MODEL STRUCTURES ON EXTRIANGULATED CATEGORIES

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Abstract. A model structure was originally introduced by Quillen as an abstraction of the category of topological spaces. However, it also plays an important role in the representation of algebras in connection with Grothendieck-Verdier's derived category. Afterwards, Hovey proved a one-to-one correspondence theorem between model structures and cotorsion pairs in abelian categories. Šťovíček provided an exact version of Hovey's theorem which includes model structures whose homotopy categories are derived categories. In this talk, we focus an extriangulated version of Hovey's theorem which was proved by Nakaoka-Palu. Finally, we will explain the theory behind Nakaoka-Palu's approach, which serves a localization theory of extriangulated categories.

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1. Introduction

The *Gabriel-Zisman localization* serves a foundation where one would like to consider certain morphisms to be isomorphisms [GZ67]. It is defined by formally inverting such morphisms, e.g., *weak equivalences* and *quasi-isomorphisms*. But in this case, morphisms in the obtained quotient category are rather difficult to handle. Especially, given an

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additive category $\mathcal{C}$, the quotient category $\mathcal{C}[\mathcal{W}^{-1}]$ with respect to a class $\mathcal{W}$ of morphisms is not necessarily additive. In this note, we focus on the two different types of localization which allow us to get a better description of morphisms.

(1) *Model structure*
——Model structures were introduced by Quillen as an abstraction of the category of topological spaces [Qui67, Qui69]. It forms the foundation of homotopy theory. The *homotopy category* $\mathsf{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ is the quotient category with respect to the class of weak equivalences.

(2) Localization of *Extriangulated category*
——Grothendieck-Verdier introduced the derived category as the foundation of the homological algebra, which is defined to be a *Verdier quotient* of a *triangulated category* [Ver67]. Nakaoka-Palu's extriangulated category is a unification of triangulated categories and exact categories [NP19].

We shall present some relations between (1) and (2). We are particularly interested in the *admissible* model strucutre which belongs to both (1) and (2). Basic examples of admissible model structures can be found on the module category of a Frobenius algebra and the category of (chain) complexes of modules.

(1) The module category $\mathsf{Mod}\,A$ over a Frobenius algebra A admits an admissible model structure. Then, the homotopy category is equivalent to the (projectively) stable category $\underline{\mathsf{Mod}}A$.
(2) The category $\mathsf{C}(A)$ of complexes over an algebra A admits an admissible model structure. Then, the homotopy category is equivalent to the derived category $\mathsf{D}(A)$.

These examples show that the model structure will play important roles in the representation theory of algebras. A central concept to define and understand the admissible model structure is the *cotortion pair*, which were firstly introduced in [Sal79] to devide a given abelian category in two smaller pieces. A cotorsion pair on an abelian category $\mathcal{C}$ is defined to be a pair $(\mathcal{U}, \mathcal{V})$ of subcategories with some conditions involving $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$. Afterwards, a triangulated analog was considered in [IY08] and the unification of them was introduced by Nakaoka-Palu in terms of extriangulated categories. It was firstly proved by Hovey that, in the level of abelian categories, admissible model structures bijectively correspond to special cotorsion pairs. The first aim of this note is to prove an extriangulated version of Hovey's correspondence theorem stated as below.

Theorem 1.1. [NP19, Section 5] *Let $\mathcal{C}$ be an extriangulated category satisfying* (WIC). *There is a one-to-one correspondence between admissible model structures on $\mathcal{C}$ and twin cotorision pairs $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ satisfying* $\mathsf{Cone}(\mathcal{V}, \mathcal{S}) = \mathsf{CoCone}(\mathcal{V}, \mathcal{S})$.

In this case, the canonical functor $\mathcal{C} \to \mathsf{Ho}(\mathcal{C})$ induces an equivalence $\dfrac{\mathcal{T} \cap \mathcal{U}}{[\mathcal{S} \cap \mathcal{V}]} \xrightarrow{\sim} \mathsf{Ho}(\mathcal{C})$. *Furthermore, $\mathsf{Ho}(\mathcal{C})$ is naturally equipped with a triangulated structure.*

The second aim is to explain the triangulated structure on the homotopy category $\mathsf{Ho}(\mathcal{C})$ in a more conceptual framework. The extriangulated localization was formulated in [NOS21], which is a simultaneous generalization of the Verdier localization and the Serre localization. With this view in mind, the associated functor $Q : \mathcal{C} \to \mathsf{Ho}(\mathcal{C})$ can be regarded as an extriangulated localization of $\mathcal{C}$ with respect to the biresolving subcategory $\mathsf{Cone}(\mathcal{V}, \mathcal{S}) = \mathsf{CoCone}(\mathcal{V}, \mathcal{S})$.

This note is organized as follows: Section 2 will be devoted to prepare basic results on model categories, extriangulated categories, and cotorsion pairs. Section 3 proves

Theorem 1.1. In Section 4, as an example of admissible model structures, we investigate the construction of the derived categories in terms of cotorsion pairs. Finally, in Section 5, we formulate the extriangulated localization which contains the Verdier localization and Serre localization. This new framework provides a good understanding of the homotopy category with respect to an admissible model structure.

Convention 1.2. *Throughout, all categories and functors are assumed to be additive.* For a category $\mathcal{C}$, we denote by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ the morphism space from X to Y. This is also denoted by $\mathcal{C}(X, Y)$ for short. All subcategories are always full. For subcategories $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$ means $\mathrm{Hom}_{\mathcal{C}}(U, V) = 0$ for any $U \in \mathcal{U}$ and $V \in \mathcal{V}$. If $\mathcal{C}$ is an extriangulated category, the notion $\mathbb{E}(\mathcal{U}, \mathcal{V})$ is used in a similar meaning. We denote by $\mathcal{U}^{\perp}$ the full subcategory of objects X with $\mathbb{E}(\mathcal{U}, X) = 0$. The notion ${}^{\perp}\mathcal{U}$ is used in the dual meaning. $\mathrm{Mor}(\mathcal{C})$ is the category (or the class) of morphisms in $\mathcal{C}$. All algebra is assumed to be finite dimensional over a fixed field k.

2. Preliminary

This section is devoted to recall basic facts on model categories, extriangulated categories and cotorsion pairs.

2.1. **Model categories.**

A *model category* was introduced by Quillen [Qui67], and it has been modified by some authors from various viewpoints, e.g. [Qui69, DS95, BR07]. A model category is usually assumed to have all finite limits and colimits, and more often to have all limits and colimits. However, to include model structures on triangulated and exact categories, we need to drop such assumptions.

Definition 2.1. Let $\mathcal{C}$ be an additive category. A *model structure* on $\mathcal{C}$ is a triple $(\mathsf{Cof}, \mathcal{W}, \mathsf{Fib})$ of classes of morphisms in $\mathcal{C}$ satisfying the following axioms. The morphisms in Fib (resp. $\mathsf{Cof}, \mathcal{W}$) is called the *fibrations* (resp. *cofibrations, weak equivalences*). We denote by $\mathsf{wFib} = \mathsf{Fib} \cap \mathcal{W}$ the class of *trivial fibrations* and by $\mathsf{wCof} = \mathsf{Cof} \cap \mathcal{W}$ the class of *trivial cofibrations.*

(M1) [Two out of three axiom] For a composed morphism gf in $\mathcal{C}$, two of f, g and gf are weak equivalences, then so is the third.

(M2) [Retract axiom] For a morphism $f : X \to Y$ in $\mathcal{C}$ together with the following commutative diagram, $f \in \mathcal{W}$ implies $g \in \mathcal{W}$.

$$\begin{array}{ccccc} & & \overset{\mathsf{id}}{\frown} & & \\ X' & \hookrightarrow & X & \twoheadrightarrow & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Y' & \hookrightarrow & Y & \twoheadrightarrow & Y' \\ & & \underset{\mathsf{id}}{\smile} & & \end{array}$$

(M3) [Lifting axiom] For a commutative diagram in $\mathcal{C}$ of the following shape

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{g} & Y' \end{array}$$

with $i \in \mathsf{Cof}$ and $p \in \mathsf{Fib}$, if i or p belongs to $\mathcal{W}$, there exists a morphism $h : Y \to X'$ such that $h \circ i = f$ and $p \circ h = g$.

(M4) [Factorization axiom] $\mathrm{Mor}(\mathcal{C}) = \mathsf{wFib} \circ \mathsf{Cof} = \mathsf{Fib} \circ \mathsf{wCof}$ holds. Precisely, any morphism $f \in \mathrm{Mor}(\mathcal{C})$ admits factorizations $f = f_2 f_1 = f_2' f_1'$ with $f_1 \in \mathsf{Cof}, f_2 \in \mathsf{wFib}, f_1' \in \mathsf{wCof}$ and $f_2' \in \mathsf{Fib}$

We call an additive category equipped with a model structure a *model category.* Furthermore, we sometimes refer the following stronger version of (M4).

(M4') [Functorial factorization axiom] A *functorial factorization* is a pair (α, β) of functors $\mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$ such that $f = \beta(f) \circ \alpha(f)$ for all $f \in \mathrm{Mor}(\mathcal{C})$. There is two functorial factorizations (α, β) and (γ, δ) such that $\alpha(f) \in \mathsf{Cof}, \beta(f) \in \mathsf{wFib}, \gamma(f) \in \mathsf{wCof}$ and $\delta(f) \in \mathsf{Fib}$ for any $f \in \mathrm{Mor}(\mathcal{C})$.

For a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{g} & Y' \end{array}$$

a morphism $h : Y \to X'$ which satisfies $h \circ i = f$ and $p \circ h = g$ is called a *lifting.* If such a lifting exists, i is said to have the *left lifting property* against p and p is said to have the *right lifting property* against i.

Lemma 2.2. (1) *A morphism is a fibration iff it has the right lifting property against all trivial cofibrations.*
(2) *A morphism is a cofibration iff it has the left lifting property against all trivial fibrations.*
(3) *A morphism is a trivial fibration iff it has the right lifting property against all cofibrations.*
(4) *A morphism is a trivial cofibration iff it has the left lifting property against all fibrations.*
(5) $\mathcal{W} = \mathsf{wFib} \circ \mathsf{wCof}$.

Although (co)limits do not necessarily exist, the existence of initial and terminal objects allow us to define cofibrant and fibrant objects.

Definition 2.3. Let $(\mathsf{Cof}, \mathcal{W}, \mathsf{Fib})$ be a model structure. We associate the full subcategories:

- $\mathcal{C}_{\mathsf{tcof}} := \{S \in \mathcal{C} \mid 0 \to S \text{ is a trivial cofibration}\}$.
- $\mathcal{C}_{\mathsf{fib}} := \{T \in \mathcal{C} \mid T \to 0 \text{ is a fibration}\}$.
- $\mathcal{C}_{\mathsf{cof}} := \{U \in \mathcal{C} \mid 0 \to U \text{ is a cofibration}\}$.
- $\mathcal{C}_{\mathsf{tfib}} := \{V \in \mathcal{C} \mid V \to 0 \text{ is a trivial fibration}\}$.

The objects in $\mathcal{C}_{\mathsf{tcof}}$ (resp. $\mathcal{C}_{\mathsf{fib}}, \mathcal{C}_{\mathsf{tcof}}, \mathcal{C}_{\mathsf{fib}}$) are called *trivially cofibrant* (resp. *fibrant, cofibrant, trivially fibrant*) objects. In addition, we put $\mathcal{C}_{\mathsf{cf}} := \mathcal{C}_{\mathsf{cof}} \cap \mathcal{C}_{\mathsf{fib}}$. Note that, by the retract axiom (M2), it follows that the subcategories $\mathcal{C}_{\mathsf{tcof}}, \mathcal{C}_{\mathsf{fib}}, \mathcal{C}_{\mathsf{cof}}$ and $\mathcal{C}_{\mathsf{tfib}}$ are closed under direct summands.

The following lemma shows that a model structure provides nice approximations.

Definition 2.4. Let $\mathcal{C}$ be a category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$.
(1) For an object $X \in \mathcal{C}$, a morphism $f : D_X \to X$ from $D_X \in \mathcal{D}$ is called a *right $\mathcal{D}$-approximation* of X if it induces a surjective morphism $(\mathcal{D}, f)$. Dually, a left $\mathcal{D}$-approximation is defined.
(2) If any object $X \in \mathcal{C}$ admits a right $\mathcal{D}$-approximation, $\mathcal{D}$ is called a *contravariantly finite* subcategory of $\mathcal{C}$. A *covariantly finite* subcategory is defined dually.

Lemma 2.5. *For any object X:*

1. *there exists a right $\mathcal{C}_{\mathsf{tcof}}$-approximation $S_X \xrightarrow{\pi'} X$ which is a fibration;*
2. *there exists a left $\mathcal{C}_{\mathsf{fib}}$-approximation $X \xrightarrow{\iota} T^X$ which is a trivial cofibration;*
3. *there exists a right $\mathcal{C}_{\mathsf{cof}}$-approximation $U_X \xrightarrow{\pi} X$ which is a trivial fibration;*
4. *there exists a left $\mathcal{C}_{\mathsf{tfib}}$-approximation $X \xrightarrow{\iota'} V^X$ which is a cofibration.*

A morphism ι (resp. ι') is called a (resp. trivially*)* fibrant replacement *of X and a morphism π (resp.π') is called a (resp.* trivially*)* cofibrant replacement *of X.*

Proof. We only prove (1). The others are similar. For an object X, we consider a zero map $0 \to X$ and resolve it as $0 \xrightarrow{i} S_X \xrightarrow{p} X$ by the factorization system $\mathrm{Mor}(\mathcal{C}) = \mathsf{Fib} \circ \mathsf{wCof}$. Since $i : 0 \to S_X$ is a trivial cofibration, by definition, we have that S_X is trivially cofibrant. It remains to show that $p : S \to X$ is a right $\mathcal{S}_X$-approximation. To this end, consider a morphism to $S' \xrightarrow{f} X$ with $S' \in \mathcal{S}$ and a commutative square

$$\begin{array}{ccc} 0 & \longrightarrow & S_X \\ \downarrow & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & X \end{array}$$

Since $0 \to S'$ is a trivially cofibration and p is a fibration, we have a lifting $h : S' \to S_X$. □

Remark 2.6. The factorization axiom (M4) is not required to be functorial, so an assignment $X \mapsto S_X$ in Lemma 2.5 does not give rise to a functor. Similar assertions hold for other assignments in the lemma.

Definition 2.7. Let $\mathcal{C}$ be a model category. The *homotopy category* $\mathsf{Ho}(\mathcal{C})$ is defined to be the Gabriel-Zisman localization of $\mathcal{C}$ with respect to the class $\mathcal{W}$ of weak equivalences.

To provide an alternative description of the homotopy category $\mathsf{Ho}(\mathcal{C})$, we recall the homotopy relations.

Definition 2.8. Let $f, g : X \to Y$ are maps in a model category $\mathcal{C}$.

1. A *cylinder object* for X is a factorization of the summation map $(1_X \ 1_X) : X \coprod X \to X$ into a cofibration $X \coprod X \xrightarrow{(i_0 \ i_1)} X'$ followed by a weak equivalence $X' \to X$.
2. We say f and g are *left homotopic* if there is some cylinder object X' and a map $H : X' \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$. This defines a relation called *left hmotopy*, denoted $\sim^l$, on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

The *path object* and the *right homotopy* $\sim^r$ are defined dually.

In fact, if X is cofibrant and Y is fibrant, then $\sim^l = \sim^r$. In this case, we simply call it *homotopy* and denote by $\sim$. Then, by putting $\pi \mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\sim$, we obtain the natural functor $\pi : \mathcal{C}_{\mathsf{cf}} \to \mathcal{C}_{\mathsf{cf}}/\sim$ which is an identity on objects.

Theorem 2.9. *The composition $\mathcal{C}_{\mathsf{cf}} \hookrightarrow \mathcal{C} \xrightarrow{Q} \mathsf{Ho}(\mathcal{C})$ induces an equivalence $(\mathcal{C}_{\mathsf{cf}}/\sim) \xrightarrow{\sim} \mathsf{Ho}(\mathcal{C})$.*

Example 2.10. Let A be a finite dimensional self-injective k-algebra[1]. Then the category $\mathsf{mod}\, A$ of finite dimensional (right) A-modules admits a model strucutre $(\mathsf{Cof}, \mathcal{W}, \mathsf{Fib})$:

- Cof = the set of all monomorphisms;

[1]An algebra A is said to be *self-injective* if $\mathsf{proj}\, A = \mathsf{inj}\, A$. A Frobenius algebra is self-injective.

- Fib = the set of all epimorphisms;
- $\mathcal{W} = \{f \mid f$ factors through an object $P \in \mathsf{proj}\, A\}$.

In this case, the full subcategory $\mathcal{C}_{\mathsf{cf}}$ of cofibrant and fibrant objects is the whole category $\mathsf{mod}\, A$. The homotopy relation on $\mathcal{C}_{\mathsf{cf}}$ defines the (two-sided) ideal $[\mathsf{proj}\, A]$ consisting of all morphisms having a factorization through an object in $\mathsf{proj}\, A$. Hence, the homotopy category is equivalent to the stable category $\underline{\mathsf{mod}}A = \mathsf{mod}\, A/[\mathsf{proj}\, A]$.

We end this subsection by recalling the definition of the Quillen functors.

Proposition 2.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and consider an adjoint pair $(F, G) : \mathcal{C} \to \mathcal{D}$, i.e., $F : \mathcal{C} \to \mathcal{D}$ is a left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Then, the following are equivalent.*

(1) *F preserves fibrations and trivial fibrations.*
(2) *G preserves cofibrations and trivial cofibrations.*

Under the above equivalent conditions, we call F a left Quillen functor *and G a* right Quillen functor*. Moreover, the pair (F, G) is called a* Quillen adjunction*.*

2.2. **Extriangulated category.** An *extriangulated categories* was introduced by Nakaoka-Palu [NP19] as a simultaneous generalization of triangulated categories and exact categories. It is defined to be a triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ of

- an additive category $\mathcal{C}$;
- an additive bifunctor $\mathbb{E} : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Ab}$, where Ab is the category of abelian groups;
- a correspondence $\mathfrak{s}$ which associates each equivalence class of a sequence of the form $X \to Y \to Z$ in $\mathcal{C}$ to an element in $\mathbb{E}(Z, X)$ for any $Z, X \in \mathcal{C}$,

which satisfies some 'additivity' and 'compatibility'. It is simply denoted by $\mathcal{C}$ if there is no confusion. We refer to [NP19, Section 2] for its detailed definition. We only recall a part of the 'compatibility' analogous to the octahedral axiom.

(ET4) Let $\delta \in \mathbb{E}(D, A)$ and $\delta' \in \mathbb{E}(F, B)$ be $\mathbb{E}$-extensions realized by

$$\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{f'} D], \quad \mathfrak{s}(\delta') = [B \xrightarrow{g} C \xrightarrow{g'} F].$$

Then there exist an object $E \in \mathcal{C}$, a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e} \\ & & F & = & F \end{array}$$

in $\mathcal{C}$, and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(E, A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities:

(1) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $f'_* \delta'$;
(2) $d^* \delta'' = \delta$;
(3) $f_* \delta'' = e^* \delta'$.

where we put $f'_* \delta' := \mathbb{E}(F, f')(\delta'), d^* \delta'' := \mathbb{E}(d, A)(\delta'')$ and so on.

Remark 2.12. A triangulated category and an exact category are special cases of an extriangulated category.

(1) By putting $\mathbb{E} := \mathcal{C}(-, -[1])$, a triangulated category $(\mathcal{C}, [1], \Delta)$ can be regarded as an extriangulated category [NP19, Prop. 3.22]. In this case, we say that an extriangulated category *corresponds to a triangulated category.*

(2) By putting $\mathbb{E} := \mathrm{Ext}^1_{\mathcal{C}}(-,-)$, an exact category $\mathcal{C}$ can be regarded as an extriangulated category [NP19, Example 2.13]. In this case, we say that an extriangulated category *corresponds to an exact category.*

We shall use the following terminology in many places.

Definition 2.13. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

(1) We call an element $\delta \in \mathbb{E}(Z, X)$ an $\mathbb{E}$-*extension*, for any $X, Z \in \mathcal{C}$; A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ corresponding to an $\mathbb{E}$-extension $\delta \in \mathbb{E}(Z, X)$ is called an $\mathfrak{s}$-*conflation.* In addition, f and g are called an $\mathfrak{s}$-*inflation* and an $\mathfrak{s}$-*deflation*, respectively. The pair $\langle X \xrightarrow{f} Y \xrightarrow{g} Z, \delta\rangle$ is called an $\mathfrak{s}$-*triangle.*
(2) An object $P \in \mathcal{C}$ is said to be *projective* if for any deflation $g : Y \to Z$, the induced morphism $\mathcal{C}(P, g) : \mathcal{C}(P, Y) \to \mathcal{C}(P, Z)$ is surjective. We denote by $\mathsf{proj}(\mathcal{C})$ the subcategory of projectives in $\mathcal{C}$. An *injective* object and $\mathsf{inj}(\mathcal{C})$ are defined dually.
(3) We say that $\mathcal{C}$ *has enough projectives* if for any $Z \in \mathcal{C}$, there exists a conflation $X \to P \to Z$ with P projective. *Having enough injectives* are defined dually.
(4) Let $\mathcal{C}$ have enough projectives and injectives. If the class of projectives coincides with the class of injectives, $\mathcal{C}$ is said to be *Frobenius.*

Remark 2.14. (1) If an extriangulated category $\mathcal{C}$ corresponds to a triangulated category, then $\mathcal{C}$ has enough projectives and injectives. The projectives and inejctives are nothing but the zero objects.
(2) If an extriangulated category $\mathcal{C}$ corresponds to an exact category, projectives and injectives agree with usual definitions.

Keeping in mind the triangulated case, we introduce the notions cone and cocone.

Proposition 2.15. *Let $\mathcal{C}$ be an extriangulated category. For an inflation $f \in \mathcal{C}(X, Y)$, take a conflation $X \xrightarrow{f} Y \xrightarrow{g} Z$, and denote this Z by* $\mathsf{Cone}(f)$. *We call* $\mathsf{Cone}(f)$ *a* cone *of f. Then,* $\mathsf{Cone}(f)$ *is uniquely determined up to isomorphism. The dual notion* cocone $\mathsf{CoCone}(g)$ *exists.*

Furthermore, for any subcategories $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{C}$, we define a full subcategory $\mathsf{Cone}(\mathcal{V}, \mathcal{U})$ to be the one consisting of objects X appearing in a conflation $V \to U \to X$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. A subcategory $\mathsf{CoCone}(\mathcal{V}, \mathcal{U})$ is defined dually. Next, we recall the notion of weak pullback in extriangulated categories. Consider a conflation $X \xrightarrow{f} Y \xrightarrow{g} Z$ corresponding to $\mathbb{E}$-extension $\delta \in \mathcal{E}(Z, X)$ and a morphism $z : Z' \to Z$. Put $\delta' := \mathbb{E}(z, X)(\delta)$ and consider a corresponding conflation $X \to E \to Z'$. Then, there exists a commutative diagram of the following shape

$$\begin{array}{ccccc} X & \longrightarrow & E & \xrightarrow{g'} & Z' \\ \| & & {\scriptstyle z'}\downarrow & \text{(Pb)} & \downarrow{\scriptstyle z} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

The commutative square (Pb) is called a *weak pullback of g along z* which is a generalization of the pullback in exact categories and the homotopy pullback in triangulated categories. An induced sequence $E \xrightarrow{\binom{-g'}{z'}} Z' \oplus Y \xrightarrow{(z\ g)} Z$ is a coflation. The dual notion *weak pushout* exists and it will be denoted by (Po).

We often refer the following condition, analogous to the weak idempotent completeness ([Buh, Prop. 7.6])

Condition 2.16 (WIC)**.** For an extriangulated category $\mathcal{C}$, we consider the following conditions.

(1) Let $g \circ f$ be a composed morphism in $\mathcal{C}$. If $g \circ f$ is an inflation, then so is f.
(2) Let $g \circ f$ be a composed morphism in $\mathcal{C}$. If $g \circ f$ is an deflation, then so is g.

Remark 2.17. (1) If $\mathcal{C}$ corresponds to a triangulated category, (WIC) is automatically satisfied.
(2) If $\mathcal{C}$ corresponds to an abelian category, (WIC) is automatically satisfied.
(3) If $\mathcal{C}$ corresponds to an exact category, (WIC) is equivalent to the usual condition so that $\mathcal{C}$ is weakly idempotent complete.

We end this subsection by mentioning that the class of extriangulated categories is closed under certain operations.

Proposition 2.18. [NP19, Rem. 2.18, Prop. 3.30] *Let $\mathcal{C}$ be an extriangulated category.*

(1) *Any extension-closed subcategory admits an extriangulated structure induced from that of $\mathcal{C}$.*
(2) *Let $\mathcal{I}$ be a full additive subcategory, closed under isomorphisms which satisfies $\mathcal{I} \subseteq \mathsf{proj}(\mathcal{C}) \cap \mathsf{inj}(\mathcal{C})$, then the ideal quotient $\mathcal{C}/[\mathcal{I}]$ has an extriangulated structure, induced from that of $\mathcal{C}$.*

2.3. **Cotorsion pairs.** The aim of this subsection is briefly recalling the definition of a cotorsion pair and explaining how it relates to the theory of derived category in the representation of algebras.

Definition 2.19. Let $\mathcal{C}$ be an extriangulated category. A (complete) *cotorsion pair* $(\mathcal{U}, \mathcal{V})$ is a pair of full subcategories on $\mathcal{C}$ which is closed under direct summands and satisfies that $\mathbb{E}(\mathcal{U}, \mathcal{V}) = 0$ and, for any $X \in \mathcal{C}$, there exist conflations

$$X \to V^X \to U^X, \quad V_X \to U_X \to X$$

with $U^X, U_X \in \mathcal{S}, V^X, V_X \in \mathcal{V}$. The above defining sequences are called *approximation sequences.*

Example 2.20. (1) Let $\mathcal{C}$ be an exact category. Then, $(\mathsf{proj}\,\mathcal{C}, \mathcal{C})$ forms a cotorsion pair if and only if $\mathcal{C}$ has enough projectives.
(2) Denote by $\mathsf{mod}\,A$ the category of finite dimensional modules over a Gorenstein algebra A, then $(\mathsf{CM}\,A, \mathsf{P}^{<\infty})$ is a cotorsion pairs of $\mathsf{mod}\,A$. Here, $\mathsf{CM}\,A$ is the subcategory of Cohen-Macaulay modules, namely,

$$\mathsf{CM}\,A := \{X \in \mathsf{mod}\,A \mid \mathrm{Ext}^i_A(X, A) = 0, i > 0\},$$

and $\mathsf{P}^{<\infty}$ is the subcategory of modules of finite projective dimension.

In the representation theory of algebras, cotorsion pairs play a central role in connection with derived categories, since, if a special cotorsion pair exists in $\mathsf{mod}\,A$, we obtain a new algebra B derived equivalent to A. In the rest of this subsection, the symbol A and B always denote finite dimensional k-algebra over a fixed field k. Let us begin with some preparations.

For a subcategory $\mathcal{C}$ of $\mathsf{mod}\,A$ we denote by $\check{\mathcal{C}}$ the subcategory of $\mathsf{mod}\,A$ consisting of objects X for which there is an exact sequence $0 \to X \to C_0 \to C_1 \to \cdots \to C_n \to 0$, with $C_i \in \mathcal{C}$. Recall that $T \in \mathsf{mod}\,A$ is *basic* if T admits an indecomposable decomposition $T = \coprod_{i=1}^n T_i$ with $T_i \ncong T_j$ for $i \neq j$.

Definition 2.21. A finite dimensional A-module $T \in \mathsf{mod}\, A$ is called a *tilting A-module*, if it satisfies the following conditions.

(1) $\mathsf{pd}\, T < \infty$, where $\mathsf{pd}\, T$ is the projective dimension of T.
(2) $\mathrm{Ext}^i_{\mathcal{C}}(T,T) = 0$ for $i > 0$.
(3) There is an exact sequence $0 \to A \to T_0 \to \cdots \to T_n \to 0$ with $T_i \in \mathsf{add}\, T$.

Theorem 2.22. [Hap88, Ch. III, Thm. 2.10] *Let $T \in \mathsf{mod}\, A$ be a tilting A-module and put $B := \mathrm{End}_A(T)$. Then, the functor $\mathrm{Hom}_A(T,-) : \mathsf{Mod}\, A \to \mathsf{Mod}\, B$ induces a triangule equivalence $\mathsf{D}(A) \xrightarrow{\sim} \mathsf{D}(B)$.*

Thus, tilting modules are basic and powerful tools to construct derived equivalent algebras. The following theorem shows that tilting modules bijectively correspond to a class of cotorsion pairs.

Definition 2.23. A subcategory $\mathcal{U}$ of an exact category $\mathcal{C}$ is called a *resolving* subcategory if, for any conflation $0 \to X \to Y \to Z \to 0$, $Y, Z \in \mathcal{U}$ implies $X \in \mathcal{U}$.

Lemma 2.24. *For a cotorsion pair $(\mathcal{U},\mathcal{V})$, the following are equivalent.*

(i) *$\mathcal{U}$ is resolving.*
(ii) *$\mathrm{Ext}^2_{\mathcal{C}}(U,V) = 0$ for any $U \in \mathcal{U}$ and $V \in \mathcal{V}$.*
(iii) *$\mathrm{Ext}^i_{\mathcal{C}}(U,V) = 0$ for any $U \in \mathcal{U}, V \in \mathcal{V}$ and $i > 0$.*

Under the above equivalent conditions, we call $(\mathcal{U},\mathcal{V})$ a resolving *cotorsion pair*[2].

Theorem 2.25. [AR91, Thm. 5.5] *There is one-to-one correspondence between basic tilting modules T and resolving cotorsion pairs $(\mathcal{U},\mathcal{V})$ with $\check{\mathcal{V}} = \mathsf{mod}\, A$, given by $T \mapsto (\widetilde{\mathsf{add}} T, (\widetilde{\mathsf{add}} T)^{\perp})$ and $(\mathcal{U},\mathcal{V}) \mapsto$ direct sum of the indecomposable modules in $\mathcal{U} \cap \mathcal{V}$.*

A trivial example of tilting A-module is A itself. The corresponding cotorsion pair is $(\mathsf{proj}\, A, \mathsf{mod}\, A)$ on $\mathsf{mod}\, A$.

Krause-Solberg showed that a cotorsion pair $(\mathcal{U},\mathcal{V})$ on $\mathsf{mod}\, A$ can give rise to that on $\mathsf{Mod}\, A$.

Theorem 2.26. [KS03, Thm. 2.4] *Let $(\mathcal{U},\mathcal{V})$ be a resolving cotorsion pair on $\mathsf{mod}\, A$. Then $(\varinjlim \mathcal{U}, \varinjlim \mathcal{V})$ is a resolving cotorsion pair on $\mathsf{Mod}\, A$. Here, $\varinjlim \mathcal{U}$ denotes the full subcategory of all A-modules which are filtered colimits of modules in $\mathcal{U}$.*

3. Model structures and Cotorsion pairs

3.1. **Hovey's correspondence theorem.** Let us start with brief observations on an abelian category $\mathcal{C}$ equipped with a model structure $(\mathsf{Cof}, \mathcal{W}, \mathsf{Fib})$. Following [BR07, VII], we firstly sharpen Lemma 2.5 and show that the cofibrant/fibrant replacements are closely related to approximation sequences obtained from cotorsion pairs.

Lemma 3.1. [BR07, VII.2.1] *For any object $X \in \mathcal{C}$:*

(1) *there exists an exact sequence $0 \to T_X \xrightarrow{f} S_X \xrightarrow{\pi'} X$ where π' is a trivially cofibrant replacement and $T_X \in \mathcal{C}_{\mathsf{fib}}$;*
(2) *there exists an exact sequence $X \xrightarrow{\iota} T^X \to S^X \to 0$ where ι is a fibrant replacement and $S^X \in \mathcal{C}_{\mathsf{tcof}}$;*
(3) *there exists an exact sequence $0 \to V_X \to U_X \xrightarrow{\pi} X$ where π is a cofibrant replacement and $V_X \in \mathcal{C}_{\mathsf{tfib}}$;*

[2] This is also called a *hereditary* cotorsion pair in the literature.

(4) *there exists an exact sequence* $X \xrightarrow{\iota'} V^X \to U^X \to 0$ *where* ι' *is a trivially fibrant replacement and* $U_X \in \mathcal{C}_{\mathsf{cof}}$.

Proof. We shall only show (1). Due to Lemma 2.5, it remains to show $T_X \in \mathcal{C}_{\mathsf{fib}}$. To this end, we consider a trivial cofibration $g : A \to B$ together with the following commutative squares.

$$\begin{array}{ccccc} A & \longrightarrow & T_X & \xrightarrow{f} & S_X \\ {\scriptstyle g}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi'} \\ B & \longrightarrow & 0 & \longrightarrow & X \end{array}$$

Since π' is a fibration, we have a lifting $h : B \to S_X$ for the whole square consisting of A, B, S_X and X. The lifting h factors through T_X, say $h = f \circ f'$. Since f is monic, the morphism f' should be a lifting for the left square. Hence $T_X \to 0$ is a fibration. □

Definition 3.2. Let $\mathcal{C}$ be an abelian (more generally, extriangulated) category with a model structure. For a given morphism f, we consider the following four conditions:

(1) f is an inflation with cone in $\mathcal{C}_{\mathsf{tcof}}$ iff $f \in \mathsf{wCof}$.
(2) f is deflation with cocone in $\mathcal{C}_{\mathsf{fib}}$ iff $f \in \mathsf{Fib}$;
(3) f is an inflation with cone in $\mathcal{C}_{\mathsf{cof}}$ iff $f \in \mathsf{Cof}$;
(4) f is a deflation with cocone in $\mathcal{C}_{\mathsf{tfib}}$ iff $f \in \mathsf{wFib}$;

The model structure of $\mathcal{C}$ is said to be *right admissible* (resp. *left admissible*) if the above (3)(4) (resp. (1)(2)) are satisfied. A left and right admissible model structure is said to be *admissible.*

The following shows that admissible model structure yields a pair of cotorsion pairs.

Definition 3.3. Let $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ be cotorsion pairs on $\mathcal{C}$. We call the pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ of them a *twin cotorsion pair* if it satisfies $\mathcal{S} \subseteq \mathcal{U}$ (or equivalently $\mathcal{T} \supseteq \mathcal{V}$).

Proposition 3.4. [BR07, VII.3.4] *Let* $\mathcal{C}$ *be an abelian category with a model structure* $(\mathsf{Cof}, \mathcal{W}, \mathsf{Fib})$.

(1) *If the model structure is left admissible, then* $(\mathcal{C}_{\mathsf{tcof}}, \mathcal{C}_{\mathsf{fib}})$ *forms a cotorsion pair.*
(2) *If the model structure is right admissible, then* $(\mathcal{C}_{\mathsf{cof}}, \mathcal{C}_{\mathsf{tfib}})$ *forms a cotorsion pair.*

In particular, if the model structure is admissible, we have a twin cotorsion pair $((\mathcal{C}_{\mathsf{tcof}}, \mathcal{C}_{\mathsf{fib}}), (\mathcal{C}_{\mathsf{cof}}, \mathcal{C}_{\mathsf{tfib}}))$.

Proof. We shall prove only part (2) since the proof of (1) is dual. Due to the assumption and Lemma 3.1, we have only to check $\mathrm{Ext}^1_{\mathcal{C}}(U, V) = 0$ for $U \in \mathcal{C}_{\mathsf{cof}}$ and $V \in \mathcal{C}_{\mathsf{tfib}}$. Consider a conflation $V \to X \xrightarrow{p} U$ and a commutative square

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & = & U \end{array}$$

Since $0 \to U$ is a cofibration and p is a trivial fibration, we have a lifting, which forces p splitting. □

It is natural to ask when a given twin cotorsion pairs induces a model structure[3]. The answer was essentially given by Hovey.

[3] Another natural question is when a single cotorsion pair corresponds to a model structure. An answer is given by Beligiannis-Reiten [BR07, VII. 4.2].

Definition 3.5. A twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ on $\mathcal{C}$ is called *Hovey twin cotorsion pair* if it satisfies $\mathsf{Cone}(\mathcal{V},\mathcal{S}) = \mathsf{CoCone}(\mathcal{V},\mathcal{S})$.

Theorem 3.6. [Hov02, Thm. 2.2] *Let $\mathcal{C}$ be an abelian category. There exists one-to-one correspondence between admissible model structures on $\mathcal{C}$ and Hovey twin cotorsion pairs on $\mathcal{C}$.*

$$\{(\mathsf{Cof},\mathcal{W},\mathsf{Fib}) : \text{admissible model strucutres on } \mathcal{C}\}$$
$$\Phi \downarrow \uparrow \Psi$$
$$\{((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) : \text{Hovey twin cotorsion pairs on } \mathcal{C}\}$$

Proof. It is a special case of Theorem 1.1. □

Remark 3.7. Theorem 3.6 has been modified and generalized by some authors:

- Gillespie and Šťovíček proved an exact version of the theorem [Gil11, Sto13];
- Yang proved a triangulated version of the theorem [Yan15].
- Nakaoka-Palu' extriangulated version contains the above cases. [NP19].

In addition, we remark that the theorem is a bit modified formulation of the original one. This type of formulations can be found in [NP19].

Example 3.8. The model structure in Example 2.10 is admissible and corresponds to a Hovey twin cotorsion pair $((\mathsf{proj}\,A,\mathsf{mod}\,A),(\mathsf{mod}\,A,\mathsf{proj}\,A))$.

3.2. **Nakaoka-Palu's correspondence theorem.** This subsection is devoted to prove Theorem 1.1, the extriangulated version of Hovey's correspondence theorem.

3.2.1. *From admissible model strucutre to Hovey twin cotorsion pair.* Let $\mathcal{C}$ be an extriangulated category with an admissible model structure $(\mathsf{Cof},\mathcal{W},\mathsf{Fib})$ and put

$$((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) := ((\mathcal{C}_{\mathsf{tcof}},\mathcal{C}_{\mathsf{fib}}),(\mathcal{C}_{\mathsf{cof}},\mathcal{C}_{\mathsf{tfib}})).$$

Proposition 3.9. [NP19, Prop. 5.6] $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ *is a twin cotorsion pair.*

Proof. A similar discussion in the proof of Proposition 3.4 still works well. So we skip the details. □

It directly follows from the next proposition that $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is a Hovey twin cotorsion pair.

Proposition 3.10. [NP19, Prop. 5.7] *The following are equivalent for any object $N \in \mathcal{C}$.*

(i) $N \in \mathsf{Cone}(\mathcal{V},\mathcal{S})$.
(ii) $(0 \to N) \in \mathcal{W}$.
(iii) $(N \to 0) \in \mathcal{W}$.
(iv) $N \in \mathsf{CoCone}(\mathcal{V},\mathcal{S})$.

Proof. (i) ⇒ (ii): Consider a conflation $V \to S \to N$ with $V \in \mathcal{V}, S \in \mathcal{S}$. Then, we have a factorization of $0 \to N$ as a trivial cofibration followed by a trivial fibration $S \to N$. Hence $(0 \to N) \in \mathcal{W}$.

(ii) ⇒ (i): Since $(0 \to N) \in \mathcal{W}$ and $\mathcal{W} = \mathsf{wFib} \circ \mathsf{wCof}$, we have a factorization of $(0 \to N)$ which shows the condition (i).

(iii) ⇔ (iv): It is a dual of (i) ⇔ (ii).

(ii) ⇔ (iii): It follows from the 2-out-of-3 axiom on $\mathcal{W}$. □

3.2.2. *From Hovey twin cotorsion pair to admissible model strucutre.* In the rest of this section, we fix an extriangulated category $\mathcal{C}$ satisfying (WIC) together with a Hovey twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$.

To construct the corresponding model structure, we define the following classes of morphisms.

- $\mathsf{wCof} = \{f \in \operatorname{Mor}\mathcal{C} \mid f$ is an inflation with $\mathsf{Cone}(f) \in \mathcal{S}\}$.
- $\mathsf{Fib} = \{f \in \operatorname{Mor}\mathcal{C} \mid f$ is a deflation with $\mathsf{CoCone}(f) \in \mathcal{T}\}$.
- $\mathsf{Cof} = \{f \in \operatorname{Mor}\mathcal{C} \mid f$ is an inflation with $\mathsf{Cone}(f) \in \mathcal{U}\}$.
- $\mathsf{wFib} = \{f \in \operatorname{Mor}\mathcal{C} \mid f$ is a deflation with $\mathsf{CoCone}(f) \in \mathcal{V}\}$.
- $\mathcal{W} = \mathsf{wFib} \circ \mathsf{wCof}$.

Lemma 3.11. $\mathsf{wCof}, \mathsf{Fib}, \mathsf{Cof}$ *and* wFib *are closed under composition.*

Proof. Since the corresponding subcategories $\mathcal{S}, \mathcal{T}, \mathcal{U}$ and $\mathcal{V}$ are extension-closed, the assertions directly follow from (ET4). □

Proposition 3.12. *The lifting axiom* (M3) *is satisfied, that is, we have the following.*

(1) wCof *satisfies the left lifting property against* Fib.
(2) wFib *satisfies the right lifting property against* Cof.

Proof. We only prove (1), because (2) is dual. We consider the following square with the columns forming conflations.

$$\begin{array}{ccc} & & T \\ & & \downarrow{\scriptstyle t} \\ A & \xrightarrow{a} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[b]{} & D \\ {\scriptstyle s}\downarrow & & \\ S & & \end{array}$$

Under the assumption $S \in \mathcal{S}$ and $T \in \mathcal{T}$, we shall construct a lifting $h : B \to C$ for the above square. Since $\mathbb{E}(S, T) = 0$, by a basic property of extriangulated categories, there exists a map $c : B \to C$ with $b = g \circ c$. However, unlike the exact case, $a = c \circ f$ does not necessarily hold. So we put $d := a - c \circ f$. Since $g \circ d = 0$, we have a map $c' : A \to T$ with $d = t \circ c'$. Again, due to $\mathbb{E}(S, T) = 0$, we get $c'' : B \to T$ with $c'' \circ f = c'$. Then the map $h := c + t \circ c'' : B \to C$ is a desired lifting. □

The next proposition shows that the factorization axiom (M4) is satisfied.

Proposition 3.13. $\operatorname{Mor}(\mathcal{C}) = \mathsf{wFib} \circ \mathsf{Cof} = \mathsf{Fib} \circ \mathsf{wCof}$.

Proof. We only show $\operatorname{Mor}(\mathcal{C}) = \mathsf{wFib} \circ \mathsf{Cof}$. Let $f \in \mathcal{C}(A, B)$ a map and resolve A to get an approximation sequence $A \xrightarrow{\iota^A} V^A \to U^A$ with $U^A \in \mathcal{U}$ and $V^A \in \mathcal{V}$. A weak pushout of ι^A along f yields a conflation $A \xrightarrow{f'} B \oplus V^A \to C$ where we put $f' := \binom{f}{\iota^A}$. Resolve C by $(\mathcal{U}, \mathcal{V})$ and obtain the following commutative diagram made of conflations.

$$\begin{array}{ccccc} & & V_C & = & V_C \\ & & \downarrow & & \downarrow \\ A & \xrightarrow{i} & M & \longrightarrow & U_C \\ \| & & \downarrow{\scriptstyle p} & \text{(Pb)} & \downarrow \\ A & \xrightarrow{f'} & B \oplus V^A & \longrightarrow & C \end{array}$$

Since $i \in \mathsf{Cof}$ and $p \in \mathsf{wFib}$, a factorization $f = (1\ 0) \circ p \circ i$, the morphism $p \circ i$ followed by the projection $(1\ 0) \in B \oplus V^A \to B$, is a desired one.

The remaining axioms (M1) and (M2) follow from Corollary 5.22 and Propsition 5.24 in [NP19], repectively. A triangulated structure is given in [NP19, Thm. 6.20]. Although the detailed proof is not included here, we shall present more conceptual proofs in Section 5 via the localization of extriangulated categories. □

By the argument so far, admissible model structures and Hovey twin cotorsion pairs correspond bijectively. Recall that, the homotopy category $\mathsf{Ho}(\mathcal{C})$ admits another description $\mathcal{C}_{\mathsf{cf}}/\sim$ in terms of homotopy relations which should be better understood. We summarize the essentials which shows homotopy relations can be interpreted in view of ideal quotients.

Proposition 3.14. [Gil11, Prop. 4.4] *Let $\mathcal{C}$ be an extriangulated category satisfying* (WIC) *with an admissible model structure.*

1. *Two morphisms $f, g : X \to Y$ in $\mathcal{C}$ are right homotopic if and only if $g - f$ factors through an object of $\mathcal{C}_{\mathsf{tcof}}$.*
2. *Two morphisms $f, g : X \to Y$ in $\mathcal{C}$ are left homotopic if and only if $g - f$ factors through an object of $\mathcal{C}_{\mathsf{tfib}}$.*

In particular, the composition $\mathcal{C}_{\mathsf{cf}} \hookrightarrow \mathcal{C} \xrightarrow{Q} \mathsf{Ho}(\mathcal{C})$ induces an equivalence $\frac{\mathcal{C}_{\mathsf{cf}}}{[\mathcal{C}_{\mathsf{tcof}} \cap \mathcal{C}_{\mathsf{tfib}}]} \xrightarrow{\sim} \mathsf{Ho}(\mathcal{C})$.

Proof. We will prove (2). The part (1) is dual. We first construct a cylinder object for X by using the corresponding cotorsion pair $(\mathcal{U}, \mathcal{V}) := (\mathcal{C}_{\mathsf{cof}}, \mathcal{C}_{\mathsf{tfib}})$. Resolving X by $(\mathcal{U}, \mathcal{V})$, we get a conflation $X \xrightarrow{j} V^X \to U^X$ with $U^X \in \mathcal{U}$ and $V^X \in \mathcal{V}$. Now we consider the following factorization of the summation $(1_X\ 1_X) : X \coprod X \to X$:

$$X \coprod X \xrightarrow{i} X \coprod V^X \xrightarrow{p} X$$

with $i = \begin{pmatrix} 1_X & 1_X \\ 0 & j \end{pmatrix}$ and $p = (1_X\ 0)$. Since $\operatorname{Ker} p \cong V_X$ is trivially fibrant, we have $p \in \mathsf{wFib}$. To claim that i is a cofibration, we consider the equation

$$\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} = \begin{pmatrix} j & -1_{V_X} \\ 0 & 1_{V_X} \end{pmatrix} \begin{pmatrix} 1_X & 1_X \\ 0 & j \end{pmatrix}.$$

Since $\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$ is an inflation, the property (WIC) guarantees $i \in \mathsf{Cof}$. Thus, $X \coprod V^X$ is a cylinder object for X. By definition, $f \sim^l g$ if and only if there is a map $(\alpha\ \beta) : X \coprod V^X \to Y$ such that the equation $(\alpha\ \beta) i = (f\ g)$ holds. Since $i = \begin{pmatrix} 1_X & 1_X \\ 0 & j \end{pmatrix}$, the equation is equivalent to $f + \beta j = g$. Thus, $g - f$ factors through $V^X \in \mathcal{C}_{\mathsf{tfib}}$.

The remaining assertion directly follows from Theorem 2.9. □

4. Derived categories and model structures

In this section, we investigate a model structure on a category $\mathsf{C}(A)$ of complexes of A-modules the homotopy category of which coincides with the derived category $\mathsf{D}(A)$. A classical example of such model structures is known as an *injective model structure* which was first constructed by Joyal and Beke [Bek00]. Based on an elegant approach in [SS11, Sto13], we shall show that such model structures can be obtained from resolving cotorsion pairs on $\mathsf{Mod}\, A$. Throughout the section, fix a finite dimensional k-algebra A and the following symbols are used in many places:

- $\mathsf{C}(A)$ - the category of complexes of A-modules;
- $\mathsf{C}_{\mathrm{ac}}(A)$ - the full subcategory of acyclic complexes of $\mathsf{C}(A)$;
- $\mathsf{K}(A)$ - the homotopy category of complexes of A-modules;
- $\mathsf{K}_{\mathrm{ac}}(A)$ - the full subcategory of acyclic complexes of $\mathsf{K}(A)$;
- $\mathsf{D}(A)$ - the derived category of $\mathsf{Mod}\,A$.

Moreover, for an extension-closed subcategory $\mathcal{U}$ of $\mathsf{Mod}\,A$, let $\mathsf{C}(\mathcal{U})$ denote the category of complexes in $\mathcal{U}$ and $\mathsf{C}_{\mathrm{ac}}(\mathcal{U})$ the full subcategory of acyclic complexes $X \in \mathsf{C}(\mathcal{U})$ such that $Z^n(X) \in \mathcal{U}$ for all $n \in \mathbb{Z}$.

4.1. **Injective model structures.** The derived category is, by definition, the Gabriel-Zisman localization of $\mathsf{C}(A)$ with respect to the quasi-isomorphisms. Let us recall another construction of the derived category which fits into a well understood framework. The homotopy category $\mathsf{K}(A)$ is a triangulated category together with a thick subcategory $\mathsf{K}_{\mathrm{ac}}(A)$. The derived category is obtained as the Verdier localization of $\mathsf{K}(A)$ with respect to $\mathsf{K}_{\mathrm{ac}}(A)$. In addition, we have $\mathsf{K}_{\mathrm{ac}}(A) \to \mathsf{K}(A) \xrightarrow{Q} \mathsf{D}(A)$ with additional property that, for any $X \in \mathsf{K}(A)$, there exists a triangle

$$V_X \to U_X \to X \to V_X[1]$$

with $U_X \in \mathsf{K}_{\mathrm{ac}}(A)$ and $V_X \in \mathsf{K}_{\mathrm{ac}}(A)^{\perp}$. It turns out that $(\mathsf{K}_{\mathrm{ac}}(A), \mathsf{K}_{\mathrm{ac}}(A)^{\perp})$ forms a cotorsion pair on $\mathsf{K}(A)$. Furthermore, it realizes the derived category as the full subcategory $\mathsf{K}_{\mathsf{ac}}(A)^{\perp}$ of $\mathsf{K}(A)$.

Lemma 4.1. *The composition* $\mathsf{K}_{\mathrm{ac}}(A)^{\perp} \hookrightarrow \mathsf{K}(A) \xrightarrow{Q} \mathsf{D}(A)$ *is a triangle equivalence.*

Now, we are in position to state that one can realize the derived category as the homotopy category with respect to an admissible model structure.

Proposition 4.2. *There exists a Hovey twin cotorsion pair*

$$\big((\mathsf{C}_{\mathrm{ac}}(A), \mathsf{C}_{\mathrm{ac}}(A)^{\perp}), (\mathsf{C}(A), \mathsf{C}_{\mathrm{ac}}(\mathsf{Inj}\,A))\big)$$

the homotopy category of which is the derived category $\mathsf{D}(A)$. *The corresponding admissible model structure is called the* injective model structure. *Furthermore, the cotorsion pairs are resolving.*

Before proving that, we recall the following well-known fact.

Lemma 4.3. *The following conditions are equivalent for* $X \in \mathsf{C}(A)$.

(i) $\mathrm{Ext}^1_{\mathsf{C}(A)}(E, X) = 0$ *for any acyclic complex* $E \in \mathsf{C}_{\mathrm{ac}}(A)$.

(ii) $\mathrm{Hom}_{\mathsf{K}(A)}(E, X) = 0$ *for any acyclic complex* $E \in \mathsf{C}_{\mathrm{ac}}(A)$.

Under the above equivalent conditions, the object X *in* $\mathsf{C}(A)$ *are called an* injectively fibrant *objects*[4] *and those in* $\mathsf{K}(A)$ *are called* K-injectives.

Proof of Proposition 4.2. It is obvious that $(\mathsf{C}(A), \mathsf{C}_{\mathrm{ac}}(\mathsf{Inj}\,A))$ forms a cotorsion pair, since any acyclic complex in $\mathsf{C}_{\mathrm{ac}}(\mathsf{Inj}\,A)$ is splitting.

Due the cotorsion pair $(\mathsf{K}_{\mathrm{ac}}(A), \mathsf{K}_{\mathrm{ac}}(A)^{\perp})$, it is easy to check that $(\mathsf{C}_{\mathrm{ac}}(A), \mathsf{C}_{\mathrm{ac}}(A)^{\perp})$ is also a cotorsion pair. In fact, for any $X \in \mathsf{C}(A)$, there exists a triangle $T_X \xrightarrow{\bar{i}} S_X \to X \to T_X[1]$ with $S_X \in \mathsf{K}_{\mathrm{ac}}(A)$ and $T_X \in \mathsf{K}_{\mathrm{ac}}(A)^{\perp}$. Any triangle in $\mathsf{K}(A)$ comes from an exact

[4]Each term X^i of an injectively fibrant object X belongs to $\mathsf{Inj}\,A$.

sequence in $\mathsf{C}(A)$. More precisely, by taking the pushout of $T_X \xrightarrow{i} S_X$ along the injective hull $T_X \to I(T_X)$, we get an exact sequence

$$0 \to T_X \to S_X \oplus I(T_X) \xrightarrow{p} X' \to 0$$

which corresponds to the triangle. We may assume that there exists a contractible complex $I \in \mathsf{C}_{\mathrm{ac}}(A)$ with an isomorphism $X' \cong X \oplus I$ in $\mathsf{C}(A)$. Taking the pullback of p along the section $X \to X'$ yields the following commutative diagram made of conflations.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & I & = & I & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & T_X & \to & S'_X & \to & X & \to & 0 \\
 & & \| & & \downarrow & \text{(Pb)} & \downarrow & & \\
0 & \to & T_X & \to & S_X \oplus I(T_X) & \xrightarrow[p]{} & X' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle row is a desired sequence. Another approximation sequence can be obtained similarly.

Since $\mathsf{C}_{\mathrm{ac}}(\mathsf{Inj}\,A)$ is the class of injectives in $\mathsf{C}(A)$, it is easy to check the equiality

$$\mathsf{C}_{\mathrm{ac}}(A) = \mathsf{Cone}(\mathsf{C}_{\mathrm{ac}}(\mathsf{Inj}\,A), \mathsf{C}_{\mathrm{ac}}(A)) = \mathsf{CoCone}(\mathsf{C}_{\mathrm{ac}}(\mathsf{Inj}\,A), \mathsf{C}_{\mathrm{ac}}(A)).$$

Hence, the given pair forms a Hovey twin cotorsion pair.

It remains to show that $\mathcal{W}$ is the class of quasi-isomorphisms. Let $f : X \to Y$ be a quasi-isomorphism and consider a factorization $f : X \xrightarrow{f_1} Z \xrightarrow{f_2} Y$ with f_1 surjective and f_2 injective. Note that $K := \operatorname{Ker} f_1, \operatorname{Cok} f_2$ belongs to $\mathsf{C}_{\mathrm{ac}}(A)$ and, in particular, $f_2 \in \mathsf{wCof}$. Since $\mathsf{C}(A)$ has enough injectives forming $\mathsf{C}_{\mathrm{ac}}(\mathsf{Inj}\,A)$, we get an exact sequence $0 \to K \to I \to K' \to 0$ with $I \in \mathsf{C}_{\mathrm{ac}}(\mathsf{Inj}\,A)$ and $K' \in \mathsf{C}_{\mathrm{ac}}(A)$ and construct the following commutative diagram with all rows and columns exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & K & \to & X & \xrightarrow{f_1} & Z & \to & 0 \\
 & & \downarrow & \text{(Po)} & \downarrow{\scriptstyle f'_1} & & \| & & \\
0 & \to & I & \to & X' & \xrightarrow[f''_1]{} & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & K' & = & K' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $f'_1 \in \mathsf{wCof}, f''_1 \in \mathsf{wFib}$, we thus conclude $f \in \mathcal{W}$. Any weak equivalence is obviously a quasi-isomorphism. Therefore, the homotopy category is the derived category. □

The injective model structure is explicitly given as follows:

- $\mathcal{W}$ = the class of quasi-isomorphisms;
- Cof = the class of monomorphisms;
- Fib = the class of epimorphisms whose kernels are injectively fibrant.

Moreover, thanks to the description of the homotopy category in Theorem 1.1, we have an equivalence

$$\mathsf{Ho}(\mathcal{C}) = \mathsf{D}(A) \simeq \frac{\mathsf{C}_{\mathrm{ac}}(A)^{\perp} \cap {}^{\perp}\mathsf{C}_{\mathrm{ac}}(\mathsf{Inj}\,A)}{[\mathsf{C}_{\mathrm{ac}}(A) \cap \mathsf{C}_{\mathrm{ac}}(\mathsf{Inj}\,A)]} = \mathsf{K}_{\mathrm{ac}}(A)^{\perp}.$$

Dually, the *projective model structure* exists. The corresponding Hovey twin cotosion pair is

$$\big((\mathsf{C}_{\mathrm{ac}}(\mathsf{proj}\,A), \mathsf{C}(A)), ({}^{\perp}\mathsf{C}_{\mathrm{ac}}(A), \mathsf{C}_{\mathrm{ac}}(A))\big),$$

and the homotopy category is also the derived category.

4.2. **Lifting cotorsion pairs to categories of complexes.** In this subsection, we shall show that a cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $\mathsf{Mod}\,A$ gives rise to a model structure on $\mathsf{C}(A)$ the homotopy category is the derived category. Such model structures contain the injective and projective model structures. Recall that $\mathsf{C}_{\mathrm{ac}}(A)$ is a thick subcategory of $\mathsf{C}(A)$ with an inherited exact structure.

Proposition 4.4. [Sto13, Prop. 7.7] *A resolving cotorsion pair* $(\mathcal{U}, \mathcal{V})$ *on* $\mathsf{Mod}\,A$ *induces a resolving cotorsion pair* $(\mathsf{C}_{\mathrm{ac}}(\mathcal{U}), \mathsf{C}_{\mathrm{ac}}(\mathcal{V}))$ *on the exact category* $\mathsf{C}_{\mathrm{ac}}(A)$.

Proof. We denote for brievity $\mathsf{C}_{\mathrm{ac}}(\mathcal{U})$ and $\mathsf{C}_{\mathrm{ac}}(\mathcal{V})$ by $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$, respectively, and put $\mathcal{N} := \mathsf{C}_{\mathrm{ac}}(A)$. We first show $\mathrm{Ext}^1_{\mathcal{N}}(\widetilde{\mathcal{U}}, \widetilde{\mathcal{V}}) = 0$. Every $U \in \widetilde{\mathcal{U}}$ can be written as an extension of stalk complexes[5] $Z^i(U)$ which is depicted as follows.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Z^0(U) & \xrightarrow{0} & Z^1(U) & \xrightarrow{0} & Z^2(U) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & U^0 & \xrightarrow{\partial^0} & U^1 & \xrightarrow{\partial^1} & U^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Z^1(U) & \xrightarrow{0} & Z^2(U) & \xrightarrow{0} & Z^3(U) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Therefore we have only to check $\mathrm{Ext}^1_{\mathsf{C}(A)}(U', V) = 0$ for any $V \in \widetilde{\mathcal{V}}$ and any stalk complex $U' \in \mathsf{C}(\mathcal{U})$ concentrated in degree n. Any extension δ in $\mathrm{Ext}^1_{\mathsf{C}(A)}(U', V) = 0$ is necessarily degreewise splitting:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & V^{n-1} & \xrightarrow{\partial^{n-1}} & V^n & \xrightarrow{\partial^n} & V^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \binom{1}{0}} & & \downarrow & & \\
\cdots & \longrightarrow & V^{n-1} & \xrightarrow{\alpha} & V^n \oplus U' & \xrightarrow{\beta} & V^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle (0\ 1)} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \xrightarrow{0} & U' & \xrightarrow{0} & 0 & \longrightarrow & \cdots
\end{array}$$

where $\alpha = \binom{\partial^{n-1}}{0}$ and $\beta = (\partial^n\ f)$ for a morphism $f : U' \to V^{n+1}$. Since $\partial^{n+1} \circ f = 0$, f factors through $Z^{n+1}(V)$. Due to $\mathrm{Ext}^1_A(\mathcal{U}, \mathcal{V}) = 0$, f is factored as $f : U' \xrightarrow{f'} Z^n(V) \xrightarrow{\partial^n}$

[5]A stalk complex X is a complex satisfying $X^i = 0$ for any $i \neq n$ for some $n \in \mathbb{Z}$.

V^{n+1}. Thus we have a splitting exact sequence $0 \to U' \xrightarrow{\binom{1}{-f'}} V^n \oplus U' \xrightarrow{(f'\ 1)} V^n \to 0$ which shows that δ splits.

Clearly, $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ are closed under direct summands since $\mathcal{U}$ and $\mathcal{V}$ are. For any $X \in \mathcal{N}$, we resolve the cocycle $Z^n(X) \in \mathsf{Mod}\, A$ by the cotorsion pair $(\mathcal{U}, \mathcal{V})$ to have approximation sequences $0 \to Z^n(X) \to V_n \to U_n \to 0$ and $0 \to V'_n \to U'_n \to Z^n(X) \to 0$. As an application of the Horseshoe lemma, we can construct desired approximation sequences for X.

It is obvious that $\widetilde{\mathcal{U}}$ is closed under epikernels. Hence we have proved that $(\widetilde{\mathcal{U}}, \widetilde{\mathcal{V}})$ forms a resolving cotorsion pair $\mathsf{C}_{\mathrm{ac}}(A)$. □

The following model structures were first found in [SS11, Thm. 4.2] and streamlined in [Sto13, Thm. 7.16].

Theorem 4.5. *Let $(\mathcal{U}, \mathcal{V})$ be a resolving cotorsion pair on $\mathsf{Mod}\, A$. Then, it induces a Hovey twin cotorsion pair*

$$((\mathsf{C}_{\mathrm{ac}}(\mathcal{U}), \mathsf{C}_{\mathrm{ac}}(\mathcal{U})^{\perp}), ({}^{\perp}\mathsf{C}_{\mathrm{ac}}(\mathcal{V}), \mathsf{C}_{\mathrm{ac}}(\mathcal{V})))$$

the homotopy category of which is the derived category $\mathsf{D}(A)$. Furthermore, the cotorsion pairs are resolving.

Proof. We shall show that $(\mathsf{C}_{\mathrm{ac}}(\mathcal{U}), \mathsf{C}_{\mathrm{ac}}(\mathcal{U})^{\perp})$ is a cotorsion pair. Any $X \in \mathsf{C}(A)$ admits an approximation sequence $0 \to X' \to E \xrightarrow{f_1} X \to 0$ with $E \in \mathsf{C}_{\mathrm{ac}}(\mathcal{U})$ and $X' \in \mathsf{C}_{\mathrm{ac}}(A)^{\perp}$. Resolve E by the cotorsion pair $(\mathsf{C}_{\mathrm{ac}}(\mathcal{U}), \mathsf{C}_{\mathrm{ac}}(\mathcal{V}))$ to get an approximation sequence $0 \to \widetilde{V} \to \widetilde{U} \xrightarrow{f_2} E \to 0$ with $\widetilde{U} \in \mathsf{C}_{\mathrm{ac}}(\mathcal{U})$ and $\widetilde{V} \in \mathsf{C}_{\mathrm{ac}}(\mathcal{V})$. The composed map $f := f_2 \circ f_1$ gives rise to the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \widetilde{V} & = & \widetilde{V} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & X'' & \to & \widetilde{U} & \xrightarrow{f} & X & \to & 0 \\
 & & \downarrow & & {\scriptstyle f_2}\downarrow & & \| & & \\
0 & \to & X' & \to & E & \xrightarrow{f_1} & X & \to & 0
\end{array}$$

Since $X'' \in \mathsf{C}_{\mathsf{ac}}(A)^{\perp}$, the middle exact row provides a desired approximation sequence. A similar method yields another approximation sequence. Thus the pair forms a resolving cotorsion pair.

Dually we can show that the pair in the righthand side also forms a resolving cotorsion pair.

Proposition 4.4 shows $\mathsf{C}_{\mathrm{ac}}(A) = \mathsf{Cone}(\mathsf{C}_{\mathrm{ac}}(\mathcal{V}), \mathsf{C}_{\mathrm{ac}}(\mathcal{U})) = \mathsf{CoCone}(\mathsf{C}_{\mathrm{ac}}(\mathcal{V}), \mathsf{C}_{\mathrm{ac}}(\mathcal{U}))$. We have thus concluded that the given pair is a Hovey twin cotorsion pair.

A similar method given in the latter part of the proof of Proposition 4.2 shows that $\mathcal{W}$ is the class of quasi-isomorphism. □

In the case of $(\mathcal{U}, \mathcal{V}) = (\mathsf{Mod}\, A, \mathsf{Inj}\, A)$, the obtained model structures in Theorem 4.5 is nothing other than the injective model structures. Dually the projective model structures correspond to the cotorsion pair $(\mathsf{Proj}\, A, \mathsf{Mod}\, A)$.

Remark 4.6. (1) The admissible model structure given in Theorem 4.5, in practice, satisfies the functorial factorization axiom (M4') (see [SS11, Thm. 4.2]).

(2) Combining Theorems 4.5, 2.25 and 2.26, we can construct a model structure from a tilting module.

4.3. **A tilting Quillen equivalence.** Let us include a short discussion on the derived equivalence via tilting modules. Fix an algebra A and a tilting A-module $T \in \mathsf{mod}\, A$. As stated in Theorem 2.22, the induced functor $\mathrm{Hom}_A(T, -) : \mathsf{C}(A) \to \mathsf{C}(B)$ gives rise to a triangle equivalence $F_T : \mathsf{D}(A) \xrightarrow{\sim} \mathsf{D}(B)$, where we put $B := \mathrm{End}_A(T)$. Note that the tilting module T induces a cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $\mathsf{Mod}\, A$. Precisely, we have a cotorsion pair $(\mathcal{U}', \mathcal{V}') := (\widecheck{\mathsf{add}} T, (\widecheck{\mathsf{add}} T)^{\perp})$ on $\mathsf{mod}\, A$ via Theorem 2.25. Due to Theorem 2.26, the pair $(\mathcal{U}', \mathcal{V}')$ gives rise to a resolving cotorsion pair $(\mathcal{U}, \mathcal{V}) := (\varinjlim \mathcal{U}', \varinjlim \mathcal{V}')$ on $\mathsf{Mod}\, A$. Similarly, the tilting module $B \in \mathsf{mod}\, B$ induces a resolving cotorsion pair $(\mathcal{X}, \mathcal{Y}) := (\mathsf{Proj}\, B, \mathsf{Mod}\, B)$ on $\mathsf{Mod}\, B$. The aim of this subsection is to regard the equivalence F_T as a right Quillen equivalence with respect to the natural model structures.

Theorem 4.7. *Consider the admissible model structures on $\mathsf{C}(A)$ and $\mathsf{C}(B)$ which are obtained from $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{X}, \mathcal{Y})$ via Theorem 4.5. Then $\mathrm{Hom}_A(T, -)$ is a left Quillen functor which induces an equivalence $F_T : \mathsf{D}(A) \to \mathsf{D}(B)$ of the homotopy categories.*

Proof. Since $\mathcal{Y} = \mathsf{Mod}\, B$, it is obvious that $\mathrm{Hom}_A(T, \mathcal{V}) \subseteq \mathcal{Y}$. Since $\mathrm{Ext}^1_A(T, \mathcal{V}) = 0$, thanks to [Rec21, Thm. 3.9], we conclude that F_T is a left Quillen functor. □

Note that the admissible model structure on $\mathsf{C}(B)$ obtained from $(\mathcal{X}, \mathcal{Y})$ is nothing other than the projective model structure. As a corollary of (the proof of) Theorem 4.7, we have the following.

Corollary 4.8. [Sto13, Thm. 3.13] *Consider the injective model structure on $\mathsf{C}(A)$ and the projective model structure on $\mathsf{C}(B)$. Then $\mathrm{Hom}_A(T, -)$ is a left Quillen functor which induces an equivalence $F_T : \mathsf{D}(A) \to \mathsf{D}(B)$ of the homotopy categories.*

5. Localization of extriangulated categories

The aim of this section is to explain how the homotopy category of an admissible model structure relates to the localization of extriangulated categories which was introduced in [NOS21].

5.1. **Multiplicative systems compatible with the extriangulation.** As stated in Introduction, the quotient category $\mathcal{C}[\mathcal{W}^{-1}]$ with respect to the class $\mathcal{W}$ of morphisms is hard to control. It is basic that, if $\mathcal{W}$ forms a multiplicative system, then $\mathcal{C}[\mathcal{W}^{-1}]$ should be additive and the morphisms admit nice descriptions. Furthermore, if $\mathcal{C}$ is triangulated and $\mathcal{W}$ is compatible with the triangulation, then the natural functor $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ should be a triangulated functor.

We shall introduce an extriangulated version of the above. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and let $\mathcal{W}$ be a class of morphisms in $\mathcal{C}$ which satisfies the following condition.

(M0) $\mathcal{W}$ contains all isomorphisms in $\mathcal{C}$, and is closed under compositions. Also, $\mathcal{W}$ is closed under taking finite direct sums. Namely, if $f_i \in \mathcal{W}(X_i, Y_i)$ for $i = 1, 2$, then $f_1 \oplus f_2 \in \mathcal{W}(X_1 \oplus X_2, Y_1 \oplus Y_2)$.

Put $\mathcal{M} := \mathrm{Mor}(\mathcal{C})$ and $\widetilde{\mathcal{C}} := \mathcal{C}[\mathcal{W}^{-1}]$. Theorem 5.3 provides a sufficient condition for $\mathcal{W}$ which makes the quotient category $\widetilde{\mathcal{C}}$ to be equipped with a natural extriangulated structure. The aim of subsections 5.1 and 5.2 is to recall some basic facts on the above extriangulated localization without proof. First, we associate a full subcategory $\mathcal{N}_{\mathcal{W}} \subseteq \mathcal{C}$ in the following way.

Definition 5.1. Let $\mathcal{W}$ be the above. Define $\mathcal{N}_{\mathcal{W}}$ to be the full subcategory of $\mathcal{C}$ consisting of objects N such that both $N \to 0$ and $0 \to N$ belong to $\mathcal{W}$.

It is obvious that $\mathcal{N}_{\mathcal{W}}$ is an additive subcategory which is not full. In the rest, we will denote the ideal quotient by $p : \mathcal{C} \to \overline{\mathcal{C}} = \mathcal{C}/[\mathcal{N}_{\mathcal{W}}]$, and $\overline{f}$ will denote a morphism in $\overline{\mathcal{C}}$ represented by $f \in \mathcal{C}(X, Y)$. Also, let $\overline{\mathcal{W}}$ be the closure of $p(\mathcal{W})$ with respect to compositions with isomorphisms in $\overline{\mathcal{C}}$.

The class $\mathcal{W}$ with (M0) is said to be *compatible with the extriangulation* if it satisfies the following conditions.

(MR1) $\overline{\mathcal{W}}$ satisfies 2-out-of-3 with respect to compositions in $\overline{\mathcal{C}}$.
(MR2) $\overline{\mathcal{W}}$ is a multiplicative system in $\overline{\mathcal{C}}$.
(MR3) Let $\langle A \xrightarrow{x} B \xrightarrow{y} C, \delta\rangle$, $\langle A' \xrightarrow{x'} B' \xrightarrow{y'} C', \delta'\rangle$ be any pair of $\mathfrak{s}$-triangles, and let $a \in \mathcal{C}(A, A'), c \in \mathcal{C}(C, C')$ be any pair of morphisms satisfying $a^*\delta = c_*\delta'$. If $\overline{a}$ and $\overline{c}$ belong to $\overline{\mathcal{W}}$, then there exists $\mathbf{b} \in \overline{\mathcal{W}}(B, B')$ which satisfies $\mathbf{b} \circ \overline{x} = \overline{x}' \circ \overline{a}$ and $\overline{c} \circ \overline{y} = \overline{y}' \circ \mathbf{b}$.
(MR4) $\overline{\mathcal{M}}_{\mathsf{inf}} := \{\mathbf{v} \circ \overline{x} \circ \mathbf{u} \mid x \text{ is an } \mathfrak{s}\text{-inflation}, \mathbf{u}, \mathbf{v} \in \overline{\mathcal{W}}\}$ is closed by composition in $\overline{\mathcal{M}}$. Dually, $\overline{\mathcal{M}}_{\mathsf{def}} := \{\mathbf{v} \circ \overline{y} \circ \mathbf{u} \mid y \text{ is an } \mathfrak{s}\text{-deflation}, \mathbf{u}, \mathbf{v} \in \overline{\mathcal{W}}\} \subseteq \overline{\mathcal{M}}$ is closed by compositions.

Exact functors between extriangulated categories are defined below (see [B-TS21, Def. 2.23]).

Definition 5.2. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ be extriangulated categories. An *exact functor* $(F, \phi) : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ is a pair of an additive functor $F : \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\phi : \mathbb{E} \Rightarrow \mathbb{E}' \circ (F^{\mathsf{op}} \times F)$ which satisfies

$$\mathfrak{s}'(\phi_{C,A}(\delta)) = [F(A) \xrightarrow{F(x)} F(B) \xrightarrow{F(y)} F(C)]$$

for any $\mathfrak{s}$-triangle $\langle A \xrightarrow{x} B \xrightarrow{y} C, \delta\rangle$ in $\mathcal{C}$.

Theorem 5.3. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{W}$ a multiplicative system compatible with the extriangulation.*

(1) *Then we obtain an extriangulated category $(\widetilde{\mathcal{C}}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}})$ together with an exact functor $(Q, \mu) : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\widetilde{\mathcal{C}}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}})$.*
(2) *The exact functor $(Q, \mu) : \mathcal{C} \to \widetilde{\mathcal{C}}$ obtained in (1) has the following universality.*
- *For any exact functor $(F, \phi) : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{D}, \mathbb{F}, \mathfrak{t})$ such that $F(s)$ is an isomorphism for any $s \in \mathcal{W}$, there exists a unique exact functor $(\widetilde{F}, \widetilde{\phi}) : (\widetilde{\mathcal{C}}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}}) \to (\mathcal{D}, \mathbb{F}, \mathfrak{t})$ with $(F, \phi) = (\widetilde{F}, \widetilde{\phi}) \circ (Q, \mu)$.*

Note that, since (Q, μ) is exact, the full subcategory $\mathcal{N} := \operatorname{Ker} Q = \{X \in \mathcal{C} \mid Q(X) \cong 0\}$ forms a thick subcategory in the following sense.

Definition 5.4. Let $\mathcal{C}$ be an extriangulated category. A full additive subcategory $\mathcal{N}$ of $\mathcal{C}$ is called a *thick* subcategory if it satisfies the following conditions.

(1) $\mathcal{N} \subseteq \mathcal{C}$ is closed under isomorphisms and direct summands.
(2) $\mathcal{N}$ satisfies 2-out-of-3 for $\mathfrak{s}$-conflations. Namely, if any two of objects A, B, C in an $\mathfrak{s}$-conflation $A \xrightarrow{x} B \xrightarrow{y} C$ belong to $\mathcal{N}$, then so does the third.

This situation will be depicted as a diagram functors of the form below:

$$\mathcal{N} \longrightarrow \mathcal{C} \xrightarrow{Q} \mathcal{C}/\mathcal{N}$$

and called the *extriangulated localization of $\mathcal{C}$ with respect to the thick subcategory $\mathcal{N}$*.

5.2. **Localization via thick subcategories.** We shall construct a multiplicative system compatible with extriangulation from a given thick subcategory.

Definition 5.5. For a thick subcategory $\mathcal{N} \subseteq \mathcal{C}$, we associate the following classes of morphisms.

$$\mathcal{L} = \{f \in \mathcal{M} \mid f \text{ is an } \mathfrak{s}\text{-inflation with } \mathsf{Cone}(f) \in \mathcal{N}\}.$$
$$\mathcal{R} = \{f \in \mathcal{M} \mid f \text{ is an } \mathfrak{s}\text{-deflation with } \mathsf{CoCone}(f) \in \mathcal{N}\}.$$

Define $\mathcal{W}_{\mathcal{N}} \subseteq \mathcal{M}$ to be the smallest subset closed by compositions containing both $\mathcal{L}$ and $\mathcal{R}$. It is obvious that $\mathcal{W}_{\mathcal{N}}$ satisfies condition (M0).

Example 5.6. (1) Let $\mathcal{C}$ be an abelian category and $\mathcal{N}$ a Serre subcategory of $\mathcal{C}$. Then, the class $\mathcal{W}_{\mathcal{N}}$ is a multiplicative system compatible with the extriangulation. The Serre localization $\mathcal{C} \to \mathcal{C}/\mathcal{N}$ is an extriangulated functor.

(2) Let $\mathcal{C}$ be a triangulated category and $\mathcal{N}$ a thick subcategory of $\mathcal{C}$. Then, the class $\mathcal{W}_{\mathcal{N}}$ is a multiplicative system compatible with the extriangulation. The Verdier localization $\mathcal{C} \to \mathcal{C}/\mathcal{N}$ is an extriangulated functor.

We are particularly interested in the localization with respect to a biresolving subcategory $\mathcal{N}$ which serves a sufficient condition so that $\mathcal{W}_{\mathcal{N}}$ forms a multiplicative system compatible with the extriangulation (see [Rum21]).

Definition 5.7. A thick subcategory $\mathcal{N} \subseteq \mathcal{C}$ is called *biresolving*, if for any $C \in \mathcal{C}$ there exist an $\mathfrak{s}$-inflation $C \to N$ and an $\mathfrak{s}$-deflation $N' \to C$ for some $N, N' \in \mathcal{N}$.

Proposition 5.8. *Let $\mathcal{C}$ be an extriangulated category and $\mathcal{N}$ a biresolving subcategory of $\mathcal{C}$.*

(1) *The class $\mathcal{W}_{\mathcal{N}}$ is a multiplicative system compatible with the extriangulation. Moreover, $\mathcal{W}_{\mathcal{N}} = \mathcal{R} \circ \mathcal{L}$ holds.*
(2) *The associated localization $Q : \mathcal{C} \to \mathcal{C}[\mathcal{W}_{\mathcal{N}}^{-1}]$ is an extriangulated functor with the universality in Theorem 5.3 (2).*
(3) *The quotient category $\mathcal{C}/\mathcal{N} := \mathcal{C}[\mathcal{W}_{\mathcal{N}}^{-1}]$ is triangulated.*

Remark 5.9. The following triangulated structure can be regarded as a localization by a biresolving subcategory.

(1) In the case that $\mathcal{C}$ is triangulated, any thick subcategory $\mathcal{N}$ automatically becomes biresolving. Thus, the associated localization is nothing but the Verdier localization $\mathcal{N} \to \mathcal{C} \to \mathcal{C}/\mathcal{N}$. Moreover, a stronger condition $\mathcal{W}_{\mathcal{N}} = \mathcal{R} = \mathcal{L}$ holds.
(2) Let $\mathcal{C}$ be a Frobenius category. The stable category $\underline{\mathcal{C}}$ is the localization of $\mathcal{C}$ with respect to the biresolving subcategory $\mathcal{N} := \mathsf{proj}\,\mathcal{C} = \mathsf{inj}\,\mathcal{C}$.

5.3. **The homotopy categories via extriangulated localizations.** Let $\mathcal{C}$ be an extriangulated category satisfying (WIC) together with an admissible model structure $(\mathsf{Cof}, \mathcal{W}, \mathsf{Fib})$. We shall construct the natural functor $Q : \mathcal{C} \to \mathsf{Ho}(\mathcal{C})$ as the extriangulated localization of $\mathcal{C}$ with respect to a biresolving subcategory $\mathcal{N}$, namely, $\mathsf{Ho}(\mathcal{C}) = \mathcal{C}/\mathcal{N}$.

By Theorem 1.1, we have the corresponding Hovey twin cotorsion pair

$$((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})) := ((\mathcal{C}_{\mathsf{tcof}}, \mathcal{C}_{\mathsf{fib}}), (\mathcal{C}_{\mathsf{cof}}, \mathcal{C}_{\mathsf{tfib}})).$$

Lemma 5.10. *The full subcategory $\mathcal{N} := \mathsf{Cone}(\mathcal{V}, \mathcal{S}) = \mathsf{CoCone}(\mathcal{V}, \mathcal{S})$ is a biresolving subcategory. Moreover, $\mathcal{N} = \operatorname{Ker} Q$ is true.*

Proof. It is shown in [NP19, Prop. 5.3] that $\mathcal{N}$ is a thick subcategory. Due to the 2-out-of-3 property, we can easily check that $\mathcal{N}$ is closed under direct summands. Since there exist approximation sequences of the given cotorsion pairs, it is, in addition, biresolving. □

Lemma 5.11. *The associated multiplicative system $\mathcal{W}_{\mathcal{N}}$ coincides with the class of weak equivalences.*

Proof. Obviously, $\mathsf{wCof} \subseteq \mathcal{L}$ and $\mathsf{wFib} \subseteq \mathcal{R}$ hold. Since $\mathcal{W}$ has the 2-out-of-3 property with respect to the composition, we have $\mathcal{W} = \mathcal{W}_{\mathcal{N}}$. □

Combining the above lemmas, we conclude the homotopy category $\mathsf{Ho}(\mathcal{C})$ is nothing other than the extriangulated localization $\mathcal{C}/\mathcal{N}$.

Corollary 5.12. *The sequence $\mathcal{N} \to \mathcal{C} \xrightarrow{Q} \mathsf{Ho}(\mathcal{C})$ is the extriangulated localization of $\mathcal{C}$ with respect to the thick subcategory $\mathcal{N}$. In particular, we have $\mathsf{Ho}(\mathcal{C}) = \mathcal{C}/\mathcal{N}$.*

We have thus obtained the triangulated structure on $\mathsf{Ho}(\mathcal{C})$ stated in Proposition 5.8.

The extriangulated localization of $\mathcal{C}$ with respect to a biresolving subcategory $\mathcal{N}$ generalizes the localization associated to a Hovey twin cotorsion pair in the following sense.

Corollary 5.13. *Let $\mathcal{C}$ be an extriangulated category and $\mathcal{N}$ a biresolving subcategory of $\mathcal{C}$ equipped with a cotorsion pair $(\mathcal{S}, \mathcal{V})$. if the induced pairs $(\mathcal{S}, \mathcal{S}^{\perp}), ({}^{\perp}\mathcal{V}, \mathcal{V})$ are cotorsion pairs, then they form a Hovey twin cotorsion pair. In this case, the homotopy category $\mathsf{Ho}(\mathcal{C})$ coincides with the extriangulated localization $\mathcal{C}/\mathcal{N}$.*

Example 5.14. The constructions of the derived category $\mathsf{D}(A)$ can be understood as follows.

$$
\begin{array}{ccccc}
\mathsf{C}_{\mathrm{ac}}(A) & \longrightarrow & \mathsf{C}(A) & \xrightarrow{\;Q\;} & \mathsf{D}(A) \\
\downarrow & & \downarrow & & \| \\
\mathsf{K}_{\mathrm{ac}}(A) & \longrightarrow & \mathsf{K}(A) & \xrightarrow[\text{Verdier quot.}]{} & \mathsf{D}(A)
\end{array}
$$

The first row is the extriangulated localization and the second row is the Verdier localization.

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