_1, ..., p_m, \chi]$, where $\deg p_j = 4j$ and $\deg \chi = 2m+2$. Moreover we see that $\iota_1^*(\chi) = 0$ and $\iota_2^*(p_m) = \chi^2$ for which ι_1 and ι_2 are inclusions $\iota_1 : SO(2m+1) \to SO(2m+2)$ and $\iota_2 : SO(2m) \to SO(2m+1)$, respectively; see [25]. Thus Theorem 1.2 yields

$$vd(SO(2m+2), S^{2m+1}) = \{2m+1\} \quad \text{and} \quad vd(SO(2m+1), S^{2m}) = \{4m-1\}.$$

The result [1, 1.1.5 Lemma] allows one to conclude that the map $SO(d+1) \to \mathrm{Diff}_1(S^d)$ induced by the left translations is injective on the homotopy group. This implies that the inclusion $\mathrm{Diff}_1(S^d) \to \mathrm{aut}_1(S^d)$ is surjective on the rational homotopy group.

The key device for the study of rational visibility is the function space model due to Brown and Szczarba [4] and Haefliger [12]. In particular, the rational model for the evaluation map $\mathrm{aut}_1(G/U) \times G/U \to G/U$, constructed in [17] and [5], plays a crucial role in constructing an explicit rational model for the map $\lambda_{G,M}$. By analyzing such elaborate models, we obtain Theorem 3.1 which gives an exact criterion for rational visibility. Applying the theorem, we have

Theorem 1.5. *Let M be the flag manifold $U(m)/U(m_1) \times \cdots \times U(m_l)$. Then $SU(m)$ is rationally visible in $\mathrm{aut}_1(M)$ with respect to the map $\lambda_{SU(m),M}$ given by the left translations; that is, $vd(SU(m), M) = n(SU(m)) = \{3, 5, ..., 2m-1\}$. In particular, the localized map*

$$(\lambda_{SU(m),U(m)/U(m-1)\times U(1)})_{\mathbb{Q}} : SU(m)_{\mathbb{Q}} \to \mathrm{aut}_1(\mathbb{C}P^{m-1})_{\mathbb{Q}}$$

is a homotopy equivalence.

This result is not new because the first assertion follows from [16, Proposition 4.8] due to Kedra and McDuff. The latter half is a particular case of the main theorem in [30]. We here emphasize that not only does our machinery developed in this manuscript work well to prove Theorem 1.5 but also it leads us to an unifying way of looking at the visibility problem explicitly as is seen in Tables 1 and 2 below. Furthermore, the same argument as in the proof of Theorem 1.5 enables us to deduce the following result.

Theorem 1.6. *Let M be the flag manifold $Sp(m)/Sp(m_1) \times \cdots \times Sp(m_l)$. Then $vd(Sp(m), M) = \{7, 11, ..., 4m-1\}$. In particular, the 3-connected cover $Sp(m)\langle 3\rangle$ is rationally visible in $\mathrm{aut}_1(M)$ with respect to $\lambda_{Sp(m),M} \circ \pi$, where $\pi : Sp(m)\langle 3\rangle \to Sp(m)$ is the projection.*

Let G be a compact connected simple Lie group and U a closed connected subgroup for which G/U is a simply-connected homogeneous space of rank one; that is, its rational cohomology is generated by a single element. Such couples (G, U) are classified by Oniscik; see [28, Theorems 2 and 4]. In order to illustrate usefulness of Theorems 1.2 and 3.1, we determine visible degrees of G in $\mathrm{aut}_1(G/U)$ for each couple (G, U) classified in [28] by applying the theorems.

In the following table, we first list such homogeneous spaces of the form G/U with a simple Lie group G and its subgroup U, which is not diffeomorphic to spheres or projective spaces, together with the sets $vd(G, G/U)$.

(G, U, index)	$(G/U)_{\mathbb{Q}}$	$vd(G, G/U)$	$n(G)$
(1) $(SO(2n+1), SO(2n-1)\times SO(2), 1)$	$\mathbb{C}P^{2n-1}$	$\{3, ..., 4n-1\}$	$\{3, ..., 4n-1\}$
(2) $(SO(2n+1), SO(2n-1), 1)$	S^{4n-1}	$\{4n-1\}$	$\{3, ..., 4n-1\}$
(3) $(SU(3), SO(3), 4)$	S^5	$\{5\}$	$\{3, 5\}$
(4) $(Sp(2), SU(2), 10)$	S^7	$\{7\}$	$\{3, 7\}$
(5) $(G_2, SO(4), (1, 3))$	$\mathbb{H}P^2$	$\{11\}$	$\{3, 11\}$
(6) $(G_2, U(2), 3)$	$\mathbb{C}P^5$	$\{3, 11\}$	$\{3, 11\}$
(7) $(G_2, SU(2), 3)$	S^{11}	$\{11\}$	$\{3, 11\}$
(6)' $(G_2, U(2), 1)$	$\mathbb{C}P^5$	$\{3, 11\}$	$\{3, 11\}$
(7)' $(G_2, SU(2), 1)$	S^{11}	$\{11\}$	$\{3, 11\}$
(8) $(G_2, SO(3), 4)$	S^{11}	$\{11\}$	$\{3, 11\}$
(9) $(G_2, SO(3), 28)$	S^{11}	$\{11\}$	$\{3, 11\}$

Table 1

Here the value of the index of the inclusion $j : U \to G$ is regarded as the integer i by which the induced map $j_* : H_3(U; \mathbb{Z}) \to H_3(G; \mathbb{Z}) = \mathbb{Z}$ is multiplication; see the proof of [28, Lemma 4]. The second column denotes the rational homotopy type of G/U corresponding a triple (G, U, i). The homogeneous spaces G/U for the cases (6)' and (7)' are diffeomorphic to those for the cases (1) and (2) with $n = 3$, respectively. Moreover, the homogeneous spaces are not diffeomorphic each other except for the cases (6)' and (7)'.

The following table describes visible degrees of a simple Lie group G in $\mathrm{aut}_1(G/U)$ for which G/U is of rank one and diffeomorphic to the sphere or the projective space, where the second column denotes the diffeomorphism type of the homogeneous space G/U for the triple (G, U, i) and $\mathcal{L}P^2$ is the Cayley plane.

(G, U, index)	G/U	$vd(G, G/U)$	$n(G)$
(10) $(SU(n+1), SU(n), 1)$	S^{2n+1}	$\{2n+1\}$	$\{3, ..., 2n+1\}$
(11) $(SU(n+1), S(U(n)\times U(1)), 1)$	$\mathbb{C}P^n$	$\{3, ..., 2n+1\}$	$\{3, ..., 2n+1\}$
(12) $(SO(2n+1), SO(2n), 1)$	S^{2n}	$\{4n-1\}$	$\{3, ..., 4n-1\}$
(13) $(SO(9), SO(7), 1)$	S^{15}	$\{15\}$	$\{3, 7, 11, 15\}$
(14) $(Spin(7), G_2, 1)$	S^7	$\{7\}$	$\{3, 7, 11\}$
(15) $(Sp(n), Sp(n-1), 1)$	S^{4n-1}	$\{4n-1\}$	$\{3, ..., 4n-1\}$
(16) $(Sp(n), Sp(n-1)\times S^1, 1)$	$\mathbb{C}P^{2n-1}$	$\{3, ..., 4n-1\}$	$\{3, ..., 4n-1\}$
(17) $(Sp(n), Sp(n-1)\times Sp(1), 1)$	$\mathbb{H}P^{n-1}$	$\{7, ..., 4n-1\}$	$\{3, ..., 4n-1\}$
(18) $(SO(2n), SO(2n-1), 1)$	S^{2n-1}	$\{2n-1\}$	$\{3, ..., 4n-5, 2n-1\}$
(19) $(F_4, Spin(9), 1)$	$\mathcal{L}P^2$	$\{23\}$	$\{3, 11, 15, 23\}$
(20) $(G_2, SU(3), 1)$	S^6	$\{11\}$	$\{3, 11\}$

Table 2

We here emphasize that Theorems 1.2 and 3.1 serve to determine explicitly the sets of visible degrees in the above tables, see Section 9 for the detail. In particular, the former half of Theorem 1.2, namely the Lannes functor argument, enables us to obtain the result in the case (6). Observe that for the cases (12) and (18) the results follow from those in Example 1.4. We aware that in the above tables G is rationally visible in $\mathrm{aut}_1(G/U)$ if and only if G/U has the rational homotopy type of the complex projective space. It should be mentioned that for the map $\lambda_* : \pi_*(F_4) \otimes \mathbb{Q} \to \pi_*(\mathrm{aut}_1(\mathcal{L}P^2)) \otimes \mathbb{Q}$, the restriction $(\lambda_*)_{15}$ is not injective though the vector space $\pi_{15}(\mathrm{aut}_1(\mathcal{L}P^2)) \otimes \mathbb{Q}$ and $\pi_{15}(F_4) \otimes \mathbb{Q}$ are non-trivial; see Section 9.

Let X be a space and $\mathcal{H}_{H,X}$ the monoid of all homotopy equivalences that act trivially on the rational homology of X. The result [16, Proposition 4.8] asserts that if X is generalized flag manifold $U(m)/U(m_1) \times \cdots \times U(m_l)$, then the map $B\psi_{SU(m)} : BSU(m) \to B\mathcal{H}_{H,X}$ arising from the left translations is injective on the rational homotopy. Let $\iota : \mathrm{aut}_1(X) \to \mathcal{H}_{H,X}$ be the inclusion. Since $B\psi_{SU(m)} = B\iota \circ B\lambda_{SU(m),X}$, the result [16, Proposition 4.8] yields Theorem 1.5. Theorem 1.7 below guarantees that the converse also holds; that is, the result due to Kedra and McDuff is deduced from Theorem 1.5; see Section 7.

Before describing Theorem 1.7, we recall an F_0-space, which is a simply-connected finite complex with finite-dimensional rational homotopy and trivial rational cohomology in odd degree. For example, a homogeneous space G/T for which G is a connected Lie group and T is a maximal torus of G is an F_0-space.

Theorem 1.7. *Let X be an F_0-space or a space having the rational homotopy type of the product of odd dimensional spheres and G a connected topological group which acts on X. Then $(B\lambda_{G,X})_* : H_*(BG) \to H_*(B\mathrm{aut}_1(X))$ is injective if and only if so is $(B\psi)_* : H_*(BG) \to H_*(B\mathcal{H}_{H,X})$. Here $\psi : G \to \mathcal{H}_{H,X}$ denotes the morphism of monoids induced by the action of G on X.*

As is seen in Remark 7.1, the induced map $(B\psi)_* : H_j(BG) \to H_j(B\mathcal{H}_{H,G/U})$ is injective for each triple (G, U, i) in Tables 1 and 2 if $j \in vd(G, G/U)$.

We now direct our attention to generators of the cohomology of the classifying space $B\mathrm{aut}_1(X)$ for a cohomologically symplectic manifold X.

Let (M, a) be a $2m$-dimensional cohomologically symplectic (c-symplectic) manifold; that is, a is a class in $H^2(M)$ such that $a^m \neq 0$; see [18]. Let $\mathcal{H}_a$ denote the group of diffeomorphisms of M that fix a. Kedra and McDuff defined in [16, Section 3] cohomology classes, which are called the μ-classes, of the classifying space of $B\mathcal{H}_a$ provided $H^1(M) = 0$. These classes are generalization of the characteristic classes of the classifying space of the group of Hamiltonian symplectomorphisms due to Reznikov [29] and Januszkiewicz and Kedra [15]. By the same way, we can define characteristic classes μ_k of $B\mathrm{aut}_1(M)$ for $2 \leq k \leq m+1$. The class μ_k is also called the kth μ-class; see Section 8 for the explicit definition of such classes.

If the cohomology algebra $H^*(M)$ is generated by a single element, then generators of $H^*(B\mathrm{aut}_1(M))$ are determined algebraically by means of the function space model due to Brown and Szczarba [4] and due to Haefliger [12]. Then we can relate such generators to the μ-classes.

Theorem 1.8. *Let (M, a) be a nilpotent connected c-symplectic manifold whose cohomology is isomorphic to $\mathbb{Q}[a]/(a^{m+1})$. Then, as an algebra,*

$$H^*(B\mathrm{aut}_1(M)) \cong \mathbb{Q}[\mu_2, ..., \mu_{m+1}],$$

where $\deg \mu_k = 2k$.

We here give a computational example. Consider the real Grassmannian manifold M of the form $SO(2m+1)/SO(2) \times SO(2m-1)$ and the map $\lambda_{SO(2m+1),M} : SO(2m+1) \to M$ arising from the left translation of $SO(2m+1)$ on M. Since $H^*(M) \cong \mathbb{Q}[\chi]/(\chi^{2m})$ as an algebra, it follows from Theorem 1.8 that

$$H^*(B\mathrm{aut}_1(M)) \cong \mathbb{Q}[\mu_2, \mu_3, \mu_4, ..., \mu_{2m}].$$

Observe that $\chi \in H^2(M)$ is the element which comes from the Euler class $\chi \in H^2(BSO(2))$ via the induced map

$$j^* : H^*(B(SO(2) \times SO(2m-1)) \cong \mathbb{Q}[\chi, p_1', ..., p_{m-1}'] \to H^*(M),$$

where j is the fibre inclusion of the fibration $M \xrightarrow{j} B(SO(2) \times SO(2m-1) \xrightarrow{Bi} BSO(2m+1)$. Recall that the rational cohomology of $BSO(2m+1)$ is a polynomial algebra generated by Pontrjagin classes; that is, $H^*(BSO(2m+1)) \cong \mathbb{Q}[p_1, ..., p_m]$, where $\deg p_i = 4i$. We relate the Pontrjagin classes to the μ-classes with the map induced by $\lambda_{SO(2m+1),M}$. More precisely, we have

Proposition 1.9. $(B\lambda_{SO(2m+1),M})^*(\mu_{2i}) \equiv p_i$ *modulo decomposable elements.*

The proof of Proposition 1.9 also allows us to deduce that the image of the kth μ-class by the induced map

$$(B\lambda_{SU(m+1),\mathbb{C}P^m})^* : H^*(B\mathrm{aut}_1(\mathbb{C}P^m)) \to H^*(BSU(m+1))$$

coincides with the kth Chern class up to sign modulo decomposable elements; see also the proof of [16, Proposition 1.7].

We now provide an overview of the rest of the paper. In Section 2, we recall briefly a model for the evaluation map of a function space from [17], [5] and [14]. In Section 3, a rational model for the map $\lambda_{G,M}$ mentioned above is constructed. Section 4 is devoted to the study of a model for the left translation of a Lie group on a homogeneous space. In Section 5, we prove Theorem 1.2. Theorem 1.5 is proved in Section 6. In Section 7, we prove Theorem 1.7. In Section 8, following Kedra-McDuff, we first define the coupling class and μ-classes. By considering the Eilenberg-Moore spectral sequence converging to the cohomology of the total space of the universal M-fibration, Theorem 1.8 and Proposition 1.9 is proved. The results on visible degrees in Tables 1 and 2 are verified in Section 9. In Appendix, Section 10, the group cohomology of $\mathrm{Diff}_1(M)$ for an appropriate homogeneous space M is discussed. By using Theorem 1.2, we find non-trivial classes in the cohomology.

2. Preliminaries

The tool for the study of the rational visibility problem is a rational model for the evaluation map $ev : \mathrm{aut}_1(M) \times M \to M$, which is described in terms of the rational model due to Brown and Szczarba [4]. For the convenience of the reader and to make notation more precise, we recall from [5] and [17] the model for the evaluation map. We shall use the same terminology as in [3] and [8].

Throughout the paper, for an augmented algebra A, we write QA for the space $\overline{A}/\overline{A} \cdot \overline{A}$ of indecomposable elements, where $\overline{A}$ denotes the augmentation ideal. For a DGA (A, d), let d_0 denote the linear part of the differential.

In what follows, we assume that a space is nilpotent and has the homotopy type of a connected CW complex with rational homology of finite type unless otherwise explicitly stated. We denote by $X_{\mathbb{Q}}$ the localization of a nilpotent space X.

Let A_{PL} be the simplicial commutative cochain algebra of polynomial differential forms with coefficients in $\mathbb{Q}$; see [3] and [8, Section 10]. Let $\mathcal{A}$ and $\Delta\mathcal{S}$ be the category of DGA's and that of simplicial sets, respectively. Let $\mathrm{DGA}(A,B)$ and $\mathrm{Simpl}(K,L)$ denote the hom-sets of the categories $\mathcal{A}$ and $\Delta\mathcal{S}$, respectively. Following Bousfield and Gugenheim [3], we define functors $\Delta : \mathcal{A} \to \Delta\mathcal{S}$ and $\Omega : \Delta\mathcal{S} \to \mathcal{A}$ by $\Delta(A) = \mathrm{DGA}(A, A_{PL})$ and by $\Omega(K) = \mathrm{Simpl}(K, A_{PL})$.

Let (B, d_B) be a connected, locally finite DGA and B_* denote the differential graded coalgebra defined by $B_q = \mathrm{Hom}(B^{-q}, \mathbb{Q})$ for $q \leq 0$ together with the coproduct D and the differential d_{B*} which are dual to the multiplication of B and to the differential d_B, respectively. We denote by I the ideal of the free algebra $\wedge(\wedge V \otimes B_*)$ generated by $1 \otimes 1_* - 1$ and all elements of the form

$$a_1a_2 \otimes \beta - \sum_i (-1)^{|a_2||\beta_i'|}(a_1 \otimes \beta_i')(a_2 \otimes \beta_i''),$$

where $a_1, a_2 \in \wedge V$, $\beta \in B_*$ and $D(\beta) = \sum_i \beta_i' \otimes \beta_i''$. Observe that $\wedge(\wedge V \otimes B_*)$ is a DGA with the differential $d := d_A \otimes 1 \pm 1 \otimes d_{B*}$. The result [4, Theorem 3.5] implies that the composite $\rho : \wedge(V \otimes B_*) \hookrightarrow \wedge(\wedge V \otimes B_*) \to \wedge(\wedge V \otimes B_*)/I$ is an isomorphism of graded algebras. Moreover, it follows that [4, Theorem 3.3] that $dI \subset I$. Thus $(\wedge(V \otimes B_*), \delta = \rho^{-1}d\rho)$ is a DGA. Observe that, for an element $v \in V$ and a cycle $e \in B_*$, if $d(v) = v_1 \cdots v_m$ with $v_i \in V$ and $D^{(m-1)}(e_j) = \sum_j e_{j_1} \otimes \cdots \otimes e_{j_m}$, then

$$\delta(v \otimes e) \quad = \quad \textstyle\sum_j \pm (v_1 \otimes e_{j_1}) \cdots (v_m \otimes e_{j_m}). \tag{2.1}$$

Here the sign is determined by the Koszul rule; that is, $ab = (-1)^{\deg a \deg b} ba$ in a graded algebra. Let F be the ideal of $E := \wedge(V \otimes B_*)$ generated by $\oplus_{i<0} E^i$ and $\delta(E^{-1})$. Then E/F is a free algebra and $(E/F, \delta)$ is a Sullivan algebra (not necessarily connected), see the proofs of [4, Theorem 6.1] and of [5, Proposition 19].

Remark 2.1. The result [4, Corollary 3.4] implies that there exists a natural isomorphism $\mathrm{DGA}(\wedge(\wedge V \otimes B_*)/I, C) \cong \mathrm{DGA}(\wedge V, B \otimes C)$ for any DGA C. Then $\wedge(\wedge V \otimes B_*)/I$ is regarded as the Lannes division functor $(\wedge V : B)$ by definition.

The singular simplicial set of a topological space U is denoted by ΔU and let $|X|$ be the geometrical realization of a simplicial set X. By definition, $A_{PL}(U)$ the DGA of polynomial differential forms on U is given by $A_{PL}(U) = \Omega\Delta U$. Given spaces X and Y, we denote by $\mathcal{F}(X,Y)$ the space of continuous maps from X to Y. The connected component of $\mathcal{F}(X,Y)$ containing a map $f : X \to Y$ is denoted by $\mathcal{F}(X,Y;f)$.

Let $\alpha : A = (\wedge V, d) \xrightarrow{\simeq} A_{PL}(Y) = \Omega\Delta Y$ be a Sullivan model (not necessarily minimal) for Y and $\beta : (B, d) \xrightarrow{\simeq} A_{PL}(X)$ a Sullivan model for X for which B is connected and locally finite. For the function space $\mathcal{F}(X,Y)$ which is considerd below, we assume that

$$\dim \oplus_{q \geq 0} H^q(X;\mathbb{Q}) < \infty \quad \text{or} \quad \dim \oplus_{i \geq 2} \pi_i(Y) \otimes \mathbb{Q} < \infty. \tag{2.2}$$

Then the proof of [17, Proposition 4.3] enables us to deduce the following lemma; see also [5].

Lemma 2.2. (i) *Let $\{b_j\}$ and $\{b_{j*}\}$ be a basis of B and its dual basis of B_*, respectively and $\widetilde{\pi} : \wedge(A \otimes B_*) \to (\wedge(A \otimes B_*)/I)/F$ denote the projection. Define* *a map* $m(ev) : A \to (\wedge(A \otimes B_*)/I)\big/F \otimes B$ *by*

$$m(ev)(x) = \sum_j (-1)^{\tau(|b_j|)} \widetilde{\pi}(x \otimes b_{j*}) \otimes b_j,$$

for $x \in A$*, where* $\tau(n) = [(n+1)/2]$*, the greatest integer in* $(n+1)/2$*. Then* $m(ev)$ *is a well-defined DGA map.*
(ii) *There exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{F}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) \times X_{\mathbb{Q}} & \xrightarrow{\ ev\ } & Y_{\mathbb{Q}} \\ {\scriptstyle \Theta \times 1}\uparrow & & \| \\ |\Delta(E/F)| \times |\Delta(B)| & \xrightarrow[|\Delta m(ev)|]{} & |\Delta(A)| \end{array}$$

in which Θ *is the homotopy equivalence described in* [4, Sections 2 and 3]*; see also* [17, (3.1)].

We next recall a Sullivan model for a connected component of a function space. Choose a basis $\{a'_k, b'_k, c'_j\}_{k,j}$ for B_* so that $d_{B_*}(a'_k) = b'_k$, $d_{B_*}(c'_j) = 0$ and $c'_0 = 1$. Moreover we take a basis $\{v_i\}_{i \geq 1}$ for V such that $\deg v_i \leq \deg v_{i+1}$ and $d(v_{i+1}) \in \wedge V_i$, where V_i is the subvector space spanned by the elements $v_1, ..., v_i$. The result [4, Lemma 5.1] ensures that there exist free algebra generators w_{ij}, u_{ik} and v_{ik} such that

(2.3) $w_{i0} = v_i \otimes 1$ and $w_{ij} = v_i \otimes c'_j + x_{ij}$, where $x_{ij} \in \wedge(V_{i-1} \otimes B_*)$,
(2.4) δw_{ij} is in $\wedge(\{w_{sl}; s < i\})$,
(2.5) $u_{ik} = v_i \otimes a'_k$ and $\delta u_{ik} = v_{ik}$.

We then have a inclusion

$$\gamma : E := (\wedge(w_{ij}), \delta) \hookrightarrow (\wedge(V \otimes B_*), \delta), \tag{2.6}$$

which is a homotopy equivalence with a retract

$$r : (\wedge(V \otimes B_*), \delta) \to E; \tag{2.7}$$

see [4, Lemma 5.2] for more details. Let q be a Sullivan representative for a map $f : X \to Y$; that is, q fits into the homotopy commutative diagram

$$\begin{array}{ccc} \wedge W & \xrightarrow{\ \simeq\ } & A_{PL}(X) \\ {\scriptstyle q}\uparrow & & \uparrow{\scriptstyle A_{PL}(f)} \\ \wedge V & \xrightarrow[\simeq]{} & A_{PL}(Y). \end{array}$$

Moreover we define a 0-simplex $\widetilde{u} \in \Delta(\wedge(\wedge V \otimes B_*)/I)_0$ by

$$\widetilde{u}(a \otimes b) = (-1)^{\tau(|a|)} b(q(a)), \tag{2.8}$$

where $a \in \wedge V$ and $b \in B_*$. Put $u = \Delta(\gamma)\widetilde{u}$. Let M_u be the ideal of E generated by the set $\{\eta \mid \deg \eta < 0\} \cup \{\delta\eta \mid \deg \eta = 0\} \cup \{\eta - u(\eta) \mid \deg \eta = 0\}$. Then the result [4, Theorem 6.1] asserts that $(E/M_u, \delta)$ is a model for a connected component of the function space of the form $\mathcal{F}(X, Y)$. The proof of [17, Proposition 4.3] and [14, Remark 3.4] allow us to deduce the following proposition; see also [5].

Proposition 2.3. *With the same notation as in Lemma 2.2, we define a map* $m(ev) : A = (\wedge V, d) \to (E/M_u, \delta) \otimes B$ *by*

$$m(ev)(x) = \sum_j (-1)^{\tau(|b_j|)} \pi \circ r(x \otimes b_{j*}) \otimes b_j,$$

for $x \in A$, where $\pi : E \to E/M_u$ denotes the natural projection. Then $m(ev)$ is a model for the evaluation map $ev : \mathcal{F}(X, Y; f) \times X \to Y$; that is, there exists a homotopy commutative diagram

$$\begin{array}{ccc} A_{PL}(Y) & \xrightarrow{A_{PL}(ev)} & A_{PL}(\mathcal{F}(X,Y;f)\times X) \\ \alpha \uparrow \simeq & & \uparrow \simeq \\ & & A_{PL}(\mathcal{F}(X,Y;f)) \otimes A_{PL}(X) \\ & & \simeq \uparrow \xi\otimes\beta \\ A & \xrightarrow[m(ev)]{} & (E/M_u, \delta) \otimes B, \end{array}$$

in which $\xi : (E/M_u, \delta) \xrightarrow{\simeq} A_{PL}(\mathcal{F}(X, Y; f))$ is the Sullivan model for $\mathcal{F}(X, Y; f)$ due to Brown and Szczarba [4].

We call the DGA $(E/M_u, \delta)$ the Brown-Szczarba model for the function space $\mathcal{F}(X, Y; f)$.

Example 2.4. Let M be a space whose rational cohomology is isomorphic to the truncated algebra $\mathbb{Q}[x]/(x^m)$, where $\deg x = l$. Recall the model $(E/M_u, \delta)$ for $\mathrm{aut}_1(M)$ mentioned in [14, Example 3.6]. Since the minimal model for M has the form $(\wedge(x, y), d)$ with $dy = x^m$, it follows that

$$E/M_u = \wedge(x \otimes 1_*, y \otimes (x^s)_*; 0 \leq s \leq m-1)$$

with $\delta(x\otimes 1_*) = 0$ and $\delta(y\otimes(x^s)_*) = (-1)^s \begin{pmatrix} m \\ s \end{pmatrix} (x\otimes 1_*)^{m-s}$, where $\deg x\otimes 1_* = l$ and $\deg(y\otimes(x^s)_*) = lm - ls - 1$. Then the rational model $m(ev)$ for the evaluation map $ev : \mathrm{aut}_1(M) \times M \to M$ is given by $m(ev)(x) = (x \otimes 1_*) \otimes 1 + 1 \otimes x$ and

$$m(ev)(y) = \sum_{s=0}^{m-1} (-1)^s (y \otimes (x^s)_*) \otimes x^s + 1 \otimes y.$$

Remark 2.5. We here describe variants of the function space model due to Brown and Szczarba model.
(i) Let $\wedge\widetilde{V} \xrightarrow{\simeq} A_{PL}(Y)$ be a Sullivan model (not necessarily minimal) and $B \xrightarrow{\simeq} A_{PL}(X)$ a Sullivan model of finite type. We recall the homotopy equivalence $\gamma : E \to \widetilde{E} = \wedge(\wedge V \otimes B_*)/I$ mentioned in (2.6). Let $\widetilde{u} \in \Delta(\widetilde{E})_0$ be a 0-simplex and u a 0-simplexes of E defined by composing $\widetilde{u}$ with the quasi-isomorphism γ. Then the induced map $\overline{\gamma} : E/M_u \to \widetilde{E}/M_{\widetilde{u}}$ is a quasi-isomorphism. In fact the results [4, Theorem 6.1] and [5, Proposition 19] imply that the projections onto the quotient DGA's E/M_u and $\widetilde{E}/M_{\widetilde{u}}$ induce homotopy equivalences $\Delta(p) : \Delta(E/M_u) \to \Delta(E)_u$ and $\Delta(\widetilde{p}) : \Delta(\widetilde{E}/M_{\widetilde{u}}) \to \Delta(\widetilde{E})_{\widetilde{u}}$, respectively. Then we have a commutative diagram

$$\begin{array}{ccc} \pi_*(|\Delta(E/M_u)|) & \xrightarrow[\cong]{|\Delta(p)|} & \pi_*(|\Delta(E)|, |u|) \\ |\Delta(\overline{\gamma})|_* \uparrow & & \uparrow |\Delta(\gamma)|_* \\ \pi_*(|\Delta(\widetilde{E}/M_{\widetilde{u}})|) & \xrightarrow[|\Delta(\widetilde{p})|]{\cong} & \pi_*(|\Delta(\widetilde{E})|, |\widetilde{u}|) \end{array}$$

Since γ is a homotopy equivalence, it follows that $|\Delta(\gamma)|_*$ is an isomorphism and hence so is $|\Delta(\overline{\gamma})|_*$. This yields that $|\Delta(\overline{\gamma})|$ is homotopy equivalence. By virtue of the Sullivan-de Rham equivalence Theorem [3, 9.4], we see that $\overline{\gamma}$ is a quasi-isomorphism.

As in Lemma 2.2, we define the DGA map $\widetilde{m(ev)} : (\wedge V, d) \to \widetilde{E}/\widetilde{F} \otimes B$ and let $m(ev) : (\wedge V, d) \to \widetilde{E}/M_{\widetilde{u}} \otimes B$ be the DGA defined by $m(ev) = \pi \otimes 1 \circ \widetilde{m(ev)}$. We then have a homotopy commutative diagram

$$\begin{array}{ccc} & \xrightarrow{m(ev)} & E/M_u \otimes B \\ \wedge V & & \simeq \big\downarrow \overline{\gamma}\otimes 1 \\ & \xrightarrow{m(ev)} & \widetilde{E}/M_{\widetilde{u}} \otimes B. \end{array}$$

In fact the homotopy between $id_{\widetilde{E}}$ and $\gamma \circ r$ defined in [4, Lemma 5.2] induces a homotopy between $id_{\widetilde{E}/\widetilde{F}}$ and $\gamma \circ r : \widetilde{E}/\widetilde{F} \to E/F \to \widetilde{E}/\widetilde{F}$. It is immediate that $r \circ \gamma = id_{E/F}$. Let $m(ev)' : \wedge V \to E/F \otimes B$ be the DGA defined as in Proposition 2.3. Then it follows that

$$\begin{aligned} \overline{\gamma} \otimes 1 \circ m(ev) &= \overline{\gamma} \otimes 1 \circ \pi \otimes 1 \circ m(ev)' \\ &= \pi \otimes 1 \circ \gamma \otimes 1 \circ r \otimes 1 \circ \widetilde{m(ev)} \\ &\simeq \pi \otimes 1 \circ \widetilde{m(ev)} = m(ev). \end{aligned}$$

(ii) In the case where X is formal, we have a more tractable model for $\mathcal{F}(X, Y; f)$. Suppose that X is a formal space with a minimal model $(B, d_B) = (\wedge W', d)$. Then there exists a quasi-isomorphism $k : (\wedge W', d) \to H^*(B)$ which is surjective; see [7, Theorem 4.1]. With the notation mentioned above, let $\{e_j\}_j$ be a basis for the homology $H(B_*)$ of the differential graded coalgebra $B_* = (\wedge W')_*$ and $\{v_i\}_i$ a basis for V. Then it follows from the proof of [4, Theorem 1.9] that the subalgebra $\mathbb{Q}\{v_i \otimes e_j\}$ is closed for the differential δ and that the inclusion $\mathbb{Q}\{v_i \otimes e_j\} \to \wedge(W \otimes B_*) = \widetilde{E}$ gives rise to a homotopy equivalence

$$\gamma : E' := (\wedge(v_i \otimes e_j), \delta) \to (\wedge(W \otimes B_*), \delta) = \widetilde{E}.$$

In fact, the elements w_{ij} in (2.3) can be chosen so that $w_{i0} = v_i \otimes 1_*$ and $w_{ij} = v_i \otimes e_j$ for $j \geq 1$. Moreover we see that there exists a retraction $r : \wedge(W \otimes B_*) \to E'$ which is the homotopy inverse of γ. Thus Proposition 2.3 remains true after replacing E by E'. Here the 0-simplex $\widetilde{u} \in \Delta(\wedge(W \otimes B_*))_0$ needed in the construction of the model for $\mathcal{F}(X, Y; f)$ has the same form as in (2.8).

Observe that $\mathrm{aut}_1(X)$ is nothing but the function space $\mathcal{F}(X, X; id_M)$. Moreover, for a manifold M, the function space $\mathrm{aut}_1(M)$ satisfies the assumption (2.2). Thus we have explicit models for $\mathrm{aut}_1(X)$ and the evaluation map according to the procedure in this section. With the models, we construct a model for the map $\lambda_{G,M}$ mentioned in Introduction in the next section.

3. A rational model for the map λ induced by left translation

Let M be a space admitting an action of Lie group G on the left. We define the map $\lambda : G \to \mathrm{aut}_1(M)$ by $\lambda(g)(x) = gx$. The subjective in this section is to construct an algebraic model for the map

$$in \circ \lambda : G \to \mathrm{aut}_1(M) \to \mathcal{F}(M, M),$$

where $in : \mathrm{aut}_1(M) \to \mathcal{F}(M, M)$ denotes the inclusion. To this end we use a model for the evaluation map

$$ev : \mathcal{F}(X, Y) \times X \to Y$$

defined by $ev(f)(x) = f(x)$ for $f \in \mathcal{F}(X, Y)$ and $x \in X$, which is considered in [17] and [5].

Let G be a connected Lie group, U a closed subgroup of G and K a closed subgroup which contains U. Let $(\wedge V_G, d)$ and $(\wedge W, d)$ denote a minimal model for G and a Sullivan model for the homogeneous space G/U, respectively. Let $\lambda : G \to \mathcal{F}(G/U, G/K)$ be the adjoint of the composite of the left translation $G \times G/U \to G/U$ and projection $p : G/U \to G/K$. Observe that the map λ coincides with the composite

$$p_* \circ in \circ \lambda_{G,G/U} : G \to \mathrm{aut}_1(G/U) \to \mathcal{F}(G/U, G/U) \to \mathcal{F}(G/U, G/K).$$

We shall construct a model for λ by using a Sullivan representative

$$\zeta' : \wedge W \to \wedge V_G \otimes \wedge W'$$

for the composite $G \times G/U \to G/K$ of the left translation $G \times G/U \to G/U$ and the projection $p : G/U \to G/K$. Let A, B and C be DGA's. Recall from [4, Section 3] the bijection $\Psi : (A \otimes B_*, C)_{DG} \stackrel{\cong}{\to} (A, C \otimes B)_{DG}$ defined by

$$\Psi(w)(a) = \sum_j (-1)^{\tau(|b_j|)} w(a \otimes b_{j*}) \otimes b_j.$$

Consider the case where $A = (\wedge W, d)$, $B = (\wedge W', d)$ and $C = (\wedge V_G, d)$. Moreover define a map $\widetilde{\mu} : \wedge(A \otimes B_*) \to \wedge V_G$ by

$$\widetilde{\mu}(y \otimes b_{j*}) = (-1)^{\tau(|b_j|)} \langle \zeta'(y), b_{j*} \rangle, \tag{3.1}$$

where $\langle \ \ , b_{j*} \rangle : \wedge V_G \otimes \wedge W' \to \wedge V_G$ is a map defined by $\langle x \otimes a, b_{j*} \rangle = x \cdot \langle a, b_{j*} \rangle$. Then we see that $\Psi(\widetilde{\mu}) = \zeta'$. Hence it follows from [4, Theorem 3.3] that

$$\widetilde{\mu} : \widetilde{E} := \wedge(A \otimes B_*)/I \to \wedge V_G$$

is a well-defined DGA map. We define an augmentation $\widetilde{u} : \widetilde{E} \to \mathbb{Q}$ by $\widetilde{u} = \varepsilon \circ \widetilde{\mu}$, where $\varepsilon : \wedge V_G \to \mathbb{Q}$ is the augmentation. It is readily seen that that $\widetilde{\mu}(M_{\widetilde{u}}) = 0$. Thus we see that $\widetilde{\mu}$ induces a DGA map $\widetilde{\widetilde{\mu}} : \widetilde{E}/M_{\widetilde{u}} \to \wedge V_G$. We have an exact criterion for rational visibility.

Theorem 3.1. *Let $\{x_i\}_i$ be a basis for the image of the induced map*

$$H^*(Q(\widetilde{\widetilde{\mu}})) : H^*(Q(\widetilde{E}/M_{\widetilde{u}}), \delta_0) \to H^*(Q(\wedge V_G), d_0) = V_G.$$

Then there exists a map $\rho : \times_{j=1}^s S^{\deg x_i} \to G$ such that the map

$$(\lambda_{\mathbb{Q}} \circ \rho_{\mathbb{Q}})_* : \pi_*((\times_{j=1}^s S^{\deg x_i})_{\mathbb{Q}}) \to \pi_*(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}), e \circ p)$$

is injective. Moreover $\lambda_{\mathbb{Q}} : \pi_i(G_{\mathbb{Q}}) \to \pi_i(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}), e \circ p)$ is injective if and only if $H^i(Q(\widetilde{\widetilde{\mu}}))$ is surjective.

In order to prove Theorem 3.1, we first observe that the diagram

$$\begin{array}{ccc} \wedge V_G \otimes \wedge W' & \xleftarrow{\widetilde{\mu} \otimes 1} & (\wedge(A \otimes B_*)/I)/F \otimes \wedge W' \\ & {}_{\zeta'}\nwarrow \quad \wedge W \quad \nearrow_{m(ev)} & \end{array} \tag{3.2}$$

is commutative. Thus Lemma 2.2 enables us to obtain a commutative diagram (3.3)

$$\begin{array}{ccc} |\Delta \wedge V_G| \times |\Delta \wedge W'| & \xrightarrow{(\Theta\circ|\Delta\widetilde{\mu}|)\times 1} & \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}) \times (G/U)_{\mathbb{Q}} \\ & {\scriptstyle |\Delta\zeta'|=\text{action}_{\mathbb{Q}}} \searrow \quad \swarrow {\scriptstyle ev} & \\ & |\Delta \wedge W| = (G/K)_{\mathbb{Q}}. & \end{array}$$

Observe that the assumption (2.2) is satisfied in the case where we here consider.

Since the restriction $|\Delta\zeta'||_{*\times|\Delta\wedge W|}$ is homotopic to $p_{\mathbb{Q}}$, it follows from the commutativity of the diagram (3.3) that $p_{\mathbb{Q}} \simeq \Theta \circ |\Delta\widetilde{\mu}|(*)$. This implies that $\Theta \circ |\Delta\widetilde{\mu}|$ maps $G_{\mathbb{Q}}$ into the function space $\mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; p_{\mathbb{Q}})$. The result [13, Theorem 3.11] asserts that $e_\sharp : \mathcal{F}(G/U, (G/K); p) \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ is a localization. We then have the localization $\lambda_{\mathbb{Q}} : G_{\mathbb{Q}} \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$. Observe that $\lambda_{\mathbb{Q}}$ fits into the homotopy commutative diagram

$$\begin{array}{ccc} G_{\mathbb{Q}} & \xrightarrow{\lambda_{\mathbb{Q}}} & \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p) \\ {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle e_\sharp} \\ G & \xrightarrow[\lambda]{} & \mathcal{F}(G/U, (G/K); p), \end{array}$$

where e denotes the localization map.

Lemma 3.2. *Let $\lambda_{\mathbb{Q}} : G_{\mathbb{Q}} \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ be the localized map of λ mentioned above and $e^\sharp : \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; p_{\mathbb{Q}}) \to \mathcal{F}((G/U), (G/K)_{\mathbb{Q}}; e \circ p)$ the map induced by the localization $e : (G/U) \to (G/U)_{\mathbb{Q}}$. Then*

$$e^\sharp \circ \Theta \circ |\Delta\widetilde{\mu}| \simeq \lambda_{\mathbb{Q}} : G_{\mathbb{Q}} \to \mathcal{F}((G/U), (G/K)_{\mathbb{Q}}; e \circ p).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} [G \times G/U, G/K] & \underset{\approx}{\xrightarrow{\theta}} & [G, \mathcal{F}(G/U, G/K)] \\ {\scriptstyle e_*}\downarrow & & \downarrow{\scriptstyle (e_\sharp)_*} \\ [G \times G/U, (G/K)_{\mathbb{Q}}] & \underset{\approx}{\xrightarrow{\theta}} & [G, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}})] \\ {\scriptstyle (e\times e)^*}\uparrow\approx & & \uparrow{\scriptstyle e^*} \\ [G_{\mathbb{Q}} \times (G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}] & & [G_{\mathbb{Q}}, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}] \\ & {\scriptstyle \theta}\searrow{\scriptstyle\approx} & \approx\uparrow{\scriptstyle (e^\sharp)_*} \\ & & [G_{\mathbb{Q}}, \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}})] \end{array} \tag{3.4}$$

in which θ is the adjoint map and e stands for the localization map. It follows from the diagram (3.3) that $\theta(\text{action}_{\mathbb{Q}}) = \Theta \circ |\Delta\widetilde{\mu}|$. Moreover we have $\theta(\text{action}) = e_\sharp \circ \lambda = \lambda_K \circ e$. Thus the commutativity of the diagram (3.4) implies that $e^*([e^\sharp \circ \Theta \circ |\Delta\widetilde{\mu}|]) = e^*([\lambda_{\mathbb{Q}}])$ in $[G, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}})]$. Since G is connected, it follows that $(e^\sharp) \circ \Theta \circ |\Delta\widetilde{\mu}| \circ e \simeq \lambda_{\mathbb{Q}} \circ e : G \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$. The fact that $e_\sharp : \mathcal{F}(G/U, (G/K); p) \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ is the localization yields that the induced map $e^* : [G_{\mathbb{Q}}, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)] \to [G, \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)]$ is bijective. This completes the proof. $\square$

Before proving Theorem 3.1, we recall some maps. For a simplicial set K, there exists a natural homotopy equivalence $\xi_K : K \to \Delta|K|$, which is defined by $\xi_K(\sigma) =$

$t_\sigma : \Delta^n \to \{\sigma\} \times \Delta \to |K|$. This gives rise to a quasi-isomorphism $\xi_A : \Omega\Delta A \xrightarrow{\simeq} \Omega\Delta|\Delta A|$. Moreover, we can define a bijection $\eta : \mathrm{DGA}(A, \Omega K) \xrightarrow{\cong} \mathrm{Simp}(K, \Delta A)$ by $\eta : \phi \mapsto f; f(\sigma)(a) = \phi(a)(\sigma)$, where $a \in A$ and $\sigma \in K$. We observe that $\eta^{-1}(id) : A \to \Omega\Delta A$ is a quasi-isomorphism if A is a connected Sullivan algebra; see [3, 10.1. Theorem].

Proof of Theorem 3.1. Let $p : \widetilde{E} \to \widetilde{E}/M_u$ be the projection. With the same notation as above, we then have a commutative diagram

(3.5)

$$\begin{array}{ccccccc}
 & & |\Delta(\wedge(W \otimes B_*)/F)| & & \xrightarrow[\simeq]{\Theta} & & \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}) \\
 & \nearrow^{|\Delta(\widetilde{\mu})|} & \uparrow{\scriptstyle |\Delta p|} & & & & \uparrow \\
|\Delta(\wedge V_G)| & \xrightarrow[|\Delta(\widetilde{\widetilde{\mu}})|]{} & |\Delta(\widetilde{E}/M_u)| & \xrightarrow[|\Delta p|]{\simeq} & |(\Delta\widetilde{E})_u| & \xrightarrow[\Theta]{\simeq} & \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; \Theta([(0,u)])),
\end{array}$$

where $[(1,u)] \in |\Delta\widetilde{E}|$ is the element whose representative is $(1,u) \in \Delta^0 \times (\Delta\widetilde{E})_0$. Lemma 3.2 yields that

$$e^\sharp \circ \Theta \circ |\Delta p| \circ |\Delta\widetilde{\widetilde{\mu}}| \simeq e^\sharp \circ \Theta \circ |\Delta\widetilde{\mu}| \simeq \lambda_{\mathbb{Q}}. \tag{3.6}$$

Thus we see that $e^\sharp$ maps $\mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; \Theta([(1,u)]))$ to $\mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; e^\sharp \circ \Theta([(1,u)]))$, which is the connected component containing $\mathrm{Im}(\lambda_{\mathbb{Q}})$. This implies that $\mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; e^\sharp \circ \Theta([(1,u)])) = \mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; e \circ p)$. Therefore, by the naturality of maps η and ξ_A, we have a diagram

(3.7)

$$\begin{array}{ccc}
A_{PL}(G_{\mathbb{Q}}) & \xleftarrow{A_{PL}(\lambda_{\mathbb{Q}})} & A_{PL}(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)) \\
\| & & \downarrow{\scriptstyle ((e^\sharp))^*} \\
\| & & A_{PL}(\mathcal{F}((G/U)_{\mathbb{Q}}, (G/K)_{\mathbb{Q}}; \Theta([(1,u)]))) \\
\| & & \downarrow{\scriptstyle \Theta^*} \\
A_{PL}(|\Delta \wedge V_G|) & \xleftarrow{|\Delta\widetilde{\widetilde{\mu}}|^*} & A_{PL}(|\Delta(\widetilde{E}/M_{\widetilde{u}})|) = \Omega\Delta(\widetilde{E}/M_{\widetilde{u}}) \\
{\scriptstyle t' := (\xi_{\wedge V_G})\eta^{-1}(id)}\uparrow\simeq & & \simeq\uparrow{\scriptstyle \xi_{\widetilde{E}/M_{\widetilde{u}}}\eta^{-1}(id) =: t} \\
\wedge V_G & \xleftarrow{\widetilde{\widetilde{\mu}}} & \widetilde{E}/M_{\widetilde{u}}
\end{array}$$

in which the upper square is homotopy commutative and the lower square is strictly commutative. Lifting Lemma allows us to obtain a DGA map $\varphi : \widetilde{E}/M_{\widetilde{u}} \to A_{PL}(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}))$ such that $\Theta^* \circ ((e^\sharp)_*)^* \circ \varphi \simeq t$ and hence $A_{PL}(\lambda_{\mathbb{Q}}) \circ \varphi \simeq t' \circ \widetilde{\mu}$.

Given a space X, let $u : A \to A_{PL}(X)$ be a DGA map from a Sullivan algebra A. Let $[f]$ be an element of $\pi_n(X)$ and $\iota : (\wedge Z, d) \xrightarrow{\simeq} A_{PL}(S^n)$ the minimal model. By taking a Sullivan representative $\widetilde{f} : A \to \wedge Z$ with respect to u, namely a DGA map satisfying the condition that $\iota \circ \widetilde{f} \simeq A_{PL}(f) \circ u$, we define a map $\nu_u : \pi_n(X) \to \mathrm{Hom}(H^nQ(A), \mathbb{Q})$ by $\nu_u([f]) = H^nQ(\widetilde{f}) : H^nQ(A) \to H^nQ(\wedge Z) = \mathbb{Q}$. By virtue of [3, 6.4 Proposition], in particular, we have a commutative diagram

$$\begin{array}{ccc}
\pi_n(G_{\mathbb{Q}}) & \xrightarrow{\lambda_{\mathbb{Q}}} & \pi_n(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}); e \circ p) \\
{\scriptstyle \nu_{t'}}\downarrow\cong & & \cong\downarrow{\scriptstyle \nu_\varphi} \\
\mathrm{Hom}((V_G)^n, \mathbb{Q}) & \xrightarrow[HQ(\widetilde{\widetilde{\mu}})^*]{} & \mathrm{Hom}(H^nQ(\widetilde{E}/M_{\widetilde{u}}), \mathbb{Q}).
\end{array}$$

in which $\nu_{t'}$ and ν_φ are an isomorphism; see [3, 8.13 Proposition]. There exists an element $[f_i] \otimes q$ in $\pi_*(G) \otimes \mathbb{Q}$ which corresponds to the dual element x_i^* via the isomorphism $\pi_*(G) \otimes \mathbb{Q} \cong \pi_*(G_\mathbb{Q}) \overset{\nu_{t'}}{\to} \mathrm{Hom}((V_G)^n, \mathbb{Q})$ for any $i = 1, ..., s$. The required map $\rho : \times_{j=1}^s S^{\deg x_i} \to G$ is defined by the composite of the map $\times_{j=1}^s f_i$ and the product $\times_{j=1}^s G \to G$. □

4. A model for the left translation

In order to prove Theorems 1.2 and 1.5, a more explicit model for the map $\lambda_{G,M} : G \to \mathrm{aut}_1(M)$ is required. To this end, we refine the model of the left translation described in the proof of Theorem 3.1.

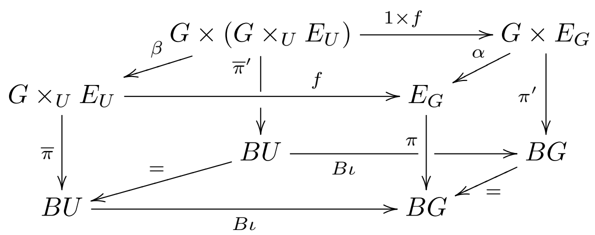
We first observe that the cohomology $H^*(BU; \mathbb{Q})$ is isomorphic to a polynomial algebra with finite generators, say $H^*(BU; \mathbb{Q}) \cong \mathbb{Q}[h_1, ..., h_l]$. We consider a commutative diagram of fibrations

$$\begin{array}{ccccc} & & G & = & G \\ & & {\scriptstyle i}\downarrow & & \downarrow \\ G/U & \underset{\simeq}{\overset{h}{\longleftarrow}} & G \times_U E_U & \longrightarrow & E_G \\ & & {\scriptstyle \overline{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ & & BU & \underset{B\iota}{\longrightarrow} & BG \end{array}$$

in which $h : G \times_U E_U \to G/U$ is a homotopy equivalence defined by $h([g, e]) = [g]$. This diagram yields a Sullivan model $(\wedge W, d)$ for G/U which has the form $(\wedge W, d) = (\wedge(h_1, ..., h_l, x_1, ..., x_k), d)$ with $dx_j = (B\iota)^* c_j$; see [8, Proposition 15.16] for the details. Moreover we have a model $(\wedge V_G, d)$ for G of the form $(\wedge(x_1, ..., x_k), 0)$. Since $h \circ i$ is nothing but the projection $\pi : G \to G/U$, it follows that the natural projection $\rho : (\wedge(h_1, ..., h_l, x_1, ..., x_k), d) \to (\wedge(x_1, ..., x_k), 0)$ is a Sullivan model for the map π.

Let $\beta : G \times (G \times_U E_U) \to G \times_U E_G$ be the action of G on $G \times_U E_U$. Then the left translation $tr : G \times G/U \to G/U$ coincides with β up to the homotopy equivalence $h : (G \times_U E_U) \to G/U$ mentioned above. Thus in order to obtain a model for the linear action, it suffices to construct a model for β. Recall the fibration $G \to G \times_U E_U \overset{\overline{\pi}}{\to} BU$ and the universal fibration $G \to E_G \overset{\pi}{\to} BG$. We here consider a commutative diagram

(4.1)

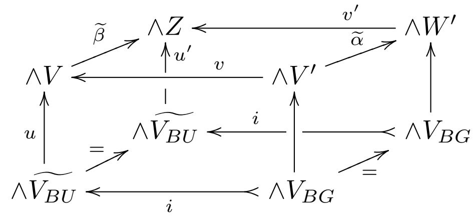


in which π' and $\overline{\pi}'$ are fibrations with the same fibre $G \times G$ and the restrictions $\alpha_{|\text{fibre}} : G \times G \to G$ and $\beta_{|\text{fibre}} : G \times (G \times_U E_U) \to (G \times_U E_U)$ are the multiplication on G and the action of G, respectively. Let $i : (\wedge V_{BG}, 0) \rightarrowtail \wedge(\widetilde{V_{BU}}, d)$ be a Sullivan model for $B\iota$. In particular, we can choose such a model so that

$$\wedge \widetilde{V_{BU}} = \wedge(c_1, ..., c_m) \otimes \wedge(h_1, ..., h_l) \otimes \wedge(\tau_1, ..., \tau_m)$$

and $d(\tau_i) = B\iota(c_i) - c_i$. By the construction of a model for pullback fibration mentioned in [8, page 205], we obtain a diagram

(4.2)



in which vertical arrows are Sullivan models for the fibrations in the diagram (4.1). Observe that squares are commutative except for the top square. Let $\Psi : \wedge Z \to A_{PL}(G \times (G \times_U E_U))$ be the Sullivan model with which Sullivan representatives in (4.2) are constructed. The argument in [8, page 205] allows us to choose homotopies, which makes maps v, $\widetilde{\beta}$, v' and $\widetilde{\alpha}$ Sullivan representatives for the corresponding maps, so that all of them are relative with respect to $\wedge V_{BG}$. This implies that $\Psi \circ \beta \circ v \simeq \Psi \circ v' \circ \widetilde{\alpha}$ rel $\wedge V_{BG}$. By virtue of Lifting lemma [8, Proposition 14.6], we have a homotopy $H : \widetilde{\beta} \circ v \simeq v' \circ \widetilde{\alpha}$ rel $\wedge V_{BG}$. Thus we have a homotopy commutative diagram

$$\begin{array}{ccc} \wedge V' \otimes_{\wedge V_{BG}} \wedge \widetilde{V_{BU}} & \xrightarrow{u\cdot v} & \wedge V \\ {\scriptstyle \widetilde{\alpha}\otimes 1}\downarrow & & \downarrow{\scriptstyle \widetilde{\beta}} \\ \wedge W' \otimes_{\wedge V_{BG}} \wedge \widetilde{V_{BU}} & \xrightarrow[u'\cdot v']{} & \wedge Z \end{array}$$

in which horizontal arrows are quasi-isomorphisms; see [8, (15.9) page 204]. In fact the homotopy $K : \wedge\widetilde{V_{BU}} \otimes_{\wedge V_{BG}} \wedge V' \to \wedge W \otimes \wedge(t, dt)$ is given by $K = (\widetilde{\beta} \circ u) \cdot H$. Observe that $\widetilde{\beta} \circ u = u'$. Thus we have a model $\widetilde{\alpha} \otimes 1$ for $\widetilde{\beta}$ and hence for the left translation.

The model $\widetilde{\alpha} \otimes 1$ can be replaced by more tractable one. In fact, recalling the model $(\widetilde{V_{BU}}, d)$ for BU mentioned above, it is readily seen that the map $s : \wedge\widetilde{V_{BU}} \to \wedge V_{BU} = \wedge(h_1, ..., h_l)$, which is defined by $s(c_i) = (B\iota)^*(c_i)$, $s(h_i) = h_i$ and $s(\tau_j) = 0$, is a quasi-isomorphism and is compatible with $\wedge V_{BG}$-action. Observe that the Sullivan representative for $B\iota : BU \to BG$ is also denoted by $(B\iota)^*$. Thus we have a commutative diagram

$$\begin{array}{ccc} \wedge V' \otimes_{\wedge V_{BG}} \wedge V_{BU} & \xleftarrow{1\otimes s} & \wedge V' \otimes_{\wedge V_{BG}} \wedge \widetilde{V_{BU}} \\ {\scriptstyle \zeta := \widetilde{\alpha}\otimes 1}\downarrow & & \downarrow{\scriptstyle \widetilde{\alpha}\otimes 1} \\ \wedge W' \otimes_{\wedge V_{BG}} \wedge V_{BU} & \xleftarrow[1\otimes s]{} & \wedge W' \otimes_{\wedge V_{BG}} \wedge \widetilde{V_{BU}} \end{array}$$

in which the DGA maps $1 \otimes s$ are quasi-isomorphisms. As usual, the Lifting lemma enables us to deduce the following lemma.

Lemma 4.1. *The DGA map* $\zeta := \widetilde{\alpha} \otimes 1$ *is a Sullivan representative for the left translation* $tr : G \times G/U \to G/U$.

In order to construct a model for tr more explicitly, we proceed to construct that for α.

Lemma 4.2. *There exists a Sullivan representative ψ for α such that a diagram*

$$\begin{array}{ccc} & & \wedge(x_1,...,x_l)\otimes\wedge V_{BG} = \wedge V' \\ & \overset{i_1}{\nearrow} & \downarrow \psi \\ \wedge V_{BG} & \underset{i_2}{\searrow} & \\ & & \wedge(x_1,...,x_l)\otimes\wedge(x_1,...,x_l)\otimes\wedge V_{BG} = \wedge W' \end{array}$$

is commutative and $\psi(x_i) = x_i\otimes 1\otimes 1 + 1\otimes x_i\otimes 1 + \sum_n X_n\otimes X'_nC_n$ for some monomials $X_n \in \wedge(x_1,...,x_l)$, $X'_n \in \wedge^+(x_1,...,x_l)$ and monomials $C_n \in \wedge^+V_{BG}$. Here i_1 and i_2 denote Sullivan models for p and p', respectively.

Proof. We first observe that $d(x_i\otimes 1) = 0$ and $d(1\otimes x_i) = c_i \in \wedge(c_1,...,c_l) = \wedge V_{BG}$ in $\wedge W'$. It follows from [8, 15.9] that there exists a DGA map ψ which makes the diagram commutative. We write

$$\psi(x_i) = x_i\otimes 1\otimes 1 + 1\otimes x_i\otimes 1 + \sum_n X_n\otimes X'_nC_n + \sum_n \tilde{X}_n\otimes\tilde{X}'_n + \sum_n X''_n\otimes C''_n$$

with monomial bases, where $X_n, X''_n \in \wedge(x_1,...,x_l)\otimes 1\otimes 1$, $X'_n \in 1\otimes\wedge^+(x_1,...,x_l)\otimes 1$, $\tilde{X}_n\otimes\tilde{X}'_n \in \wedge(x_1,...,x_l)\otimes\wedge(x_1,...,x_l)\otimes 1$ and $C_n, C''_n \in \wedge^+V_{BG}$. The map $\wedge(x_1,...,x_l)\to\wedge(x_1,...,x_l)\otimes\wedge(x_1,...,x_l)$ induced by ψ is a Sullivan representative for the product of G. This allows us to conclude that $\tilde{X}_n$ and $\tilde{X}'_n$ are in $\wedge^+(x_1,...,x_l)$. Since ψ is a DGA map, it follows that

$$dx_i = \psi(dx_i) = dx_i + \sum_n X_n\otimes d(X'_n)C_n + \sum_n \tilde{X}_n\otimes d(\tilde{X}'_n).$$

This implies that $\sum_n X_n\otimes d(X'_n)C_n = 0$ and $\sum_n \tilde{X}_n\otimes d(\tilde{X}'_n) = 0$. Since the map $d : \wedge^+(x_1,...,x_l)\to\wedge(x_1,...,x_l)\otimes\wedge V_{BG}$ is a monomorphism, it follows that $\sum_n \tilde{X}_n\otimes\tilde{X}'_n = 0$. We write $C''_n = c_{i_n}^{k_n}\tilde{C}_n$, where $k_n \geq 1$. Define a homotopy

$$H : \wedge(x_1,...,x_l)\otimes\wedge V_{BG}\to\wedge(x_1,...,x_l)\otimes\wedge(x_1,...,x_l)\otimes\wedge V_{BG}\otimes\wedge(t,dt)$$

by $H(c_i) = c_i\otimes 1$ and

$$\begin{aligned} H(x_i) &= x_i\otimes 1\otimes 1 + 1\otimes x_i\otimes 1 + \sum_n X_n\otimes X'_nC_n \\ &\quad -\sum_n X''_n\otimes x_{i_n}\otimes c_{i_n}^{k_n-1}\tilde{C}_n\otimes dt + \sum_n X''_n\otimes 1\otimes c_{i_n}^{k_n}\tilde{C}_n\otimes t. \end{aligned}$$

Put $\widetilde{\psi} = (\varepsilon_0\otimes 1)\circ\psi$. We see that $\widetilde{\psi}\simeq\psi$ rel $\wedge V_{BG}$. This completes the proof. □

5. Proof of Theorem 1.2

We prove Theorem 1.2 by means of the model for the left translation described in the previous section.

Proof of Theorem 1.2. We adapt Theorem 3.1. We recall the Sullivan model $(\wedge W, d)$ for G/U mentioned in Section 4. Observe that $(\wedge W, d)$ has the form

$$(\wedge W, d) = (\wedge(h_1,...,h_l,x_1,...,x_k), d)$$

with $dx_j = (B\iota)^*c_j$.

Let $l : (H^*(BU), 0)\to(\wedge W, d)$ be the inclusion and

$$k : (\wedge W, d)\longrightarrow(\wedge(h_1,...,h_l)/(dx_1,...,dx_l), 0)\rightarrowtail(H^*(G/U), 0)$$

the DGA map defined by $k(h_i) = (-1)^{\tau(|h_i|)}h_i$ and $k(x_i) = 0$. Recall the DGA $\widetilde{E} = \wedge(\wedge W \otimes (\wedge W)_*)/I$ and the DGA map $\widetilde{\mu} : \widetilde{E} \to \wedge V_G$ mentioned in Section 3, where we use the model $\zeta : \wedge W \to \wedge V_G \otimes \wedge W$ for the action $G \times G/U \to G/U$ constructed in Lemmas 4.1 and 4.2 in order to define $\widetilde{\mu}$; see (3.1). Consider the composite

$$\theta : (H^*(BU){:}H^*(G/U)) = \wedge(H^*(BU) \otimes H_*(G/U))/I$$

$$\xrightarrow{l\otimes 1} \wedge(\wedge W \otimes H_*(G/U))/I \xrightarrow{1\otimes k^\sharp} \wedge(\wedge W \otimes (\wedge W)_*)/I = \widetilde{E}.$$

Let $\widetilde{u} : \widetilde{E} \to \mathbb{Q}$ be an augmentation defined by $\widetilde{u} = \varepsilon \circ \widetilde{\mu}$, where $\varepsilon : \wedge V_G \to \mathbb{Q}$ is the augmentation. Then we that $\theta(M_u) \subset M_{\widetilde{u}}$. In fact, since $i^*(h_i) = (-1)^{\tau(|h_i|)}k \circ l(h_i)$ and $\langle h_i, k^\sharp b_*\rangle = \langle \zeta h_i, b_*\rangle$ for $h_i \in H^*(BU)$, it follows that

$$\begin{aligned}
\theta(h_i \otimes b_* - u(h_i \otimes b_*)) &= h_i \otimes k^\sharp b_* - \langle i^* h_i, b_*\rangle \\
&= h_i \otimes k^\sharp b_* - (-1)^{\tau(|h|)}\langle k h_i, b_*\rangle \\
&= h_i \otimes k^\sharp b_* - (-1)^{\tau(|h|)}\langle \zeta h_i, b_*\rangle \\
&= h_i \otimes k^\sharp b_* - \widetilde{u}(h_i \otimes k^\sharp b_*).
\end{aligned}$$

Consider an element $z := x_{i_t} \otimes 1_* - (-1)^{\tau(|u_{t*}|)}x_{j_t} \otimes k^\sharp(u_{t*}) \in Q(\widetilde{E}/M_{\widetilde{u}})$. For any $\alpha \in \wedge W$, $\langle \alpha, d^\sharp k^\sharp u_{t*}\rangle = \langle k d\alpha, u_{t*}\rangle = 0$. Therefore we see that, in $Q(\widetilde{E}/M_{\widetilde{u}})$,

$$\delta_0(z) = dx_{i_t}\otimes 1_* - (-1)^{\tau(|u_{t*}|)}dx_{j_t}\otimes k^\sharp(u_{t*}) = \theta((B\iota)^*(c_{i_t})\otimes 1_* - (B\iota)^*(c_{j_t})\otimes u_{t*}) = 0.$$

The last equality follows from the assumption that $(B\iota)^*(c_{i_t}) \otimes 1_* \equiv (B\iota)^*(c_{j_t}) \otimes u_{t*}$ modulo decomposable elements in $(H^*(BU) : H^*(G/U))/M_u$. By using the notation in Lemma 4.2, we see that

$$\begin{aligned}
H^*Q(\widetilde{\widetilde{\mu}})(z) &= \langle \zeta x_{i_t}, 1_*\rangle - \langle \zeta x_{j_t}, k^\sharp u_{t*}\rangle \\
&= \langle x_{j_t} \otimes 1, 1_*\rangle - \langle \sum X_n \otimes X'_n C_n, k^\sharp u_{t*}\rangle \\
&= x_{t_i} - \sum X_n \langle k(X'_n)C_n, u_{t*}\rangle \;=\; x_{i_t}.
\end{aligned}$$

Observe that $k(X'_n) = 0$. By virtue of Theorem 3.1, we have the result. □

Remark 5.1. As for the latter half of Theorem 3.1, namely, in the case where $(B\iota)^*(c_{i_1})$, ..., $(B\iota)^*(c_{i_s})$ are decomposable, we have a very simple proof of the assertion. In fact, the composite of the evaluation map $ev_0 : \mathrm{aut}_1(G/U) \to G/U$ and the map $\lambda : G \to \mathrm{aut}_1(G/U)$ is nothing but the projection $\pi : G \to G/U$. We consider the model $\eta : (\wedge W, d) \to (\wedge V_G, 0)$ for π mentioned in the proof of Theorem 1.2. Then we see that $HQ(\rho)(x_{i_t}) = x_{i_t}$ for the map $HQ(\rho) : HQ(\wedge W) \to HQ(\wedge V_G) = V_G$. Observe that $x_{i_t} \in HQ(\wedge W)$ since $(B\iota)^*(c_{i_t})$ is decomposable. The same argument as the proof of Theorem 3.1 enables us to conclude that there is a map $\rho : \times_{t=1}^s S^{\deg c_{i_t}-1} \to G$ such that $\pi_* \circ \rho_* : \pi_*(\times_{t=1}^s S_{\mathbb{Q}}^{\deg c_{i_t}-1}) \to \pi_*(G_{\mathbb{Q}})$ is injective. Thus $\lambda_* \circ \rho_*$ is injective in the rational homotopy.

Remark 5.2. In the proof of Theorem 1.2, we construct a model for G of the form $(\wedge(x_1, ..., x_k), 0)$. By virtue of [8, Proposition 15.13], we can choose the elements x_j so that $\sigma^*(c_j) = x_j$, where $\sigma^* : H^*(BG) \xrightarrow{\pi^*} H^*(E_G, G) \underset{\cong}{\xleftarrow{\delta}} H^*(G)$ denotes the cohomology suspension.

In the rest of this section, we describe a suitable model for $\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e\circ p)$ for proving Theorems 1.5 and 1.6.

Let G be a connected Lie group, U a connected maximal rank subgroup and K another connected maximal rank subgroup which contains U. We recall from Section 2 a Sullivan model for the connected component $\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ containing the composite $e \circ p$ of the function space $\mathcal{F}(G/U, (G/K)_{\mathbb{Q}})$, where $e : G/K \to (G/K)_{\mathbb{Q}}$ is the localization map.

Let $\iota_1 : K \to G$ and $\iota_2 : U \to K$ be the inclusions and put $\iota = \iota_1 \circ \iota_2$. Let $\varphi_U : (\wedge W', d) \overset{\simeq}{\to} \Omega\Delta(G/U)$ and $\varphi_K : (\wedge \widetilde{W}, d) \overset{\simeq}{\to} \Omega\Delta(G/K)$ be the Sullivan models for G/U and G/K, respectively, mentioned in the proof of Theorem 1.2; that is, $(\wedge W', d) = (\wedge(h_1, ..., h_l, x_1, ..., x_k), d)$ with $d(x_i) = (B\iota)^*(c_i)$ and $(\wedge\widetilde{W}, d) = (\wedge(e_1, ..., e_s, x_1, ..., x_k), d)$ with $d(x_i) = (B\iota_1)^*(c_i)$. By applying Lifting Lemma to the commutative diagram

$$\begin{array}{ccccc} \wedge V_{BK} & \xrightarrow{(B\iota_2)^*} & \wedge V_{BU} & \rightarrowtail & \wedge W' \\ \downarrow & & & & \downarrow \varphi_U \\ \wedge\widetilde{W} & \xrightarrow[\varphi_K]{} & \Omega\Delta(G/K) & \xrightarrow[\Omega\Delta(p)]{} & \Omega\Delta(G/U), \end{array}$$

we have a diagram

$$(5.1)\qquad \begin{array}{ccccc} H^*(G/U) & \xleftarrow[\simeq]{k} & \wedge W' & \xrightarrow[\simeq]{} & \Omega\Delta(G/U) \\ {\scriptstyle p^*}\uparrow & & {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle \Omega\Delta(p)} \\ H^*(G/K) & \xleftarrow[l]{\simeq} & \wedge\widetilde{W} & \xrightarrow{\simeq} & \Omega\Delta(G/K) \end{array}$$

in which the right square is homotopy commutative and the left that is strictly commutative. In particular, $k(x_i) = 0$, $l(x_i) = 0$ and $\varphi(e_i) = (B\iota_2)^*e_i$.

Let $w : \wedge W \to \wedge\widetilde{W}$ be a minimal model for $(\wedge\widetilde{W}, d)$ and $k^\sharp : (H^*(G/U))^\sharp \to (\wedge W')^\sharp$ the dual to the map k.

As in Remark 2.5(ii), we construct the DGA E' by using $(\wedge W', d) = (B, d_B)$ and $(\wedge W, d)$. Then we have a sequence of quasi-isomorphisms

$$E' \xrightarrow[\simeq]{\gamma:=1\otimes k^\sharp} \wedge(\wedge W \otimes (\wedge W')_*)/I \xrightarrow[\simeq]{w\otimes 1} \wedge(\wedge\widetilde{W} \otimes (\wedge W')_*)/I = \widetilde{E}.$$

Moreover, we choose a model ζ' for the action $G \times G/U \overset{tr}{\to} G/U \overset{p}{\to} G/K$ defined by the composite $\zeta' : \wedge\widetilde{W} \overset{\zeta}{\to} \wedge V_G \otimes \wedge\widetilde{W} \overset{1\otimes\varphi}{\to} \wedge V_G \otimes \wedge W'$, where ζ is the Sullivan representative for the left translation tr mentioned in Lemmas 4.1 and 4.2. Then the map ζ' deduces a model

$$(5.2)\qquad \widetilde{\widetilde{\mu}} : E'/M_u \to \wedge V_G$$

for $\lambda : G \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e \circ p)$ as in Theorem 3.1. Observe that

$$(5.3)\qquad \widetilde{\mu}(v_i \otimes e_j) = (-1)^{\tau(|e_j|)}\langle(1 \otimes \varphi)\zeta w(v_i), k^\sharp e_j\rangle \quad \text{and} \quad u = \varepsilon \circ \widetilde{\mu},$$

where $\varepsilon : \wedge V_G \to \mathbb{Q}$ denotes the augmentation. In the next section, we shall prove Theorem 1.5 by using the model $\widetilde{\widetilde{\mu}} : E'/M_u \to \wedge V_G$.

6. Proof of Theorem 1.5

Let G and U be the Lie group $U(m+k)$ and a maximal rank subgroup of the form $U(m_1) \times \cdots \times U(m_s) \times U(k)$, respectively. Without loss of generality, we can assume that $m_1 \geq \cdots \geq m_s \geq k$. Let K be the subgroup $U(m) \times U(k)$ of U, where $m = m_1 + \cdots + m_s$. Then the Leray-Serre spectral sequence with coefficients in the rational field for the fibration $p : G/U \to G/K$ with fibre K/U collapses at the E_2-term because the cohomologies of G/K and of K/U are algebras generated by elements with even degree. Therefore it follows that the induced map $p^* : H^*(G/K) \to H^*(G/U)$ is a monomorphism. In order to prove Theorem 1.5, we apply Theorem 3.1 to the function space $\mathcal{F}(G/U, G/K, p)$.

Let $P = \{S_1, ..., S_n\}$ be a family consisting of subsets of the finite ordered set $\{1, ..., s\}$ which satisfies the condition that $x < y$ whenever $x \in S_i$ and $y \in S_{i+1}$. Define $\sharp^l P$ to be the number of elements of the set $\{S_j \in P \mid |S_j| = l\}$. Let k be a fixed integer. We call such the family P a $(i_1, ..., i_k)$-type block partition of $\{1, ..., s\}$ if $\sharp^l P = i_l$ for $1 \leq l \leq k$. Let $Q^{(s)}_{i_1,...,i_k}$ denote the number of $(i_1, ..., i_k)$-type block partitions of $\{1, ..., s\}$.

We construct a minimal model explicitly for the homogeneous space $U(m+k)/U(m) \times U(k)$. Assume that $m \geq k$. As in the proof of Theorem 1.2, we have a Sullivan model for $U(m+k)/U(m) \times U(k)$ of the form

$$(\wedge \widetilde{W}, d) = (\wedge(\tau_1, ..., \tau_{m+k}, c_1, ..., c_k, c'_1, ..., c'_m), d)$$

with $d\tau_l = \sum_{i+j=l} c'_i c_j$.

Lemma 6.1. *There exists a sequence of quasi-isomorphisms*

$$\wedge \widetilde{W} \xleftarrow{\simeq} \wedge W_{(1)} \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} \wedge W_{(s)} \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} \wedge W_{(m)}$$

in which, for any s, $(\wedge W_{(s)}, d_{(s)})$ is a DGA of the form

$$\wedge W_{(s)} = \wedge(\tau_{s+1}, ..., \tau_{m+k}, c_1, ..., c_k, c'_{s+1}, ..., c'_m) \quad \textit{with}$$

$$\begin{aligned} d_{(s)}\tau_l &= c'_l + c'_{l-1}c_1 + \cdots + c'_{s+1}c_{l-(s+1)} \\ &+ \sum_{i_1+2i_2+\cdots+ki_k=s} (-1)^{i_1+\cdots+i_k} Q^{(s)}_{i_1,...,i_k} c_1^{i_1} \cdots c_k^{i_k} c_{l-s} \\ &+ \sum_{i_1+2i_2+\cdots+ki_k=s-1} (-1)^{i_1+\cdots+i_k} Q^{(s-1)}_{i_1,...,i_k} c_1^{i_1} \cdots c_k^{i_k} c_{l-(s-1)} \\ &+ \cdots + (-c_1)c_{l-1} + c_l \end{aligned}$$

for $s+1 \leq l \leq m+k$, where $c_i = 0$ for $i < 0$ or $i > k$.

Proof. We shall prove this lemma by induction on the integer s. We first observe that $d\tau_2 = c'_2 - c_1c_1 + c_2$ in $\wedge W_{(1)}$ because $Q^{(1)}_{i_1} = 1$. Define a map $\varphi : \wedge W_{(1)} \to \wedge \widetilde{W}$ by $\varphi(c_i) = c_i$, $\varphi(c'_j) = c'_j$ and $\varphi(\tau_2) = \tau_2 - \tau_1 c_1$. Since $d\tau_1 = c'_1 + c_1$ in $\wedge W$, it follows that φ is a well-defined quasi-isomorphism. Suppose that $(\wedge W_{(s)}, d_{(s)})$ in the lemma can be constructed for some $s \leq m-1$. In particular, we have

$$d_{(s)}\tau_{s+1} = c'_{s+1} + \sum_{0 \leq j \leq s} \sum_{i_1+2i_2+\cdots+ki_k=j} (-1)^{i_1+\cdots+i_k} Q^{(j)}_{i_1,...,i_k} c_1^{i_1} \cdots + c_k^{i_k} c_{s+1-j}.$$

Claim 1.

$$Q^{(s+1)}_{i_1,...,i_k} = Q^{(s)}_{i_1-1,i_2,...,i_k} + Q^{(s-1)}_{i_1,i_2-1,...,i_k} + \cdots + Q^{(s+1-k)}_{i_1,...,i_k-1}.$$

Claim 1 implies that

$$d_{(s)}\tau_{s+1} = c'_{s+1} - \sum_{i_1+2i_2+\cdots+ki_k=s+1} (-1)^{i_1+\cdots+i_k} Q^{(s+1)}_{i_1,\ldots,i_k} c_1^{i_1}\cdots c_k^{i_k}.$$

We define $d_{(s+1)}\tau_{l+1}$ in $\wedge W_{(s+1)}$ by replacing the factor c'_{s+1} which appears in $d_{(s)}\tau_{l+1}$ with $c'_{s+1} - d_{(s)}\tau_{s+1}$, namely,

$$\begin{aligned} d_{(s+1)}\tau_{l+1} &= c'_{l+1} + c'_l c_1 + \cdots + c'_{s+2}c_{(l+1)-(s+2)} \\ &\quad + \sum_{i_1+2i_2+\cdots+ki_k=s+1} (-1)^{i_1+\cdots+i_k} Q^{(s+1)}_{i_1,\ldots,i_k} c_1^{i_1}\cdots c_k^{i_k} c_{l-s} \\ &\quad + \cdots + (-c_1)c_l + c_{l+1}. \end{aligned}$$

Moreover define a map $\varphi : \wedge W_{(s+1)} \to \wedge W_{(s)}$ by $\varphi(c_i) = c_i$, $\varphi(c'_j) = c'_j$ and $\varphi(\tau_{l+1}) = \tau_{l+1} - \tau_{s+1}c_{l+1-(s+1)}$. It is readily seen that φ is a well-defined DGA map. The usual spectral sequence argument enables us to deduce that φ is a quasi-isomorphism. This finishes the proof. □

Proof of Claim 1. Let $\{P_l\}$ denote the family of all $(i_1, ..., i_k)$-type block partitions of $\{1, ..., s+1\}$. We write $P_l = \{S^{(l)}_1, ..., S^{(l)}_{n(l)}\}$. Then $\{P_l\}$ is represented as the disjoint union of the families of $(i_1, ..., i_k)$-type block partitions whose last sets $S^{(l)}_{n(l)}$ consist of j elements, namely, $\{P_l\} = \amalg_{1\le j\le k}\{P_l \mid |S_{n(l)}| = j\}$. It follows that

$$\Big|\{P_l \mid |S_{n(l)}| = j\}\Big| = Q^{(s+1-j)}_{i_1,\ldots,i_{j-1},i_j-1,i_{j+1},\ldots,i_k}.$$

We have the result. □

Recall the minimal model $(\wedge W_{(m)}, d)$ for G/K in Lemma 6.1. We see that $\deg d\tau_{m+1} = \deg c_1^m c_1 = 2(m+1)$ and that $d\alpha = 0$ for any element α with $\deg \alpha \le 2m+1$. This yields that $c_1^m \ne 0$ in $H^*(G/K;\mathbb{Q})$. As mentioned before Lemma 6.1, the induced map $p^* : H^*(G/K) \to H^*(G/U)$ is injective. Therefore we have $(p^*c_1)^s \ne 0$ for $s \le m$.

Let $\widetilde{\widetilde{\mu}} : \widetilde{E}/M_u \to \wedge V_G$ be the model for the map $\lambda : G \to \mathcal{F}(G/U, (G/K)_{\mathbb{Q}}; e\circ p)$ mentioned in the previous section; see (5.2) and (5.3). The following four lemmas are keys to proving Theorem 1.5. The proofs are deferred to the end of this section.

Lemma 6.2. *$\delta_0(\tau_{m+(m-s+1)} \otimes ((p^*c_1)^m)_*) = (-1)^m c_{m-s+1}$ if $m \ne s$.*

Lemma 6.3. *$\widetilde{\widetilde{\mu}}(\tau_{m+(m-s+1)} \otimes ((p^*c_1)^m)_*) = 0$ if $m \ne s$.*

Lemma 6.4. *$\delta_0(\tau_{m+1} \otimes ((p^*c_1)^s)_*) = (-1)^s s c_{m-s+1}$.*

Lemma 6.5. *$\widetilde{\widetilde{\mu}}(\tau_{m+1} \otimes ((p^*c_1)^s)_*) = \tau_{m-s+1}$.*

Proof of Theorem 1.5. By virtue of Lemmas 6.2, 6.3, 6.4 and 6.5, we have

$$\begin{aligned} &\delta_0((-1)^m\tau_{m+(m-s+1)} \otimes ((p^*c_1)^m)_* - \frac{(-1)^s}{s}\tau_{m+1} \otimes ((p^*c_1)^s)_*) \\ &= (-1)^m(-1)^m c_{m-s+1} - \frac{(-1)^s}{s}(-1)^s s c_{m+s-1} = 0 \quad \text{and} \\ &\widetilde{\widetilde{\mu}}((-1)^m\tau_{m+(m-s+1)} \otimes ((p^*c_1)^m)_* - \frac{(-1)^s}{s}\tau_{m+1} \otimes ((p^*c_1)^s)_*) \\ &= -\frac{(-1)^s}{s}\tau_{m-s+1}, \end{aligned}$$

where $s \le m-1$. Theorem 3.1 implies that

$$(\lambda_{\mathbb{Q}})_i : \pi_i(G_{\mathbb{Q}}) \to \pi_i(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}, e\circ p))$$

is injective for $i = \deg \tau_1, ..., \deg \tau_m$. Since $d\tau_l = \sum_{i+j} c_i' c_j$ in $(\wedge W)$, it follows that $d\tau_l$ is decomposable for $l \ge M+1$. Therefore Theorem 1.2 yields that $(\lambda_{\mathbb{Q}})_i$ is also injective for $i = \deg \tau_{m+1}, ..., \deg \tau_{m+k}$.

The latter half of Theorem 1.5 is obtained by comparing the dimension of rational homotopy groups. In fact, it follows from the rational model for $\mathrm{aut}_1(\mathbb{C}P^{m-1})$ mentioned in Example 2.4 that

$$\begin{aligned}\pi_*(\mathrm{aut}_1(\mathbb{C}P^{m-1})\otimes\mathbb{Q})^\sharp &\cong H_*(Q(\widetilde{E}/M_u), \delta_0)\\ &\cong \mathbb{Q}\{y\otimes 1_*, y\otimes (x^1)_*, ..., y\otimes (x^{m-2})_*\}.\end{aligned}$$

This implies that $\dim \pi_i(\mathrm{aut}_1(\mathbb{C}P^{m-1}))\otimes\mathbb{Q} = 1 = \dim \pi_i(SU(m))\otimes\mathbb{Q}$ for $i = 3, ..., 2m-1$. The result follows from the first assertion. This completes the proof. □

We conclude this section with proofs of Lemmas 6.2, 6.3, 6.4 and 6.5.

Proof of Lemma 6.2. We regard the free algebra $\wedge(c_1, ..., c_l)$ as a primitively generated Hopf algebra. Observe that $(c_i^s)_* = \frac{1}{s!}((c_i)_*)^s$. Recall the 0-simplex u in $\Delta E'$ mentioned in (5.3). We have $u(c_j\otimes(p^*c_1)_*) = 0$ if $j \ne 1$ and

$$\begin{aligned}u(c_1\otimes(p^*c_1)_*) &= (-1)^{\tau(|p^*(c_1)|)}k^\sharp(p^*(c_1)_*)(\varphi\circ w(c_1))\\ &= (-1)((p^*(c_1)_*)k\circ\varphi\circ w(c_1) = (-1)((p^*(c_1)_*)p^*c_1) = -1.\end{aligned}$$

For the map k and q, see the diagram (5.1) and the ensuing paragraph. Thus it follows that

$$\begin{aligned}&\delta_0(\tau_{m+(m-s+1)}\otimes((p^*c_1)^m)_*)\\ =\ & c_1^m c_{m-s+1}\cdot D^{(m)}(p^*c_1^m)_* = c_1^m c_{m-s+1}\cdot\frac{1}{m!}D^{(m)}(p^*c_1)_*^m\\ =\ & \frac{1}{m!}c_1^m c_{m-s+1}\cdot\Big((p^*c_1)_*\otimes 1\otimes\cdots\otimes 1 + 1\otimes(p^*c_1)_*\otimes 1\otimes\cdots\otimes 1\\ &\qquad + \cdots + 1\otimes\cdots\otimes 1\otimes(p^*c_1)_*\Big)^m\\ =\ & \frac{1}{m!}c_1^m c_{m-s+1}\cdot(\cdots + m!(p^*c_1)_*\otimes\cdots\otimes(p^*c_1)_*\otimes 1 + \cdots)\\ =\ & u(c_1\otimes(p^*c_1)_*)\cdots u(c_1\otimes(p^*c_1)_*)c_{m-s+1} = (-1)^m c_{m-s+1}.\end{aligned}$$

□

Proof of Lemma 6.3. Recall the quasi-isomorphism $\varphi_{s+1} : \wedge W_{(s+1)} \to \wedge W_{(s)}$ in the proof of Lemma 6.1 which is defined by $\varphi(\tau_{l+1}) = \tau_{s+1} - \tau_{l+1}c_{l+1-(s+1)}$. Let w denote the composite $\varphi_1\circ\cdots\circ\varphi_m : \wedge W = \wedge W_{(m)} \to \wedge\widetilde{W}$. It is readily seen that $w(\tau_{m+(m-s+1)})$ does not have the element c_1^m as a factor if $s \ne m$. Hence using the DGA map ζ' in Lemma 4.1, we have

$$\widetilde{\widetilde{\mu}}(\tau_{m+(m-s+1)}\otimes((p^*c_1)^m)_*) = (-1)^{\tau(|p^*c_1^m|)}\langle(1\otimes\varphi)\zeta w(\tau_{m+(m-s+1)}), k^\sharp(p^*c_1^m)_*\rangle = 0.$$

See (5.1) for the notations. Observe that $H^*(G/K) \cong H^*(\wedge W) \cong \mathbb{Q}[c_1, ..., c_k]$ for $* \le 2m$. This completes the proof. □

Proof of Lemma 6.4. From Lemma 6.1, we see that in $\wedge W_{(m)}$,

$$\begin{aligned} d\tau_{m+1} &= \sum_{i_1+2i_2+\cdots+ki_k=m} (-1)^{i_1+\cdots+i_k} Q^{(m)}_{i_1,\ldots,i_k} c_1^{i_1}\cdots c_k^{i_k} c_1 \\ &\quad + \sum_{i_1+2i_2+\cdots+ki_k=m-1} (-1)^{i_1+\cdots+i_k} Q^{(m-1)}_{i_1,\ldots,i_k} c_1^{i_1}\cdots c_k^{i_k} c_2 \\ &\quad + \cdots + \sum_{i_1+2i_2+\cdots+ki_k=l} (-1)^{i_1+\cdots+i_k} Q^{(l)}_{i_1,\ldots,i_k} c_1^{i_1}\cdots c_k^{i_k} c_{m-l+1} \\ &\quad + \cdots \end{aligned}$$

Suppose that $c_1^{i_1}\cdots c_k^{i_k}c_{m-l+1}\otimes((p^*c_1)^s)_* \neq 0$ in $Q(\widetilde{E}/M_u)$, where $i_1+2i_2+\cdots ki_k = l$. Then we have
(1) $l = m$ and $c_1^{i_1}\cdots c_k^{i_k} = c_1^{s-1}c_{m-s+1}$ or
(2) $l \neq m$, $l = s$ and $c_1^{i_1}\cdots c_k^{i_k} = c_1^s$.
It follows that $(-1)^{i_1+\cdots+i_k}Q^{(m)}_{i_1,\ldots,i_k}c_1^{s-1}c_{m-s+1}c_1 = (-1)^{s-1+1}(s-1)c_1^s c_{m-s+1}$ if $(i_1,...,i_k) = (s-1,0,...,0,1,0,...,0)$ with $i_{m-s+1}=1$ and that $Q^{(s)}_{i_1,\ldots,i_k}c_1^s c_{m-s+1} = (-1)^s\cdot 1\cdot c_1^s c_{m-s+1}$ if $(i_1,...,i_k) = (s,0,...,0)$. This fact allows us to conclude that $\delta_0(\tau_{m+1}\otimes((p^*c_1)^s)_*) = (-1)^s(s-1)c_{m-s+1} + (-1)^s c_{m-s+1} = (-1)^s s c_{m-s+1}$. We have the result. □

Proof of Lemma 6.5. In order to compute $\widetilde{\widetilde{\mu}}$, we determine $\langle(1\otimes\varphi)\zeta w(\tau_{m+1}), k^\sharp(p^*c_1^s)_*\rangle$. With the the same notation as in the proof of Lemma 6.3, we have $w(\tau_{m+1}) = \cdots + (-1)^s\tau_{m-s+1}c_1^s + \cdots$. Lemmas 4.1 and 4.2 imply that

$$\begin{aligned} \zeta(\tau_{m-s+1}c_1^s) &= \psi\otimes 1(\tau_{m-s+1}\otimes c_1^s) \\ &= (\tau_{m-s+1}\otimes 1\otimes 1 + 1\otimes\tau_{m-s+1}\otimes 1 + \sum_n X_n\otimes X_n' C_n)c_1^s. \end{aligned}$$

Thus it follows that

$$\begin{aligned} \widetilde{\widetilde{\mu}}(\tau_{m+1}\otimes((p^*c_1)^s)_*) &= (-1)^{\tau(|p^*c_1^s|)}\langle(1\otimes\varphi)\zeta w(\tau_{m+1}), k^\sharp(p^*c_1^s)_*\rangle \\ &= (-1)^{s+s}\langle(1\otimes\varphi)\zeta(\tau_{m-s+1}c_1^s), k^\sharp(p^*c_1^s)_*\rangle \\ &= \tau_{m-s+1}\langle\varphi(c_1^s), k^\sharp(p^*c_1^s)_*\rangle + \langle\varphi(\tau_{m-s+1}c_1^s), k^\sharp(p^*c_1^s)_*\rangle \\ &\quad + \sum_n X_n\langle\varphi(X_n'C_nc_1^s), k^\sharp(p^*c_1^s)_*\rangle \\ &= \tau_{m-s+1}\langle k\varphi(c_1^s), (p^*c_1^s)_*\rangle + \langle k\varphi(\tau_{m-s+1}c_1^s), (p^*c_1^s)_*\rangle \\ &\quad + \sum_n X_n\langle k\varphi(X_n'C_nc_1^s), (p^*c_1^s)_*\rangle \\ &= \tau_{m-s+1}. \end{aligned}$$

The last equality is extracted from the commutativity of the diagram (5.1). This completes the proof. □

7. Proof of Theorem 1.7.

This section is devoted to proving Theorem 1.7. The inclusion $\iota : \mathrm{aut}_1(X) \to \mathcal{H}_{H,X}$ induces the map $B\iota : B\mathrm{aut}_1(X) \to B\mathcal{H}_{H,X}$ with $B\iota\circ B\lambda_{G,X} = B\psi$. Therefore if $B\psi$ is injective on homology, then so is $B\lambda_{G,X}$.

We shall prove the "only if" part by using the general categorical construction of a classifying space due to May [19, Section 12] and by applying a part of the argument in the proof of [20, Theorem 3.2] to our case.

We here recall briefly the notion of a $\mathcal{O}$-graph; see [20, page 68] for more detail. Let $\mathcal{O}$ be a discrete topological space. Define a $\mathcal{O}$-graph to be a space $\mathcal{A}$ together with maps $S : \mathcal{A} \to \mathcal{O}$ and $T : \mathcal{A} \to \mathcal{O}$. The space $\mathcal{O}$ itself is regarded as $\mathcal{O}$-graph with arrows S and T the identity map. Let $\mathcal{O}Gr$ be the category of $\mathcal{O}$-graphs whose morphisms are maps $h : \mathcal{A} \to \mathcal{A}'$ compatible with maps S and T. Observe that the pullback construction with respect to S and T makes $\mathcal{O}Gr$ a monoidal category. In fact, for $\mathcal{O}$-graphs $\mathcal{A}$ and $\mathcal{A}'$, $\mathcal{A}\Box\mathcal{A}'$ is defined by $\{(a, a') \in \mathcal{A} \times \mathcal{A}' | Sa = Ta'\}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be a left $\mathcal{O}$-graph and a right $\mathcal{O}$-graph, respectively; that is, $\mathcal{X}$ is a space with a map $T : \mathcal{X} \to \mathcal{O}$ and the space $\mathcal{Y}$ admits only a map $S : \mathcal{Y} \to \mathcal{O}$.

Let $\mathcal{M}$ be a monoid in $\mathcal{O}Gr$ the category of $\mathcal{O}$-graphs and $B(\mathcal{Y}, \mathcal{M}, \mathcal{X})$ denote the two-sided bar construction in the sense of May [19, Section 12], which is the geometric realization of the simplicial space B_* with $B_j = \mathcal{Y}\Box\mathcal{M}^{\Box j}\Box\mathcal{X}$. We regard a topological monoid G as that in $\mathcal{O}Gr$ with $\mathcal{O} = \{x\}$ the space of a point. Then the classifying space BG we consider here is regarded as the bar construction $B(x, G, x)$.

Proof of the "only if" part of Theorem 1.7. Let $\iota' : \mathcal{H}_{H,X} \to \mathcal{F}(X, X)$ be the inclusion and $e_* : \mathcal{F}(X, X) \to \mathcal{F}(X, X_{\mathbb{Q}})$ the map induced by the localization $e : X \to X_{\mathbb{Q}}$. Since X is an F_0-space or a space having the rational homotopy type of the product of odd dimensional spheres by assumption, it follows from [2, 3.6 Corollary] and [8, Proposition 32.16] that the natural map $[X, X_{\mathbb{Q}}] \to \mathrm{Hom}(H^*(X_{\mathbb{Q}}; \mathbb{Q}), H^*(X; \mathbb{Q}))$ is bijective. We see that $e \circ \varphi \simeq e$ for any $\varphi \in \mathcal{H}_{H,X}$ Therefore the composite $e_* \circ \iota'$ factors through the connected component $\mathcal{F}(X, X_{\mathbb{Q}}; e)$ of $\mathcal{F}(X, X_{\mathbb{Q}})$. We have a commutative diagram

$$\begin{array}{ccccc} \mathcal{H}_{H,X} & \xrightarrow{e_*\circ\iota'} & & & \\ \uparrow{\scriptstyle \iota} & & \mathcal{F}(X, X_{\mathbb{Q}}; e) & \xleftarrow[\simeq]{e^*} & \mathrm{aut}_1(X_{\mathbb{Q}}) \\ \mathrm{aut}_1(X) & \xrightarrow{e_*} & & & \end{array}$$

in which the induced map e^* is a homotopy equivalence.

Define $\mathcal{O}$ to be the discrete space with two points x and y. Let $\mathcal{M}$ be the monoid in $\mathcal{O}Gr$ defined by $\mathcal{M}(x, x) = \mathrm{aut}_1(X)$, $\mathcal{M}(y, y) = \mathrm{aut}_1(X_{\mathbb{Q}})$ and $\mathcal{M}(x, y) = \mathcal{F}(X, X_{\mathbb{Q}}; e)$ with $\mathcal{M}(y, x)$ empty. Arrows $S, T : \mathcal{M}(a, b) \to \mathcal{O}$ are defined by $S(z) = a$ and $T(z) = b$ for $z \in \mathcal{M}(a, b)$. Moreover we define another monoid $\mathcal{M}'$ in $\mathcal{O}Gr$ by $\mathcal{M}'(x, x) = \mathcal{H}_{H,X}$, $\mathcal{M}'(y, y) = \mathrm{aut}_1(X_{\mathbb{Q}})$, $\mathcal{M}'(x, y) = \mathcal{F}(X, X_{\mathbb{Q}}; e)$ and $\mathcal{M}'(y, x) = \phi$ with arrows defined immediately as mentioned above.

The inclusions $i : \mathrm{aut}_1(X) \to \mathcal{M}$, $j : \mathrm{aut}_1(X_{\mathbb{Q}}) \to \mathcal{M}$, $i' : \mathcal{H}_{H,X} \to \mathcal{M}'$ and $j' : \mathrm{aut}_1(X_{\mathbb{Q}}) \to \mathcal{M}'$ induce the maps between classifying spaces which fit into the commutative diagram

$$(7.1) \qquad \begin{array}{ccccccc} & \nearrow{\scriptstyle B\psi} & B\mathcal{H}_{H,X} & \xrightarrow{Bi'} & B(\mathcal{O}, \mathcal{M}', \mathcal{O}) & \xleftarrow[\simeq]{Bj'} & \\ BG & & \uparrow{\scriptstyle B\iota} & & \uparrow{\scriptstyle B\widetilde{\iota}} & & B\mathrm{aut}_1(X_{\mathbb{Q}}), \\ & \searrow{\scriptstyle B\lambda_{G,X}} & B\mathrm{aut}_1(X) & \xrightarrow[Bi]{} & B(\mathcal{O}, \mathcal{M}, \mathcal{O}) & \xleftarrow[Bj]{\simeq} & \end{array}$$

where $\widetilde{\iota} : \mathcal{M} \to \mathcal{M}'$ is the morphism of monoids in $\mathcal{O}Gr$ induced by the inclusion $\iota : \mathrm{aut}_1(X) \to \mathcal{H}_{H,X}$. The proof of [20, Theorem 3.2] enables us to conclude that maps Bj and Bj' are homotopy equivalences. The map $\Omega((Bj)^{-1} \circ (Bi))$ coincides with the composite $(e^*)^{-1} \circ e_* : \mathrm{aut}_1(X) \to \mathcal{F}(X, X_{\mathbb{Q}}; e) \to \mathrm{aut}_1(X_{\mathbb{Q}})$ up to weak equivalence; see [20, Theorem 3.2(i)]. Moreover the map $e_* : \mathrm{aut}_1(X) \to$

$\mathcal{F}(X, X_{\mathbb{Q}}; e)$ is a localization; see [13]. These facts yield that $\pi_*(\Omega Bi) \otimes \mathbb{Q}$ is an isomorphism and hence so is $\pi_*(Bi) \otimes \mathbb{Q}$. Thus the localized map $(Bi)_{\mathbb{Q}}$ is a weak equivalence. This implies that $(Bi)_* : H_*(B\mathrm{aut}_1(X); \mathbb{Q}) \to H_*(B(\mathcal{O}, \mathcal{M}, \mathcal{O}); \mathbb{Q})$ is an isomorphism. The commutative diagram (7.1) enables us to conclude that $H_*(B\psi; \mathbb{Q})$ is injective if so is $H_*(B\lambda_{G,X}; \mathbb{Q})$. This completes the proof. □

As we pointed out in the introduction, [16, Proposition 4.8] follows from Theorems 1.5 and 1.7. In fact, suppose that M is the flag manifold $U(m)/U(m_1) \times \cdots \times U(m_l)$ and $G = SU(m)$. Then as is seen in Remark 7.1 below $(\lambda_{G,M})_* : \pi_*(BG) \otimes \mathbb{Q} \to \pi_*(B\mathrm{aut}_1(M)) \otimes \mathbb{Q}$ is injective if and only if $(B\lambda_{G,M})^* : H^*(BG) \to H^*(B\mathrm{aut}_1(M))$ is surjective.

Remark 7.1. Suppose that M is a homogeneous space of the form G/H for which rank G = rank H. The main theorem in [31] due to Shiga and Tezuka implies that $\pi_{2i}(\mathrm{aut}_1(M)) \otimes \mathbb{Q} = 0$ for any i. Thus $H^*(B\mathrm{aut}_1(M); \mathbb{Q})$ is a polynomial algebra generated by the graded vector space $(sV)^\sharp$, where $(sV)_l = \pi_{l-1}(\mathrm{aut}_1(M))$. Therefore the dual map to the Hurewicz homomorphism $\Xi^\sharp : H^*(B\mathrm{aut}_1(M); \mathbb{Q}) \to \mathrm{Hom}(\pi_*(B\mathrm{aut}_1(M)), \mathbb{Q})$ induces an isomorphism on the vector space of indecomposable elements; see [8, page 173] for example. Thus the commutative diagram

$$\begin{array}{ccc} H^*(BG; \mathbb{Q}) & \xleftarrow{(B\lambda_{G,M})^*} & H^*(B\mathrm{aut}_1(M); \mathbb{Q}) \\ \Xi^\sharp \downarrow & & \downarrow \Xi^\sharp \\ \mathrm{Hom}(\pi_*(BG), \mathbb{Q}) & \xleftarrow[((B\lambda_{G,M})_*)^\sharp]{} & \mathrm{Hom}(\pi_*(B\mathrm{aut}_1(M)), \mathbb{Q}) \end{array}$$

yields that the map $(B\lambda_{G,M})^*$ is surjective if G is rationally visible in $\mathrm{aut}_1(M)$.

8. A function space model description of the Kedra-McDuff μ-classes

In this section, $H^*(-)$ denotes the cohomology with coefficients in rational field unless otherwise explicitly mentioned. In order to define μ-classes due to Kedra and McDuff, we first recall the coupling class.

Let M be a k-dimensional manifold. Consider the Leray-Serre spectral sequence $\{E_r, d_r\}$ for a fibration $M \xrightarrow{i} E \xrightarrow{\pi} B$ for which $\pi_1(B)$ act trivially on $H^k(M) = \mathbb{Q}$. Let $\{F^pH^*\}_{p \geq 0}$ denote the filtration of $\{E_r, d_r\}$. Then the integration along the fibre (the cohomology push forward) $\pi! : H^{p+k}(E) \to H^p(B)$ is defined by the composite

$$\begin{array}{ccc} H^{p+k}(E; \mathbb{Q}) = F^0H^{p+k} = F^pH^{p+k} \twoheadrightarrow E_\infty^{p,q} \rightarrowtail \cdots \rightarrowtail E_2^{p,q} & \cong & H^p(B; H^k(M; \mathbb{Q})) \\ & & \cong \downarrow \\ & & H^p(B; \mathbb{Q}). \end{array}$$

Let (M, a) be a $2m$-dimensional c-symplectic manifold and $\mathcal{G}$ denote the monoid $\mathcal{H}_a$ or $\mathrm{aut}_1(M)$. Let $M \xrightarrow{i} M_{\mathcal{G}} \xrightarrow{\pi} B\mathcal{G}$ be the universal M-fibration; see [19, Proposition 7.9]. Proposition 8.1 below follows from the proofs of [15, Proposition 2.4.2] and [16, Proposition 3.1].

Proposition 8.1. *Suppose that $H^1(M) = 0$, then the element $a \in H^2(M)$ is extendable to an element $\overline{a} \in H^2(M_{\mathcal{G}})$. Moreover, there exists a unique element*

$\widetilde{a} \in H^2(M_{\mathcal{G}})$ that restricts to $a \in H^2(M)$ and such that $\pi!(\widetilde{a}^{m+1}) = 0$. In fact the element $\widetilde{a}$ has the form

$$\widetilde{a} = \overline{a} - \frac{1}{n+1}\pi^*\pi!(\overline{a}^{m+1}).$$

The class $\widetilde{\omega}$ in Proposition 8.1 is called *the coupling class.*

Definition 8.2. [16, Section 3.1] [15, Section 2.4] [29] We define $\mu_k \in H^{2k}(B\mathcal{G})$, which is called kth μ-class, by

$$\mu_k := \pi!(\widetilde{a}^{m+k}),$$

where $\widetilde{a}$ is the coupling class.

Remark 8.3. Let $(\mathcal{H}_a)_1$ be the identity component of the group $\mathcal{H}_a$ of diffeomorphisms which fix the class a. The naturality of the integration along the fibre implies that the kth Kedra-McDuff μ-class of $B(\mathcal{H}_a)_1$ is extendable to the class μ_k in $H^{2k}(B\mathrm{aut}_1(M))$.

In order to prove Theorem 1.8, we first introduce a spectral sequence. Let $C_*(X)$ denote the normalized chain complex of a space X. By definition, the total space $M_{\mathrm{aut}_1(M)}$ of the universal M-fibration is regarded as the realization $|B_*(*, \mathrm{aut}_1(M), M)|$ of the geometric bar construction $B_*(*, \mathrm{aut}_1(M), M)$, which is a simplicial topological space with $B_i(*, \mathrm{aut}_1(M), M) = * \times \mathrm{aut}_1(M)^{\times i} \times M$; see [19, Proposition 7.9]. The result [19, Theorem 13.9] allows us to obtain natural quasi-isomorphisms which connect with $C_*(|B(*, \mathrm{aut}_1(M), M)|)$ and the algebraic bar construction of the form $B(C_*(*), C_*(\mathrm{aut}_1(M)), C_*(M))$ for which

$$B(C_*(*), C_*(\mathrm{aut}_1(M)), C_*(M))_k = \oplus_{i+j=k}(C_*(*) \otimes C_*(\mathrm{aut}_1(M))^{\otimes i} \otimes C_*(M))_j.$$

Moreover the Eilenberg-Zilber map gives rise to a quasi-isomorphism from the bar complex to the total complex $\mathrm{Total}C_*(B_*(*, \mathrm{aut}_1(M), M))$. Observe that

$$\mathrm{total}C_*(B_*(*, \mathrm{aut}_1(M), M))_k = \oplus_{i+j=k}C_jB_i(*, \mathrm{aut}_1(M), M).$$

In consequence, by virtue of [8, Corollary 10.10], we have natural quasi-isomorphisms which connect $C^*(M_{\mathrm{aut}_1(M)}) = C^*(|B_*(*, \mathrm{aut}_1(M), M)|)$ with the total complex of a double complex $\mathcal{B} = \{\mathcal{B}^{i,j}, d_i, \delta_j\}$ of the form

$$\mathcal{B}^{i,j} = \left(A_{PL}(\mathrm{aut}_1(M)^{\times i} \times M)\right)^j.$$

In particular, $d_0 : \mathcal{B}^{0,*} \to \mathcal{B}^{1,*}$ is regarded as the map

$$(pr_2)^* - ev^* : A_{PL}(M) \to A_{PL}(\mathrm{aut}_1(M) \times M),$$

where the maps pr_2 and ev form $\mathrm{aut}_1(M) \times M \to M$ to M are the second projection and the evaluation map, respectively.

We define a double complex $\mathcal{C} = \{\mathcal{C}^{i,j}, d_i, \delta_j\}$ by truncating the double complex $\{\mathcal{B}^{i,j}\}$ for $i \geq 2$; that is, $\mathcal{C}^{i,j} = \mathcal{B}^{i,j}$ for $i \leq 0, 1$ and $\mathcal{C}^{i,j} = 0$ for $i \geq 2$.

Let $\{E_r, d_r\}$ be the Eilenberg-Moore spectral sequence converging to $H^*(M_{\mathrm{aut}_1(M)})$ with

$$E_2^{*,*} \cong \mathrm{Cotor}^{*,*}_{H^*(\mathrm{aut}_1(M))}(\mathbb{Q}, H^*(M))$$

as an algebra. Observe that this spectral sequence is constructed with the double complex $\mathcal{B}$. The double complex $\mathcal{C}$ gives rise to a spectral sequence $\{\widetilde{E}_r, \widetilde{d}_r\}$ converging to $H^*(\mathrm{Total}(\mathcal{C}))$. Moreover, we see that the projection $q : \mathcal{B} \to \mathcal{C}$ induces the morphism of the spectral sequences

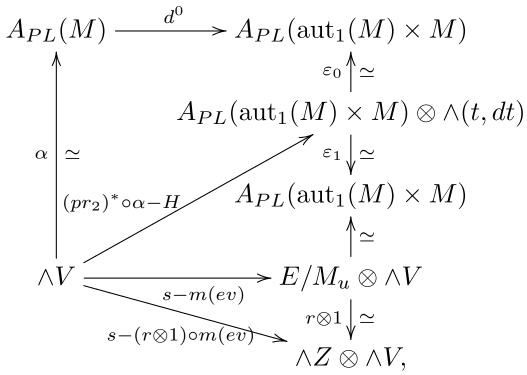
$$\{q_r\} : \{E_r, d_r\} \to \{\widetilde{E}_r, \widetilde{d}_r\}$$

and the morphism $\widehat{q} : H^*(M_{\mathrm{aut}_1(M)}) \to H^*(\mathrm{Total}(\mathcal{C}))$ of algebras.

Lemma 8.4. *For any $\alpha \in H^*(M_{\mathrm{aut}_1(M)})$, $\widehat{q}(\alpha) = 0$ if and only if $\alpha \in F^2H^*$. Here $\{F^pH^*\}_{p\geq 0}$ denotes the filtration of $H^*(M_{\mathrm{aut}_1(M)})$ associated with the spectral sequence $\{E_r, d_r\}$.*

Proof. By construction, we see that $q_2 : E_2^{p,*} \to \widetilde{E}_2^{p,*}$ is bijective for $p = 0$ and injective for $p = 1$. Since the truncated spectral sequence $\{\widetilde{E}_r, \widetilde{d}_r\}$ collapses at the E_1-term, it follows that the map $q_r^{p,*}$ for $3 \leq r \leq \infty$ and $p \leq 2$ is injective. This completes the proof. □

Let $(\wedge V, d) \overset{\simeq}{\to} A_{PL}(M)$ be a minimal model for M. Recall from Proposition 2.3 the DGA map $m(ev)$ which is a model for the evaluation map. Observe that, in the construction of the Brown-Szczarba model E/M_u, the identity map of $\wedge V$ is chosen as a Sullivan representative for the identity map of M; see (2.8). Then we have a commutative diagram



where $H : \wedge V \to A_{PL}(\mathrm{aut}_1(M) \times M) \otimes \wedge(t, dt)$ denotes the homotopy between the model $m(ev)$ for the evaluation map ev and the induced map $A_{PL}(ev)$ up to quasi-isomorphisms, $r : (E/M_u, \delta) \to (\wedge Z, \delta)$ is a retraction to a minimal model $(\wedge Z, \delta)$ for $\mathrm{aut}_1(M)$ and s stands for the inclusion into the second factor.

Let $\mathcal{D}$ be the double complex associated with the DGA map

$$s - (r \otimes 1) \circ m(ev) : (\wedge V, d_{\wedge V}) \to (\wedge Z, \delta) \otimes (\wedge V, d_{\wedge V}).$$

The usual spectral sequence argument allows us to conclude that $H^*(\mathrm{total}\mathcal{C}) \cong H^*(\mathrm{total}\mathcal{D})$ as an algebra. By using this identification, we shall prove Theorem 1.8.

Proof of Theorem 1.8. We take a minimal model of the form $(\wedge V, d) = (\wedge(y, a), d)$ with $d(y) = a^{m+1}$. Recall from Example 2.4 the model $(E/M_u, \delta)$; that is,

$$E/M_u = \wedge(a \otimes 1_*, y \otimes (a^s)_*; 0 \leq s \leq m),$$

$\delta(a \otimes 1_*) = 0$ and $\delta(y \otimes (a^s)_*) = (-1)^s \begin{pmatrix} m+1 \\ s \end{pmatrix} (a \otimes 1_*)^{m+1-s}$. Therefore we can define a retraction

$$r : E/M_u \overset{\simeq}{\to} (\wedge Z, d) = \big(\wedge (y \otimes 1_*, y \otimes (a)_*, ..., y \otimes (a^{m-1})_*), 0\big)$$

by $r(a \otimes 1_*) = 0 = r(y \otimes (a^m)_*)$. Thus we have

$$H^*(\mathrm{aut}_1(M)) \cong \wedge(y \otimes 1_*, y \otimes (a)_*, ..., y \otimes (a^{m-1})_*).$$

This yields that, in the Leray-Serre spectral sequence for the universal fibration $\mathrm{aut}_1(M) \to E\mathrm{aut}_1(M) \xrightarrow{\pi'} B\mathrm{aut}_1(M)$, the element $y \otimes (a^{m-i})_*$ is transgressive for $1 \le i \le m$. In fact we have a commutative diagram

$$\begin{array}{ccccc}
\pi_*(\mathrm{aut}_1(M))^\sharp & \underset{\cong}{\xrightarrow{\delta^\sharp}} & \pi_{*+1}(E_{\mathrm{aut}_1(M)}, \mathrm{aut}_1(M))^\sharp & \underset{\cong}{\xleftarrow{\pi'^{\sharp}_*}} & \pi_{*+1}(B\mathrm{aut}_1(M))^\sharp \\
\uparrow h_G^\sharp & & \uparrow (-1)^{*+1}h^\sharp & & \uparrow (-1)^{*+1}h_{BG}^\sharp \\
H^*(\mathrm{aut}_1(M)) & \underset{\cong}{\xrightarrow{\delta}} & H^{*+1}(E_{\mathrm{aut}_1(M)}, \mathrm{aut}_1(M)) & \xleftarrow{\pi'^*} & H^{*+1}(B\mathrm{aut}_1(M)),
\end{array}$$

where h_G, h and h_{BG} denote the duals to Hurewicz maps. Since $H^*(\mathrm{aut}_1(M))$ and $H^*(B\mathrm{aut}_1(M))$ are exterior algebra and a polynomial algebra, respectively, it follows that h_G and h_{BG} are isomorphisms on subvector spaces of indecomposable elements. Thus we see that the element $y \otimes (a^{m-i})_*$ is transgressive because the map $\delta^{-1}\pi^*$ is the transgression by definition. Hence the element $y \otimes (a^{m-i})_*$ is primitive for $1 \le i \le m$; see [25, Section 7 (2.27)].

Let $\{\widehat{E}_r, \widehat{d}_r\}$ be the Eilenberg-Moore spectral sequence converging to the cohomology $H^*(B\mathrm{aut}_1(M))$ with $\widehat{E}_2^{*,*} \cong \mathrm{Cotor}^{*,*}_{H^*(\mathrm{aut}_1(M))}(\mathbb{Q}, \mathbb{Q})$. Since the element $y \otimes (a^{m-i})_*$ is primitive for $1 \le i \le m$, it follows that

$$\widehat{E}_2^{*,*} = \mathbb{Q}\big[[y \otimes 1_*], [y \otimes (a)_*], ..., [y \otimes (a^{m-1})_*]\big],$$

where bideg $[y \otimes (a^{m-i})_*] = (1, 2i+1)$. This implies that, as algebras,

$$H^*(B\mathrm{aut}_1(M)) \cong \mathrm{Total}(\widehat{E}_2^{*,*}) \cong \mathbb{Q}\big[[y \otimes 1_*], [y \otimes (a)_*], ..., [y \otimes (a^{m-1})_*]\big].$$

Recall the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ converging to $H^*(M_{\mathrm{aut}_1(M)})$. We see that

$$\begin{aligned}
E_2^{*,*} &\cong \mathrm{Cotor}^{*,*}_{H^*(\mathrm{aut}_1(M))}(\mathbb{Q}, H^*(M)) \\
&\cong \mathbb{Q}\big[[y \otimes 1_*], ..., [y \otimes (a^{m-1})_*]\big] \otimes \mathbb{Q}[[\ ]a]/\big(([\ ]a)^{m+1}\big)
\end{aligned}$$

as algebras. For dimensional reasons, we see that the spectral sequence $\{E_r, d_r\}$ collapses at the E_2-term. Let $M \xrightarrow{i} M_{\mathrm{aut}_1(M)} \xrightarrow{\pi} B\mathrm{aut}_1(M)$ be the universal M-fibration. The naturality of the spectral sequence enables us to conclude that $\pi^*([y \otimes (a^{m-i})_*]) = [y \otimes (a^{m-i})_*]$ in $H^*(M_{\mathrm{aut}_1(M)})$. In the total complex $\mathcal{D}$, we have

$$\begin{aligned}
(d_{\wedge V} \pm (s - (r \otimes 1) \circ m(ev)))y &= d_{\wedge V}(y) + (-1)^{\deg y}(s - (r \otimes 1) \circ m(ev))(y) \\
&= ([\ ]a)^{m+1} + \sum_{i=1}^{m} (-1)^{m-i}[y \otimes (a^{m-i})_*] \otimes a^{m-i}.
\end{aligned}$$

This implies that $\widehat{q}\big(([\ ]a)^{m+1} + \sum_{i=1}^m (-1)^{m-i}[y \otimes (a^{m-i})_*] \otimes ([\ ]a)^{m-i}\big) = 0$ in $H^*(\mathrm{Total}(\mathcal{C}))$. Therefore it follows form Lemma 8.4 that

$$([\ ]a)^{m+1} \equiv \sum_{i=1}^{m} (-1)^{m-i+1}[y \otimes (a^{m-i})_*] \otimes ([\ ]a)^{m-i}$$

modulo the ideal generated by $\pi^*(H^+(B_{\mathrm{aut}_1(M)}))\cdot\pi^*(H^+(B_{\mathrm{aut}_1(M)}))$ in $H^*(M_{\mathrm{aut}_1(M)})$. Since $\pi!(([\ ]a)^{m+1}) = 0$, we can choose the element $([\ ]a)$ as the coupling class $\widetilde{a}$

mentioned in Proposition 8.1. By definition, for $2 \leq k \leq m+1$, we see that

$$\begin{aligned}\mu_k &= \pi!(\widetilde{a}^{m+k}) \;=\; \pi!(\widetilde{a}^{m+1}\cdot\widetilde{a}^{k-1}) \\ &= \pi!\Big((\sum_{i=1}^{m}(-1)^{m-i+1}[y\otimes(a^{m-i})_*]\,\widetilde{a}^{m-i})\cdot\widetilde{a}^{k-1}\Big) \\ &= \pi!(\cdots+(-1)^{m-k}[y\otimes(a^{m-k+1})_*]\,\widetilde{a}^{m}+\cdots) \\ &= (-1)^{m-k}[y\otimes(a^{m-k+1})_*]\end{aligned}$$

modulo decomposable elements. We have the result. □

Remark 8.5. Let M be a c-symplectic manifold of the form $(\mathbb{C}P^m\times\mathbb{C}P^n, a_1+a_2)$. We see that the subalgebra of $H^*(B\mathrm{aut}_1(M))$ generated by μ-classes is proper. To see this, we choose minimal models $(\wedge(y_1,a_1), dy_1=a_1^{m+1})$ and $(\wedge(y_2,a_2), dy_2=a_1^{n+1})$ for $\mathbb{C}P^m$ and $\mathbb{C}P^n$. Suppose that $m\geq n$. Then the same argument as in [14, Example 3.6] allows us to conclude that $\mathrm{aut}_1(M)$ admits a minimal model of the form

$$\begin{aligned}\wedge(y_1\otimes 1_*, y_1\otimes(a_1)_*, ..., y_1\otimes(a_1^{m-1})_*, y_2\otimes 1_*, y_2\otimes(a_2)_*, ..., y_2\otimes(a_2^{n-1})_*, \\ y_1\otimes(a_2^{n-i})_*, y_2\otimes(a_2^{m-j})_*; 0\leq i\leq n-1, m-n\leq j\leq m-1)\end{aligned}$$

with the trivial differential. This yields that

$$H^2(B\mathrm{aut}_1(M))\cong\begin{cases}\mathbb{Q}\{[y_2\otimes(a_1^n)_*]\} & \text{if } m>n\\ \mathbb{Q}\{[y_1\otimes(a_2^m)_*],[y_2\otimes(a_1^n)_*]\} & \text{if } m=n\end{cases}$$

Thus any μ-class does not detect an element in $H^2(B\mathrm{aut}_1(M))$ since the degrees of the μ-classes are greater than 4.

In order to give topological description to algebraic generators of $B\mathrm{aut}_1(M)$ which come from the Brown-Szczarba model, we need other construction of characteristic classes, for example the similar way to that of the Miller-Morita-Mumford classes; see [16, page 147]. The consideration in this direction is not pursued in this paper.

Proof of Proposition 1.9. It is well-known that $(B\iota)^*(p_i)=(-1)^i(\chi^2p'_{i-1}+p'_i)$ for the induced map $(B\iota)^*: H^*(BSO(2m+1))\to H^*(B(SO(2)\times SO(2m-1))$, where p'_i is the ith Pontrjagin class in $H^*(B(SO(2m-1))\cong\mathbb{Q}[p'_1,...,p'_{m-1}]$; see [25].

As in the proof of Theorem 1.2, we can construct a Sullivan model $(\wedge W, d)$ for the Grassmannian manifold $M := SO(2m+1)/SO(2)\times SO(2m-1)$ by using the induced map $(B\iota)^*$. It follows that $\wedge W=\wedge(\chi,p'_1,...,p'_{m-1},\tau_2,\tau_4,...,\tau_{2m})$ and $d(\tau_{2i})=(-1)^i(\chi^2p'_{i-1}+p'_i)$ for $1\leq i\leq m$. We see that there exists a quasi-isomorphism $w:(\wedge(\chi,\tau_{2m}), d\tau_{2m}=-\chi^{2m})\to(\wedge W,d)$ such that $w(\chi)=\chi$ and

$$w(\tau_{2m})=\chi^{2(m-1)}\tau_2+\cdots+\chi^2\tau_{2(m-1)}+\tau_{2m}.$$

In view of the model for $\lambda_{G,M}$ in (5.2), it follows from Lemma 4.2 that

$$\begin{aligned}Q(\widetilde{\widetilde{\mu}})(\tau_{2m}\otimes(\chi^{2l})_*)) &= (-1)^{\tau(|\chi^{2l}|)}\langle\zeta'\circ w(\tau_{2m}),(\chi^{2l})_*\rangle \\ &= \langle\chi^{2(m-1)}\tau_2+\cdots+\chi^2\tau_{2(m-1)}+\tau_{2m},\ (\chi^{2l})_*\rangle \\ &= \tau_{2(m-l)}.\end{aligned}$$

This implies that $SO(2m+1)$ is rationally visible in M with respect to the map $\lambda_{SO(2m+1),M}$. Thus the naturality of the Eilenberg-Moore spectral sequence allows us to deduce that $(B\lambda_{SO(2m+1),M})^*([\tau_{2m}\otimes(\chi^{2l})_*]) = [\tau_{2(m-l)}]$. The description of the μ-classes in the proof of Theorem 1.8 yields that $(B\lambda_{SO(2m+1),M})^*(\mu_{2(m-l)}) \equiv (B\lambda_{SO(2m+1),M})^*([\tau_{2m}\otimes(\chi^{2l})_*])$ modulo decomposable elements. It follows form Remark 5.2 that $\sigma^*(p_{m-l}) = \tau_{2(m-l)}$. By virtue of [11, Corollary 3.12], we have $\sigma^*([\tau_{2(m-l)}]) = \tau_{2(m-l)}$. In the case where $G = SO(2m+1)$, the cohomology suspension σ^* is injective on the vector subvector space of indecomposable elements. This yields that $[\tau_{2(m-l)}] = p_{m-l}$. We have the result. $\square$

9. The sets $vd(G, G/U)$ of visible degrees in Tables 1 and 2

In this section, we deal with the visible degrees described in Tables 1 and 2 in Introduction.

For the cases (1) and (16), we have the results from the proof of Proposition 1.9. For the case where the homogeneous space G/U has the rational homotopy type of the sphere, the assertion on the visible degrees follow from the latter half of Theorem 1.2. In fact, the argument in Example 1.4 does work well to obtain such results. The details are left to the reader. The results for (11) and for (17) follow from Theorems 1.5 and 1.6, respectively. We are left to verify the visible degrees for the cases (5), (6), (6)' and (19).

(19). Let $\iota : Spin(9) \to F_4$ be the inclusion map. Without loss of generality, we can assume that the induce map

$$(B\iota)^* : H^*(BF_4;\mathbb{Q}) = \mathbb{Q}[y_4, y_{12}, y_{16}, y_{24}] \to H^*(BSpin(9);\mathbb{Q}) = \mathbb{Q}[y_4, y_8, y_{12}, y_{16}]$$

satisfies the condition that $(B\iota)^*(y_i) = y_i$ for $i = 4, 12, 16$ and $(B\iota)^*(y_{24}) = y_8^3$, where $\deg y_i = i$. This fact follows from a usual argument with the Eilenberg-Moore spectral sequence for the fibration $\mathcal{L}P^2 \to BSpin(9) \xrightarrow{B\iota} BF_4$. By virtue of Lemmas 4.1 and 4.2, we see that there exists a model for the linear action $F_4 \times \mathcal{L}P^2 \to \mathcal{L}P^2$ of the form

$$\zeta : (\wedge(x'_{23}) \otimes \wedge(y_8), d) \to (\wedge(x_3, x_{11}, x_{15}, x_{23}) \otimes \wedge(x'_{23} \otimes \wedge(y_8), d')$$

with $\zeta(x'_{23}) = x_{23} \otimes 1 \otimes 1 + 1 \otimes x'_{23} \otimes 1$, where $d(x'_{23}) = y_8^3$, $d'(x_j) = 0$ for $j = 3, 11, 15, 23$. In fact, for dimensional reasons, we write $\zeta(x'_{23}) = 1 \otimes x'_{23} \otimes 1 + x_{23} \otimes 1 \otimes 1 + cx_{15} \otimes 1 \otimes y_8$ with a rational number c. By definition, we see that $\zeta = \psi \otimes 1$, where ψ denotes the DGA map in Lemma 4.2. Since the image of each element of degree less than 24 by $(B\iota)^*$ does not have the element y_8 as a factor, it follows that $c = 0$. Observe that $\wedge V_{BF_4}$-action on $\wedge V_{BSpin(9)}$ is induced by the map $(B\iota)^*$. The dual to the map $(\lambda_*)_i : \pi_i(F_4) \otimes \mathbb{Q} \to \pi_i(\mathrm{aut}_1(F_4/Spin(9))) \otimes \mathbb{Q}$ is regarded as the induced map

$$H(Q(\widetilde{\mu})) : H^*(Q(\widetilde{E}/M_u), \delta_0) \to V_G = \mathbb{Q}\{x_3, x_{11}, x_{15}, x_{23}\}$$

in Theorem 3.1. We see that

$$Q(\widetilde{E}/M_u) = \mathbb{Q}\{y_8 \otimes 1_*, x'_{23} \otimes 1_*, x'_{23} \otimes (y_8)_*, x'_{23} \otimes (y_8^2)_*\},$$

$\delta_0(x'_{23} \otimes (y_8^2)_*) = 3y_8 \otimes 1_*$, $\delta_0(x'_{23} \otimes 1_*) = \delta_0(x'_{23} \otimes (y_8^1)_*) = 0$; see Example 2.4. Moreover the direct computation with (3.1) shows that $Q(\widetilde{\mu})(x'_{23} \otimes 1_*) = \pm x_{23}$ and $Q(\widetilde{\mu})(x'_{23} \otimes (y_8)_*) = 0$. This implies that $vd(F_4, \mathcal{L}P^2) = \{23\}$.

(5). The inclusion $\iota : SO(4) \to G_2$ induces the ring homomorphism

$$(B\iota)^* : H^*(BG_2) \cong \mathbb{Q}[y_4, y_{12}] \to H^*(BSO(4)) \cong \mathbb{Q}[p_1, \chi],$$

where $\deg p_1 = 4$ and $\deg \chi = 4$. It is immediate that $(B\iota)^*(y_{12})$ is decomposable for dimensional reasons. Form Example 2.4, we see that $\pi_*(\mathrm{aut}_1(\mathbb{H}P^2)) \cong \mathbb{Q}\{y \otimes 1_*, y \otimes (x^1)_*\}$, where $\deg y \otimes 1_* = 11$ and $\deg y \otimes (x^1)_* = 7$. It follows from Theorem 1.2 that $vd(G_2, G_2/SO(4)) = \{11\}$.

(6). Let T^2 be the standard maximal torus of $U(2)$. We assume that $G_2 \supset U(2) \supset T^2$ without loss of generality. Then the inclusion $W(G_2) \supset W(U(2))$ of Weyl groups gives the inclusions

$$\begin{array}{ccccc} \mathbb{Q}[t_1,t_2]^{W(G_2)} & \rightarrowtail & \mathbb{Q}[t_1,t_2]^{W(U(2))} & \rightarrowtail & \mathbb{Q}[t_1,t_2] \\ \cong\uparrow & & \cong\uparrow & & \cong\uparrow \\ H^*(BG_2) & & H^*(BU(2)) & & H^*(BT^2). \end{array}$$

The result [32, page 212, Example 3] implies that there exist generators y_4, y_{12} of $H(BG_2)$ such that $H(BG_2) \cong \mathbb{Q}[y_4, y_{12}]$ and $y_4 = t_1^2 - t_1t_2 + t_2^2$, $y_{12} = (t_1t_2^2 - t_1^2t_2)^2$ in $\mathbb{Q}[t_1,t_2]^{W(G_2)}$. Since the Chern classes $c_1, c_2 \in H^*(BU(2))$ are regarded as $t_1 + t_2$ and t_1t_2, respectively in $\mathbb{Q}[t_1,t_2]^{W(U(2))}$, it follows that

$$(B\iota)^*(y_4) = c_1^2 - 3c_2 \quad \text{and} \quad (B\iota)^*(y_{12}) = c_1^2c_2^2 - 4c_2^3,$$

where $\iota : U(2) \to G_2$ is the inclusion. Put $\tilde{c}_2 = -\frac{1}{3}c_1^2 + c_2$. Then we see that $(B\iota)^*(-\frac{1}{3}y_4) = \tilde{c}_2$ and

$$(B\iota)^*(y_{12}) = -\frac{1}{27}c_1^6 - \frac{2}{3}c_1^4\tilde{c}_2 - 3c_1^2\tilde{c}_2^2 - 4\tilde{c}_2^3.$$

By the direct computation implies that

$$\begin{aligned} & (B\iota)^*(-\frac{1}{3}y_4) \otimes 1_* - (B\iota)^*(y_{12}) \otimes (-\frac{3}{2})(c_1^4)_* \\ = \ & \tilde{c}_2 \otimes 1_* + \frac{3}{2}\big(-\frac{1}{27}c_1^6 - \frac{2}{3}c_1^4\tilde{c}_2 - 3c_1^2\tilde{c}_2^2 - 4\tilde{c}_2^3\big) \otimes (c_1^4)_* \\ \equiv \ & \tilde{c}_2 \otimes 1_* - \tilde{c}_2 \otimes 1_* \equiv 0 \end{aligned}$$

modulo decomposable elements in $(H^*(BU(2)) : H^*(G_2/U(2)))/M_u$. It is immediate that $(B\iota)^*(y_{12})$ is decomposable. By virtue of Theorem 1.2, we have $vd(G_2, G_2/U(2)) = \{3, 11\}$. The same argument works well to obtain the result for the case (6)'.

Acknowledgments. I am particularly grateful to Kojin Abe, Kohhei Yamaguchi and Masaki Kameko for valuable comments on this work and to Hiroo Shiga for drawing my attention to this subject. I thank the anonymous referee for showing me a very simple proof of the latter half of Theorem 1.2, which is described in Remark 5.1.

10. Appendix. Extensions of characteristic classes

For a space X, let X^δ denote the space with the discrete topology whose underlying set is the same as that of X. Let M be a homogeneous space admitting an action of a connected Lie group G. In this section, we consider cohomology classes of $B(\mathrm{Diff}_1(M))^\delta$ as well as those of $B(\mathrm{aut}_1(M))^\delta$, which detect familiar characteristic classes via the induced map

$$(B\lambda)^* : H^*(B(\mathrm{Diff}_1(M))^\delta; \mathbb{Q}) \to H^*(BG^\delta; \mathbb{Q}).$$

Let G be a real semi-simple connected Lie group with finitely many components and $h : G \to G_{\mathbb{C}}$ the complexification of G. One has a commutative diagram

$$\begin{array}{ccccc} H^*(BG_{\mathbb{C}}) & \xrightarrow{h^*} & H^*(BG) & \xrightarrow{j^*} & H^*(BG^{\delta}) \\ & \nearrow_{B\lambda^*} & \uparrow & & \uparrow \\ H^*(B\mathrm{aut}_1(G/U)) & \longrightarrow & H^*(B\mathrm{Diff}_1(G/U)) & \longrightarrow & H^*(B(\mathrm{Diff}_1(G/U))^{\delta}) \end{array} ,$$

where $j : G^{\delta} \to G$ stands for the natural map. The result [22, THEOREM 2] asserts that the kernel of j^* is equal to the ideal generated by the positive dimensional elements in $\mathrm{Im}h^*$.

As an example, we consider the case where $G = SL(2m;\mathbb{R})$ and U is a maximal rank subgroup of $SO(2m)$ with $(QH^*(BU;\mathbb{Q}))^{2m} = 0$, for example U is a maximal torus of $SO(2m)$. Then Milnor's result mentioned above allows us to conclude that the Euler class χ of $H^*(BSL(2m;\mathbb{R}))$ survives in $H^*(B(G^{\delta}))$; see [24]. Moreover Theorem 1.2 yields that $(B\lambda)^* : H^i(B\mathrm{aut}_1(G/U)) \to H^i(BG)$ is surjective for $i = 2m$; see also Remark 7.1. Thus the class $\chi \in H^*(B(G)^{\delta})$ is extendable to an element $\widetilde{\chi}$ of $H^*(B(\mathrm{Diff}_1(G/U))^{\delta})$. This implies that the rational cohomology algebra $H^*(B(\mathrm{Diff}_1(G/U))^{\delta})$ contains the polynomial algebra $\mathbb{Q}[\widetilde{\chi}]$ generated by the extended element $\widetilde{\chi}$. In particular, it follows that

$$H^{2mi}(B(\mathrm{Diff}_1(G/U))^{\delta})) \neq 0$$

for $i \geq 0$.

Remark 10.1. The result [22, Corollary] yields that the induced homomorphism $(Bj)^* : H^*(BG;\mathbb{Z}) \to H^*(BG^{\delta};\mathbb{Z})$ is injective. Thus the same argument as above does work well to find nontrivial elements in the cohomology $H^*(B(\mathrm{Diff}_1(G/U))^{\delta}))$ for an appropriate subgroup U of G if $H^*(BG;\mathbb{Z})$ is torsion free.

References

[1] P. L. Antonelli, D. Burghelea and P. J. Kahl, The non-finite homotopy type of some diffeomorphism groups, Topology **11**(1972), 1-49.

[2] M. Arkowitz and G. Lupton, On finiteness of subgroups of self-homotopy equivalences, Contemp. Math. **181**(1995), 1-25.

[3] A. K. Bousfield and V. K. A. M. Gugenheim, On PL de Rham theory and rational homotopy type, Memoirs of AMS **179**(1976).

[4] E. H. Brown Jr and R. H. Szczarba, On the rational homotopy type of function spaces, Trans. Amer. Math. Soc. **349**(1997), 4931-4951.

[5] U. Buijs and A. Murillo, Basic constructions in rational homotopy theory of function spaces, Ann. Inst. Fourier (Grenoble) **56**(2006), 815-838.

[6] F. T. Farrell and W. C. Hsiang, On the rational homotopy groups of the diffeomorhism groups of discs, spheres and aspherical manifolds, Proceedings of Symposia in Pure Math. **32**(1978), 325-337.

[7] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, Real homotopy theory of Kähler manifolds, Invent. Math. **29**(1975), 245-274.

[8] Y. Félix, S. Halperin and J. -C. Thomas, Rational Homotopy Theory, Graduate Texts in Mathematics **205**, Springer-Verlag.

[9] Y. Félix and J. -C. Thomas, The monoid of self-homotopy equivalences of some homogeneous spaces, Exposiotiones Math. **12**, 305-322.

[10] Gottlieb, D. H.: On fibre spaces and the evaluation map, Ann. of Math. (2)**87**(1968), 42-55.

[11] V. K. A. M. Gugenheim and J. P. May, On the theory and applications of differential torsion products, Mem. Amer. Math. Soc. 142, 1974.

[12] A. Haefliger, Rational homotopy of space of sections of a nilpotent bundle, Trans. Amer. Math. Soc. **273**(1982), 609-620.

[13] P. Hilton, G. Mislin and J. Roitberg, Localization of nilpotent groups and spaces, North Holland Mathematics Studies 15, North Holland, New York, 1975.
[14] Y. Hirato, K. Kuribayashi and N. Oda, A function space model approach to the rational evaluation subgroups, Math. Z. **258**(2008), 521-555.
[15] T. Januszkiewicz and J. Kedra, Characteristic classes of smooth fibrations, preprint (2002) `arXiv:math/0209288v1`.
[16] J. Kedra and D. McDuff, Homotopy properties of Hamiltonian group actions, Geometry & Topology, **9**(2005), 121-162.
[17] K. Kuribayashi, A rational model for the evaluation map, Georgian Mathematical Journal **13**(2006), 127-141.
[18] G. Lupton and J. Oprea, Cohomologically symplectic space: Toral actions and the Gottlieb group, Trans. Amer. Math. Soc. **347**(1995), 261-288.
[19] J.P.May, Classifying spaces and fibrations, Mem. Amer. Math. Soc. 155, 1975.
[20] J.P.May, Fiberwise localization and completion, Trans. Amer. Math. Soc. **258**(1980), 127-146.
[21] D. McDuff and D. Salamon, Introduction to Symplectic Topology, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1995.
[22] J. Milnor, On the homology of Lie groups made discrete, Comment. Math. Helvetici **58**(1983), 72-85.
[23] J. Milnor and J. -C. Moore, On the structure of Hopf algebras. Ann. of Math. **81** (1965), 211-264.
[24] J. Milnor and J. D. Stasheff, Characteristic classes, Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, 1974.
[25] M. Mimura and H. Toda, Topology of Lie groups. II. Translations of Mathematical Monographs, 91. American Mathematical Society, Providence, RI, 1991.
[26] D. Notbohm and L. Smith, Fake Lie groups and maximal tori. III, Math. Ann **290**(1991), 629-642.
[27] D. Notbohm and L. Smith, Rational Homotopy of the space of homotopy equivalences of a flag manifold, Algebraic topology, Homotopy and Group Cohomology, Proceedings barcelona 1990, J. Aguadé, M. Castellt and F. R. Cohen editors, Lecture Notes in Math. 1509 (1992), 301-312.
[28] A. L. Oniscik, Transitive compact transformation groups, Mat. Sb. **60**(1963), 447-485 [Russian], Amer. Math. Soc. Transl. **55**(1966), 153-194.
[29] A. G. Reznikov, Characteristic classes in symplectic topology, Selecta Math. **3**(1997), 601–642.
[30] S. Sasao, The homotopy of MAP$(\mathbb{C}P^m, \mathbb{C}P^n)$, J. London Math. Soc. **8**(1974), 193-197.
[31] H. Shiga and M. Tezuka, Rational fibrations, homogeneous spaces with positive Euler characteristics and Jacobians. Ann. Inst. Fourier (Grenoble) **37**(1987), 81-106.
[32] L. Smith, Polynomial invariants of finite groups. Research Notes in Math., 6. A K Peters, Ltd., Wellesley, MA, 1995.
[33] S. B. Smith, Rational evaluation subgroups, Math. Z. **221**(1996), 387-400.
[34] S. B. Smith, Rational classification of simple function space components for flag manifolds, Canad. J. Math. **49**(1997), 855–864.
[35] K. Yamaguchi, On rational homotopy of MAP$(\mathbb{H}P^m, \mathbb{H}P^n)$, Kodai Math. J. **6**(1983), 279-288.