Derived equivalences and Gorenstein dimension

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A ring A is said to be left (resp., right) coherent if every finitely generated left (resp., right) ideal of it is finitely presented (see [3]). Let A, B be derived equivalent left and right coherent rings (see [5]). In [4] Kato showed that a standard derived equivalence $\mathcal{D}^{\mathrm{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}B)$ induces an equivalence between the triangulated categories consisting of complexes of finite Gorenstein dimension and that an equivalence $\mathcal{D}^{\mathrm{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}B)$ induces an equivalence between the projectively stable categories of modules of Gorenstein dimension zero (see [1] and [2]) if either inj dim $A < \infty$ or inj dim $A^{\mathrm{op}} < \infty$. In this talk, we provide alternative proofs of these results from another point of view. Also, we do not assume the existence of standard derived equivalence or finiteness of selfinjective dimension.

Denote by $\hat{\mathcal{G}}_A$ the full additive subcategory of mod-A consisting of modules $X \in$ mod-A with $\mathrm{Ext}^i_A(X, A) = 0$ for $i \neq 0$. We provide a characterization of modules of Gorenstein dimension zero as follows. A module $X \in \hat{\mathcal{G}}_A$ has Gorenstein dimension zero if and only if for each $i > 0$ there exists $Y_i \in \hat{\mathcal{G}}_A$ such that $X \cong Y_i[-i]$ in $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$, the quotient category of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ over the épaisse subcategory $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$. Using this fact, we show that an equivalence of triangulated categories $F : \mathcal{D}^{\mathrm{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}B)$ induces an equivalence between the triangulated categories consisting of complexes of finite Gorenstein dimension. As corollaries we show that F induces an equivalence between the projectively stable categories of modules of Gorenstein dimension zero and that G-dim $X = 0$ for all $X \in \hat{\mathcal{G}}_A$ if and only if G-dim $M = 0$ for all $M \in \hat{\mathcal{G}}_B$.

References

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